Iranian Journal of Science & Technology, Transaction A, Vol. 28, No. A2
Printed in Islamic Republic of Iran, 2004

THE STRONG LAW OF LARGE NUMBERS FOR PAIRWISE NEGATIVELY DEPENDENT RANDOM VARIABLES*

H. R. NILI SANI, H. A. AZARNOOSH AND A. BOZORGNIA**

Department of Statistics, School of Mathematical Sciences, Ferdowsi University, Mashhad, I. R. of Iran
Nili@math.um.ac.ir, Azarnush@math.um.ac.ir, Bozorg@math.um.ac.ir

Abstract – In this paper, strong laws of large numbers (SLLN) are obtained for the sums $\sum_{i=1}^{n} X_i$, under certain conditions, where $\{X_n, n \geq 1\}$ is a sequence of pairwise negatively dependent random variables.

Keywords – Strong law of large numbers, pairwise negatively dependent random variables

1. INTRODUCTION AND PRELIMINARIES

In many stochastic models, the assumption of independence among random variables (henceforth r.v.'s) is not plausible. In fact, increases in some r.v.'s are often related to decreases in other r.v.'s, and the assumption of pairwise negative dependence is more appropriate than the independence assumption. Let $\{X_n, n \geq 1\}$ be a sequence of integrable r.v.'s defined on the same probability space, and put $S(n) = \sum_{i=1}^{n} X_i$ and $\overline{X}_n = S(n)/n$. Chandra and Goswami [1] modified Kolmogrov 's SLLN (Theorem 5.4.2 of Chung [2]) and the SLLN of Landers and Rogge [3] for pairwise independent r.v.'s which are not necessarily identically distributed and satisfy certain moment conditions. Matula [4] has proved the SLLN for pairwise negatively dependent r.v.'s with the same distribution. Bozorgnia et al. [5] obtained the SLLN for weighted sums of an array of rowwise negatively dependent r.v.'s under certain moment conditions. Amini [6] has proved the SLLN for special negatively dependent r.v.'s and for weighted sums of uniformly bounded negatively dependent r.v.'s. He has also proved the WLLN for special pairwise negatively dependent r.v.'s. In this paper, we extend some of the theorems of SLLN of Chandra and Goswami [1] for pairwise negatively dependent r.v.'s which are not necessarily identically distributed, but satisfy certain moment conditions.

---

*Received by the editor February 16, 2002 and in final revised form September 20, 2003
**Corresponding author

Definition 1: The random variables $X_1, \cdots, X_n (n \geq 2)$ are said to be pairwise negatively dependent (henceforth pairwise *ND*) if

$$P(X_i > x_i, X_j > x_j) \leq P(X_i > x_i) P(X_j > x_j), \tag{1}$$

for all $x_i,\ x_j \in R\ , i \neq j$. It can be shown that (1) is equivalent to

$$P(X_i \leq x_i, X_j \leq x_j) \leq P(X_i \leq x_i) P(X_j \leq x_j), \tag{2}$$

for all $x_i,\ x_j \in R\ , i \neq j$.

Definition 2: The random variables $X_1, \cdots, X_n (n \geq 2)$ are said to be negatively associated (*NA* for short) if for every pair of disjoint nonempty subsets A_1, A_2 of $\{1, \ldots, n\}$,

$$Cov(f_1(X_i, i \in A_1), f_2(X_i, i \in A_2)) \leq 0 \tag{3}$$

whenever f_1 and f_2 are coordinatewise increasing (or decreasing) such that this covariance exists.

An infinite collection of $\{X_n\ , n \geq 1\}$ is said to be pairwise *ND* (negatively associated) if every finite subcollection is pairwise *ND* (negatively associated).

It can be shown that *NA* implies pairwise *ND* and for $n = 2$, pairwise *ND* is equivalent to *NA* (See Property P_3 of Joag-Dav and Proschan [7]).

Lemma 1([6]): Let $\{X_n\ , n \geq 1\}$ be a sequence of pairwise *ND* r.v.'s. If $\{f_n\ , n \geq 1\}$ is a sequence of Borel functions, all of which are monotone increasing (or all are monotone decreasing), then $\{f_n(X_n), n \geq 1\}$ is a sequence of pairwise *ND* r.v.'s.

Corollary 1: Let $\{X_n\ , n \geq 1\}$ be a sequence of pairwise *ND* r.v.'s. Then $\{X_n^+, n \geq 1\}$ and $\{X_n^-, n \geq 1\}$ are two sequences of pairwise *ND* r.v.'s where X_n^+ and X_n^- are the positive and the negative parts, respectively, of the random variable X_n.
The theorem below can be obtained from the arguments of Csörgo et al. [8].

Theorem 1([1]): Let $\{X_n\ , n \geq 1\}$ be a sequence of non-negative r.v.'s with finite $Var(X_n)$. If

(i) $$\sup_{n \geq 1} \left[\sