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REAL GROUP ALGEBRAS*

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Abstract – In this paper we initiate the study of real group algebras and investigate some of its aspects. Let $L^1(G)$ be a group algebra of a locally compact group $G, \tau : G \to G$ be a group homeomorphism such that $\tau^2 = \tau o \tau = 1$, the identity map, and $L^p(G,\tau) = \{f \in L^p(G) : fo\tau = \bar{f}\}$ $(p \geq 1)$. In this paper, among other results, we clarify the structure of $L^p(G,\tau)$ and characterize amenability of $L^1(G,\tau)$ and identify its multipliers.

Keywords – Real Banach algebra, amenability, multiplier, derivation, group involution

1. INTRODUCTION

In 1965, Ingelstam [1] introduced the theory of real Banach algebras. The real function algebra theory was developed further by Kulkarni and Limaye [2]. In their excellent monograph, "Real function algebras", Kulkarni and Limaye present interesting aspects of the theory of $C(X,\tau)$. We refer to [3] for our notations.

Let G be a locally compact group. An automorphism $\tau : G \to G$ is called a topological group involution on G if τ is a homeomorphism and $\tau(\tau(x)) = x$ for all $x \in G$. For example, in group $(C,+)\tau(z) = \bar{z}$ and in $(R \setminus \{0\},.), \tau(x) = x^{-1}$ are topological group involutions. Note that we do not assume that $\tau(xy) = \tau(y)\tau(x)$.

Let $C_o(G,\tau) = \{f \in C_o(G) : fo\tau\ (x) = \overline{f(x)}, x \in G\}$, and $C_c(G,\tau) = \{ f \in C_c(G) :$ $f\ o\ \tau\ (x) = \overline{f(x)}, x \in G\}$ it is clear that, if τ is the identity map on G, then $C_o(G,\tau) = C_o^r(G), C_c(G,\tau) = C_c^r(G)$. If $1 \leq p \leq \infty$, we define $f\ o\ \tau$ (x)= $\overline{f(x)}$, for all $x \in G\}$. Clearly, $L^p(G,\tau) \subseteq L^p(G)$ and if τ is the identity map, $L^p(G,\tau)$ consists of real functions.

2. THE STRUCTURE OF $L^1(G,\tau)$ AND $M(G,\tau)$

Lemma 2. 1. Let G be a locally compact group and τ be a topological group involution on G. If $\sigma : C_c(G) \to C_c(G)$ is defined by $\sigma : (f) = \overline{f} o \tau$, then (i) σ is an algebra involution on $C_c(G)$ and $C_c(G,\tau) = \{f \in C_c(G) \mid \sigma(f) = f\}$,

$$\text{(ii) } C_c(G) = C_c(G,\tau) \oplus iC_c(G,\tau).$$

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Proof. (i) We must show that whenever $f \in C_c(G)$, then $\bar{f} o \tau \in C_c(G)$. To do this, we have supp $(\bar{f} o \tau) = cl([(\bar{f} o \tau)^{-1}\{0\}]' \subseteq \tau^{-1}(\text{supp } \bar{f})$.
It follows that supp$(\bar{f} o \tau)$ is compact, i.e., $(\bar{f} o \tau) \in C_c(G)$. Hence, supp$(\bar{f} o \tau)$ is compact, i.e., $\bar{f} o \tau \in C_c(G)$. The rest of (i) is clear.

(ii) Clearly, $f = \frac{f+\sigma(f)}{2} + i\frac{(f-\sigma(f))}{2i}$. Since $\sigma^2 = i$, (=identity) $\sigma(\frac{(f+\sigma(f))}{2}) = \frac{f+\sigma(f)}{2}$ and $\sigma(\frac{f-\sigma(f)}{2i}) = \frac{f-\sigma(f)}{2i}$. It follows that $f = g + ih$ where $g, h \in C_c(G,\tau)$.
Now if $f = g + ih = g_1 + ih_1$, then $g = \frac{f+\sigma(f)}{2}$,i.e., $g = g_1$ and thus $h = h_1$.

Note. By the same argument one can conclude that $C_0(G) = C_0(G,\tau) \oplus \ iC_0(G,\tau)$. In fact it is enough to show that $\bar{f} o \tau \in C_0(G)$ whenever $f \in C_0(G)$. Since $f \in C_0(G)$, for a given $\varepsilon > 0$, there is a compact set F in G such that $|f(x)| < \varepsilon$ whenever $x \in F'$. Clearly, $\tau^{-1}(F)$ is compact, and if $x \notin \tau^{-1}(F)$, then $\tau(x) \notin F$,i.e., $|\bar{f} o \tau(x)| < \varepsilon$. Therefore, $\bar{f} o \tau \in C_0(G)$.

Let $M(G)$ be the Banach space of all complex regular Borel measures on G. For each $\mu \in M(G)$, we define $\mu_\tau = \mu o \tau$, then it is clear that $\mu_\tau \in M(G)$. Also by Lebesgue dominated convergence theorem one can show that for every bounded Borel measurable function h on G,

$$\int_G h d\mu_\tau = \int_G (h o \tau) d\mu. \tag{1}$$

Clearly, (1) is true when h is a characteristic function; by linearity it holds when h is a simple function; by continuity (1) holds when h is integrable.

Proposition 2. 2. Let $M(G,\tau) = \{\mu \in M(G) \mid \mu o \tau = \bar{\mu}\}$. Then $M(G,\tau)$ is a real Banach algebra with the convolution product $\mu * \nu(E) = \int_G \nu(x^{-1}E) d\mu(x) = \int_G \mu(Ey^{-1}) d\nu(y) \quad (\mu, \nu \in M(G,\tau))$ and $M(G) = M(G,\tau) \oplus iM(G,\tau)$.

Proof. Let $\mu, \nu \in M(G,\tau)$. Then

$$\begin{aligned}(\mu * \nu) o \tau(E) &= \int_G \nu(x^{-1}\tau(E)) d\mu(x) = \int_G \nu(\tau(\tau(x)^{-1}E)) d\mu(x). \\ &= \int_G \overline{\nu((\tau(x)^{-1})E)} d\mu(x) = \int_G \overline{\nu(x^{-1}E)} d\mu o \tau \\ &= \overline{\mu * \nu}(E)\end{aligned} \tag{2}$$

Therefore $\mu * \nu \in M(G,\tau)$. The rest of the proof follows the same line as the proof of Lemma 2.1. Therefore, it is omitted.

Remark. For a real linear space A, the real dual space of A, that is, the space of all real-valued continuous linear functional on A will be denoted by A^*.

Proposition 2. 3. Every real-valued continuous functional ϕ on $C_0(G,\tau)$ can be represented as $\phi(f) = \int_G f d\mu$, where μ is the unique measure in $M(G,\tau)$ such that $\|\psi\| = \|\mu\|$ and vice versa.

Proof. Let $f \in C_0(G,\tau)$. Then $f = g + ih$ where $g, h \in C_0(G,\tau)$. If we define $\psi(f) = \phi(g) + i\phi(h)$, then clearly $\psi \in C_0(G)^*$ and so by the Riesz representation theorem ([3, theorem (14.4)]), there exists a unique measure μ in $M(G)$ such that $\psi(f)=\int_G fd\mu(f\in C_0(G))$ and $\|\psi\|=\|\mu\|$. It follows that $\phi(h)=\int_G hd\mu$ for every h in $C_0(G,\tau)$. Now, in order to prove that $\mu\in M(G,\tau)$, we have $\overline{\psi}(\sigma(f))=\overline{\psi}(g-ih)=\overline{\phi(g)-i\phi(h)}=\psi(f)$. Therefore,

$$\int_G fd\psi=\overline{\int_G \sigma(f)d\mu}=\int_G \overline{\sigma(f)}d\overline{\mu}=\int_G fd\overline{\mu}o\tau \tag{3}$$

$(f\in C_0(G))$. Thus, $\mu=\overline{\mu}o\tau$, i.e. $\mu\in M(G,\tau)$. Also, similar to the proof of [6, Theorem 3.2.1] we can show that $\|\psi\|=\|\mu\|$.

Conversely, let $\mu\in M(G,\tau)$ and $\phi(f)=\int_G fd\mu(f\in C_0(G,\tau))$. If $f\in C_0(G,\tau)$, then $\sigma(f)=f$. Hence,

$$\overline{\phi}(f)=\overline{\phi}(\sigma(f))=\overline{\int_G\sigma(f)d\mu}=\int_G\overline{\sigma(f)}d\overline{\mu}=\int_G(fo\tau)d\overline{\mu} \tag{4}$$

$$=\int_G fd\overline{\mu}o\tau=\int_G fd\mu=\phi(f).$$

Thus $\phi(f)$ is real.

Theorem 2. 4. Let G be a locally compact group with the left Haar measure λ and τ be a topological group involution on G. Then $\lambda o\tau=\lambda$.

Proof. It is easy to show that $\lambda o\tau$ is a positive measure on G. Also if B is a Borel set, then $\lambda o\tau(xB)=\lambda(\tau(xB))=\lambda(\tau(x)\tau(B))=\lambda(\tau(B))=\lambda o\tau(B)(x\in G)$. Therefore, $\lambda o\tau$ is left invariant. So, there is a positive number c such that $\lambda o\tau(B)=c\lambda(B)$ for every Borel set B. If U is an open set, then $\lambda o\tau(\tau(U))=c\lambda(\tau(U))$,i.e., $\lambda(U)=c\lambda(\tau(U))$ which is equal to $c^2\lambda(U)$. Therefore, for every open set U we have $\lambda(U)=c^2\lambda(U)$. So, $c=1$. Hence, $\lambda o\tau=\lambda$.

For a locally compact group G and the Haar measure λ we defined $L^p(G,\tau)=\{f\in L^p(G)\mid fo\tau=\bar{f}\}(1\le p\le\infty)$. Clearly $L^p(G,\tau)\subseteq L^p(G), L^p(G,\tau)$ is a real algebra and $L^p(G)=L^p(G,\tau)\oplus iL^p(G,\tau)$.

Theorem 2. 5. (a) For $1\le p\le\infty$, $L^p(G,\tau)$ is a real Banach space, and $L^2(G,\tau)$ is a real Hilbert space with an inner product,

$$<f,g>=\int_G \overline{g}d\lambda. \tag{5}$$

(b) For each $f,g\in L^1(G,\tau), \max\{\|f\|_p,\|g\|_p\}\le\|f+ig\|\le\|f\|_p+\|g\|_p$.
(c) $L^1(G,\tau)^*=L^\infty(G,\tau)$.
(d) $L^1(G,\tau)$ has a bounded approximate identity of norm 1.

Proof. (a). Clearly, $L^p(G,\tau)$ is a real subspace of $L^p(G)$. Let $f,g\in L^p(G,\tau)$ then $f*g\in L^p(G)$, [4]. We will show that $f*g\in L^p(G,\tau)$. In order to do this, by (2.4) and (1) we have

$$(f*g)(\tau(x))=\int_G f(y)g(y^{-1}\tau(x))d\lambda(y)$$

$$=\int_G f(\tau(\tau(y)))g(\tau(\tau(y))^{-1}x))d\lambda(y) \tag{6}$$

$$=\int_G \overline{f(\tau(y))g(\tau(y^{-1})x)}d\lambda(y)$$

$$=\int_G \overline{f(y)g(y^{-1}x)}d\lambda o\tau(y)$$

$$=\int_G \overline{f(y)g(y^{-1}x)}d\lambda(y)=\overline{(f*g)}(x)$$

for every $x \in G$, hence $f*g \in L^p(G,\tau)$. We now prove that $L^p(G,\tau)$ is complete. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $L^p(G,\tau)$. Since $L^p(G)$ is complete, there exists $f \in L^p(G)$ such that $\lim_{n\to\infty}\| f_n - f \|_p = 0$. Now, there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ as $\{f_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} f_{n_k}(x)=f(x)$, λ-almost everywhere, and so $f(\tau(x))=\lim_{k\to\infty} f_{n_k}(\tau(x))=\lim_{k\to\infty}\bar{f}_{n_k}(x)=\bar{f}(x)$, λ-almost everywhere. Therefore, $f \in L^p(G,\tau)$. Hence $L^p(G,\tau)$ is a real Banach algebra and not a complex algebra.

If $<f,g>=\int_G f\overline{g}d\lambda$ for every $f,g \in L^2(G,\tau)$, then $<f,g>=\overline{<f,g>}$. Therefore $L^2(G,\tau)$ is a real Hilbert space.

(b) For $f,g \in L^1(G,\tau)$ we have $\| f \|_p \le \frac{1}{2}(\| (f+ig) \|_p + \| (f-ig) \|_p) = \| f+ig \|_p$. Similarly, $\| g \|_p \le \| f+ig \|_p$.

(c) We know that $L^1(G)^* \cong L^\infty(G)$. Let $f \in L^1(G)$. So $f = g+ih$ where $g,h \in L^1(G,\tau)$. Now, we define $\psi(f)=\phi(f)+i\phi(g)$ where $\phi \in L^1(G,\tau)^*$. It is clear that $\psi \in L^1(G)^*$ and therefore, there exists a unique $p \in L^\infty(G)$ such that $\psi(f)=\int_G f\ p d\lambda (f \in L^1(G))$.

Hence we have,

$$\overline{\psi}(\sigma(f)) = \overline{\psi}(g-ih) = \overline{\phi(g)-i\phi(h)} = \psi(f). \tag{7} \quad (*)$$

This implies that

$$\int_G f\ p d\lambda = \overline{\int_G \sigma(f) p d\lambda} = \int_G (fo\tau)\overline{p}d\lambda = \int_G f\ \overline{p}o\tau d\lambda \ \ (f \in L^1(G)). \tag{8}$$

Therefore, $po\tau = \overline{p}$, i.e., $p \in L^\infty(G,\tau)$. Also, we have $\phi(f)=\int_G f\ p d\lambda$ for every $f \in L^1(G,\tau)$ and by $(*)$ $\phi(f)$ is real. Conversely, if $\phi: L^1(G,\tau)\to R$ is defined by $\phi(f)=\int_G f$ where $p \in L^\infty(G,\tau)$ and f is an arbitrary function, then $\phi \in L^1(G,\tau)^*$ and the proof is complete.

(d) Let U be any compact neighborhood of e and (U_α) be the collection of all compact neighborhoods of e in U, which is directed by a set inclusion ($\alpha \le \beta$ if and only if $U_\alpha \supseteq U_\beta$). If we define $f_\alpha = \frac{\chi_{U\alpha}}{\lambda(U_\alpha)}$ and $g_\alpha = \frac{\chi_{\tau(U_\alpha)}}{\lambda(U_\alpha)} = \frac{\chi_{U\alpha}o\tau}{\lambda(\tau(U_\alpha))}$, then, since τ is a homeomorphism, $\{f_\alpha\}$ and $\{g_\alpha\}$ are bounded approximate identities of norm one for $L^1(G)$. If we define $e_\alpha = \frac{f_\alpha + g_\alpha}{2}$, then $\{e_\alpha\}$ is a bounded approximate identity of norm one for $L^1(G)$, and also for $L^1(G,\tau)$ since $e_\alpha \in L^1(G,\tau)$.

Lemma 2. 6. For $1 \le p \le \infty$, the linear space $C_c(G,\tau)$ is a dense subspace of $L^p(G,\tau)$.

Proof. Suppose that $f \in L^p(G,\tau)$, since $C_c(G)$ is a dense subspace of $L^p(G)$, there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in $C_c(G)$ such that $\lim_{n\to\infty} \| f_n - f \|_p = 0$. Let $g_n = \frac{f_n + \bar{f}_n o\tau}{2}$. Then $g_n \in C_c(G,\tau)$ and $\lim_{n\to\infty} \| g_n - (\frac{f+\bar{f}o\tau}{2}) \|_p = \lim_{n\to\infty} \| g_n - f \|_p = 0$.

Theorem 2. 7. For $\mu \in M(G,\tau)$ and $\psi \in L^2(G,\tau)$, let $T_\mu \psi = \mu * \psi$. Each T_μ is a bounded operator on the real Hilbert space $L^2(G,\tau)$, and the mapping $\mu \to T_\mu$ is a faithful $*-$representation of $M(G,\tau)$. Note that $M(G,\tau)$ is a $*-$Banach algebra.

Proof. The linearity of T_μ on $L^2(G,\tau)$ is obvious, and the boundedness of T_μ, with $\| T_\mu \| \leq \| \mu \|$, follows from [3,(20.12.ii)]. For $\psi \in L^1(G,\tau) \cap L^2(G,\tau)$, we have

$$(\mu * \nu) * \psi = \mu * (\nu * \psi) \tag{9}$$

[2, (19.2.iv)]. Thus $T_{\mu*\nu}(\psi) = T_\mu(T_\nu \psi)$ for all $\psi \in L^1(G,\tau) \cap L^2(G,\tau)$. Since $C_c(G,\tau) \subseteq L^1(G,\tau) \cap L^2(G,\tau)$, by Lemma (2.7), $L^1(G,\tau) \cap L^2(G,\tau)$ is dense in $L^2(G,\tau)$. It follows that $T_{\mu*\nu} = T_\mu T_\nu$. To show that $T_\mu \neq 0$ if $\mu \neq 0$, consider an $f \in C_c(G,\tau)$ such that $\int_G f^* d\mu \neq 0$. Since $\mu * f(e) = \int_G f^* d\mu \neq 0$ and $\mu * f$ is continuous; thus $T_\mu f$ is not a zero element of $L^2(G,\tau)$. Note that f^* is the involution of f.

3. AMENABILITY AND WEAK AMENABILITY OF REAL GROUP ALGEBRAS

In this section, we show that amenability of $L^1(G,\tau)$ and $L^1(G)$ are equivalent. We shall use some notions of [1].

Definition 3. 1. A Banach algebra A over F is called amenable if for every Banach A-module X over F, $H^1(A, X^*) = \{0\}$.

Let A be a Banach algebra over F, and X be a Banach A-module over F. If $F = R$, we say that X is a real Banach A-module for the real Banach algebra A. If $F = C$, we say X is a Banach A-module for the Banach algebra A.

Definition 3. 2. Let X be a real Banach space. Then $BL_R(X, C)$, consists of all complex-valued continuous real-linear functional on X, which is a real Banach space, denoted by X' and called the complex dual of X.

If A is a real Banach algebra and X is a real Banach A-module, then X' with the natural module action is also a real Banach A-module.

Note that in this case X' is isomorphic to $X^* \times X^*$.

Lemma 3. 3. Let G be a locally compact group and let τ be a topological involution on G. Suppose X is a real Banach $L^1(G,\tau)$-module. Then $H^1(L^1(G,\tau), X') = \{0\}$ if and only if $H^1(L^1(G,\tau), X^*) = \{0\}$.

Proof. It is easy to see that $Z^1(L^1(G,\tau), X') = Z^1(L^1(G,\tau), X^*) \oplus iZ^1(L^1(G,\tau), X^*)$. Now, let $H^1(L^1(G,\tau), X^*) = \{0\}$ and let $D \in Z^1(L^1(G,\tau), X')$. There exist elements a and b in X^* such that $D = \delta_a + i\delta_b$. If $c = a + ib$, then $c \in X'$ and $d = \delta_c$. Hence $H^1(L^1(G,\tau), X') = \{0\}$.

Conversely, we assume that $H^1(L^1(G,\tau),X')=\{0\}$ and let $D\in Z^1(L^1(G,\tau),X^*)$. By the assumption $D\in B^1(L^1(G,\tau),X')$. Clearly, $B^1(L^1(G,\tau),X')=.\ B^1(L^1(G,\tau),X^*)\oplus iB^1(L^1(G,\tau),X^*)$

Hence there exist unique elements D_1,D_2 in $B^1(L^1(G,\tau),X^*)$ such that $D=D_1+iD_2$. On the other hand, $D=D+i0$ where $D,0\in Z^1(L^1(G,Z),X^*)$. Therefore, we have $D_1=D$ and $D_2=0$. Hence $D\in B^1(L^1(G,\tau),X^*)$ and so $H^1(L^1(G,\tau),X^*)=\{0\}$.

Lemma 3. 4. Let $(X,\|.\|)$ be a real Banach space and $X\times X$ be the (complex) linear space under the standard operations of addition and scalar multiplication. If we equip $X\times X$ by the norm $\||.\||$, which satisfies the inequalities

$$\max\{\|x\|,\|y\|\}\le C_1\ \|| \tag{10}$$

and

$$\||(x,y)\||\le C_2\max\{\|x\|,\|y\|\}, \tag{11}$$

for constants C_1 and C_2, then
(i) $(X\times X,\||.\||)$ is a Banach space
(ii) The map $\eta: X\to X\times X$, defined by $\eta(x)=(x,0)$, is a real-linear continuous mapping.
(iii) The map $\psi: X'\to(X\times X)^*$, defined by $\psi(\lambda)(x,y)=\lambda(x)+i\lambda(y)$, is a real-linear continuous mapping onto the real Banach space $(X\times X)^*$.

Proof. (i) and (ii) are clear. (iii) ψ is a well-defined real-linear mapping. For each $\lambda\in X'$ we have

$$\begin{aligned}\|\psi(\lambda)\|&=\sup\{|\psi(x,y)|:\||(x,y)\||\le 1,x,y\in X\}\\ &\le\sup\{|\lambda(x)|+|\lambda(y)|:\|x\|\le C_1,\|y\|\le C_1,x,y\in X\}\\ &\le 2C_1\|\lambda\|.\end{aligned} \tag{12}$$

Hence ψ is continuous. On the other hand, for each $\lambda\in X'$ we have

$$\begin{aligned}\|\psi(\lambda)\|&=\sup\{|\psi(\lambda)(x,0)|:x\in X,\||(x,0)\||\le 1\}\\ &\ge\{|\lambda(x)|:x\in X,C_2\|x\|\le 1\}=C_2^{-1}\|\lambda\|.\end{aligned} \tag{13}$$

Hence ψ is one-to-one. To show that ψ is onto, let $\Lambda\in(X\times X)^*$. Then $\Lambda o\eta\in X'$ and $\psi(\Lambda o\eta)=\Lambda$.

Theorem 3. 5 Let G be a locally compact group and τ be a topological involution on G. Then $L^1(G,\tau)$ is amenable if and only if $L^1(G)$ is amenable.

Proof. Let $L^1(G,\tau)$ be amenable, X be a Banach $L^1(G)$-module and $\Delta\in Z^1(L^1(G),X^*)$. If X_R represents X as a real Banach space then it is a real Banach $L^1(G,\tau)$-module under the module actions defined by

$$f.x=(f+i0).x, \quad x.f=x.(f+i0),(f\in L^1(G,\tau),x\in X) \tag{14}$$

Now we define the map $D:L^1(G,\tau)\to X_R^*$ by $Df=\operatorname{Re}\Delta(f+i0)$. Clearly D is a real-linear mapping and since for each $f\in L^1(G,\tau)$

$$\|Df\|\le \sup\{|\Delta(f+i0)(x)|: x\in X,\|x\|\le 1\}\le\|\Delta\|\,\|f\|, \tag{15}$$

D is continuous. On the other hand, for each $f,g\in L^1(G,\tau)$,

$$(Df).g=\operatorname{Re}(\Delta(f+i0).(g+i0)), f.(Dg)=\operatorname{Re}((f+i0).\Delta(g+i0)). \tag{16}$$

Hence $D(fg)=(Df).g+f.(Dg)$ and so $D\in Z^1(L^1(G,\tau),X_R^*)$. The amenability of $L^1(G,\tau)$ implies that there exists $u\in X_R^*$ such that $D=\delta_u$. Now we define $\lambda:X\to\mathbb{C}$ by $\lambda(x)=u(x)-iu(ix)$. Clearly $\lambda\in X^*$ and for $f\in L^1(G,\tau),x\in X$ we have

$$(\lambda..(f+i0))(x)=u(f.x)-iu(f.ix),((f+i0).\lambda)(x)=u(x.f)-iu(ix.f). \tag{17}$$

We can show that $\Delta(f+ig))(x)=(\delta_\lambda(f+ig))(x)$ for every $f,g\in L^1(G,\tau)$ and $x\in X$. Hence $\Delta=\delta_\lambda$ and so Δ is an inner derivation, i.e. $H^1(L^1(G),X^*)=\{0\}$ Thus $L^1(G)$ is amenable.

Conversely, let $L^1(G)$ be amenable and let X be a real Banach $L^1(G,\tau)$-module. By Lemma 3.3 it is enough to show that $H^1(L^1(G),X')=\{0\}$. Let $D:L^1(G,\tau)\to X'$ be a continuous real derivation. By Lemma 3.4, $X\times X$ is a Banach space under the norm $|||\,(x,y)\,|||=\max\{\|x\|,\|y\|\}$. The map $\psi:X'\to(X\times X)^*$, defined by $\psi(\lambda)(x,y)=\lambda(x)+i\lambda(y)(x,y\in X,\lambda\in X')$ is a continuous real-linear mapping which is one-one and onto. The space $X\times X$ is a Banach $L^1(G)$-module under the familiar module actions. Now we define the map $\Delta:L^1(G)\to(X\times X)^*$ by $\Delta(f+ig)=\psi(Df)+i\psi(Dg)$. Clearly Δ is a complex linear mapping and for $f,g\in L^1(G)$,

$$\|\Delta(f+ig)\|\le\|\psi\|\,\|D\|\,\|f\|_1+\psi\ \|D\|\,\|g\|\le 2\|\psi\|\,\|D\|\max\{\|f\|_1,\|g\|_1\} \tag{18}$$

$$\le 2\|\psi\|\,\|D\|\,\|f+ig\|_1.$$

Hence Δ is continuous. Considering the module actions on $X\times X$ we can show that

$$\psi((Df).g)=\psi(Df).(g+i0) \tag{19}$$

and

$$\psi(f.(Dg))=(f+i0).\psi(Dg) \tag{20}$$

Since D is a X'-derivation on $L^1(G,\tau)$, by using the above equation we have

$$\Delta((f_1+ig_1).(f_2+ig_2))=(\Delta(f_1+ig_1).(f_2+ig_2)+(f_1+ig_1).(\Delta(f_2+ig_2)).$$

Therefore, $\Delta\in Z^1(L^1(G),(X\times X)^*)$ and so there exists $\Lambda\in(X\times X)^*$ such that $\Delta=\delta_\Lambda$. Since ψ is onto and one-to-one there exists a unique $\lambda\in X'$ such that $\Lambda=\psi(\lambda)$.

Now we notice that $\psi(f.\eta)=(f+i0).\psi(\eta)$ and $\psi(\eta.f)=\psi(\eta).(f+i0)$ for every $f\in L^1(G,\tau)$ and $\eta\in X'$. Hence

$$\psi(Df)=\psi(Df)+i\psi(D0)=\Delta(f+i0)=\delta_\Lambda(f+i0)$$

$$(f+i0).\Lambda-\Lambda.(f+i0)=(f+i0).\psi(\lambda)-\psi(\lambda).(f+i0) \tag{21}$$
$$=\psi(f.\lambda)-\psi(\lambda.f)=\psi(\delta_\lambda(f)).$$

Since ψ is one-to-one, it implies that $D(f)=\delta_\lambda(f)$ for each $f\in L^1(G,\tau)$ and $D=\delta_\lambda$. This completes the proof.

Theorem 3. 6. Let G be a locally compact group and let τ be a topological involution on G. Then $L^1(G,\tau)$ is weakly amenable if and only if $L^1(G)$ is weakly amenable.

Proof. Let $L^1(G,\tau)$ be a weakly amenable real Banach algebra. We show that for each $\Delta\in Z^1(L^1(G),L^1(G)^*)$ there exists $\Lambda\in L^1(G)^*$ such that $\Delta=\delta_\Lambda$. Let $\eta:L^1(G,\tau)\to L^1(G)$ be defined by $\eta(f)=f+i0$ and $\psi:L^1(G,\tau)'\to L^1(G)^*\times L^1(G)^*$ be defined by $\psi(\lambda)(f+ig)=\lambda(f)+i\lambda(g)$. By Lemma 3.4, η and ψ are continuous real-linear mapping. Also, ψ is a one-to-one and onto mapping from the real Banach space $L^1(G,\tau)'$ onto $L^1(G)^*$ as a real Banach space. By the open Mapping Theorem for real Banach spaces, $\psi^{-1}:L^1(G)^*\times L^1(G)^*\to L^1(G,\tau)'$ is a real-linear continuous mapping. Now if we define $D=\psi^{-1}o\Delta o\eta$, then it is easy to see that D is a real-linear continuous mapping. To show that D is an $L^1(G,\tau)'$-derivation on $L^1(G,\tau)$ we see that

$$\psi(D(fg))=(\psi oD)(fg)=\Delta(fg+i0)=\Delta((f+i0).(g+i0))$$
$$=(\Delta o\eta)(f).(g+i0)+(f+i0).(\Delta o\eta)(g) \tag{22}$$
$$=\psi(Df).(g+i0)+(f+i0).\psi(Dg).$$

On the other hand, $\psi(\mu)(f+i0)=\psi(\mu.f)$ and $(f+i0).\psi(\mu)=\psi(a.\mu)$ for $f\in L^1(G)$ and $\mu\in L^1(G,\tau)'$. Hence, for each $f,g\in L^1(G,\tau)$ we have

$$\psi(D(fg))=\psi(Df.g)+\psi(f.Dg)=\psi(Df.g+f.Dg). \tag{23}$$

Since ψ is one-one, we conclude that D is an $L^1(G,\tau)'$-derivation, i.e. $D\in Z^1(L^1(G,\tau),L^1(G,\tau)')$. By Lemma 3.3, the weak amenability of $L^1(G,\tau)$ implies that there exists $\lambda\in L^1(G,\tau)'$ such that $D=\delta_\lambda$. By definition of D and the above equalities it implies that $\Delta=\delta_{\psi(\lambda)}$, and so $L^1(G)$ is weakly amenable.

Conversely, let $L^1(G)$ be weakly amenable and $D\in Z^1(L^1(G,\tau),L^1(G,\tau)')$. By Lemma 3.4 the map $\psi:L^1(G,\tau)'\to L^1(G)^*\times L^1(G)^*$, defined by $\psi(\lambda)(f+ig)=\lambda(f)+i\lambda(g)$, is a real-linear continuous one-to-one mapping onto $L^1(G)^*\times L^1(G)^*$, as a real Banach space.

Now we define the map $\Delta:L^1(G)\to L^1(G)^*\times L^1(G)^*$ by $\Delta(f+ig)=\psi(Df)+i\psi(Dg)$. Similar to the proof of Theorem 3.4 we can show that Δ is a continuous derivation. Hence there exists $\Lambda\in L^1(G)^*\times L^1(G)^*$ such that $\Delta=\delta_\Lambda$. Since ψ is one-to-one and onto, there exists a unique $\lambda\in L^1(G,\tau)'$ such that $\Lambda=\psi(\lambda)$. It can be shown that $D=\delta_\lambda$ and so $L^1(G,\tau)$ is weakly amenable by Lemma 3.3.

Corollary 3. 7. Let G be a locally compact group and τ be a topological involution on G. Then

(i) $L^1(G,\tau)$ is amenable if and only if G is amenable.

(ii) $L^1(G,\tau)$ is weakly amenable.

Proof. By Theorem 3.5 and 3.6 the amenability and weak amenability of $L^1(G,\tau)$ and $L^1(G)$ are equivalent. Since $L^1(G)$ is amenable if and only if G is amenable [5], (i) follows. Since $L^1(G)$ is weakly amenable [6], we conclude that $L^1(G,\tau)$ is also weakly amenable.

4. MULTIPLIERS

In this section we characterize the multipliers of $L^1(G,\tau)$. A bounded real linear operator T on $L^1(G,\tau)$ is called a left (right) multiplier if $T(f*g)=(Tf)*g(=f*Tg)\quad f,g\in L^1(G,\tau))$.

Definition 4. 1. Let δ_x be the point mass at $x\in G$. We define $m_x=\dfrac{\delta_x+\delta_{\tau(x)}}{2}$ and $R_x(f)=m_x*f(f\in L^1(G,\tau))$.

It is clear that $m_x o\tau=m_x$ (since $\tau^2=\tau$) and $\|m_x\|=1$. Therefore, $m_x\in M(G,\tau)$.

Lemma 4. 2. Let μ be a measure in $M(G)$ such that $f*\mu\in L^1(G,\tau)$ for every $f\in L^1(G,\tau)$. Then $\mu\in M(G,\tau)$.

Proof. We have $f*\mu(x)=\int_G f(xy^{-1})d\mu((y)\ (x\in G)$. Therefore,

$$\begin{aligned} f*(\overline{\mu}o\tau)(x) &= \int_G f(xy^{-1})d(\overline{\mu}o\tau)(y) \\ &= \int_G f(x(\tau(y^{-1}))d\overline{\mu}(y) \\ &= \int_G \bar{f}(\tau(x)y^{-1})d\overline{\mu}(y) \qquad (24)\\ &= \int_G f(\tau(x)y^{-1})d\overline{\mu}(y) \\ &= \overline{f*\mu(\tau(x))}=f*\mu(x). \end{aligned}$$

So $f*(\overline{\mu}o\tau)=f*\mu$ for every f in $L^1(G,\tau)$. Since $L^1(G,\tau)$ has a bounded approximate identity, we have $\overline{\mu}o\tau=\mu$. Hence $\mu\in M(G,\tau)$.

Theorem 4. 3. Let T be a left multiplier on $L^1(G,\tau)$. Then there exists a unique $\mu\in M(G,\tau)$ such that $Tf=f*\mu(f\in L^1(G,\tau))$ and $\|\mu\|=\|T\|$.

Proof. We define $T_0:L^1(G)\to L^1(G)$ by $T_0(f)=T(g)+iT(h)$ where $f=g+ih$. We have

$$\begin{aligned} T_0(f_1*f_2) &= T_0((g_1+ih_1)*(g_2+ih_2)) \\ &= T_0(g_1*g_2+ih_1*g_2+ig_1*h_2-h_1*h_2) \\ &= T_0(g_1*g_2-h_1*h_2)+iT(h_1*g_2+g_1*h_2) \qquad (25)\\ &= g_1*Tg_2-h_1*Th_2+ih_1*Tg_2+ig_1*Th_2 \\ &= (g_1+ih_1)*(Tg_2+iTh_2) \\ &= f_1*T_0f_2. \end{aligned}$$

Hence, T_0 is a left multiplier on $L^1(G)$. Therefore, by [7] there exists a unique $\mu \in M(G,\tau)$ such that $T_0 h = h * \mu$ for every h in $L^1(G)$ and $\| T_0 \| = |\mu|(G)$. Consequently, $Tf = f * \mu$ for every $L^1(G,\tau)$ and $\| T \| \le \| T_0 \| = |\mu|(G)$. Now, since $f * \mu \in L^1(G,\tau)$ for all $f \in L^1(G,\tau)$, $\mu \in M(G,\tau)$ by Lemma 4.2. But by the proof of [7, Theorem 1] we have $\mu = \omega^* - \lim_\alpha T_0 e_\alpha = \omega^* - \lim_\alpha Te_\alpha$ .Now, since $\| Te_\alpha \| \le \| T \|$,

$$|\mu|(G) \le \| T \|. \tag{26}$$

Therefore, $\| T \| = \| \mu \|$.

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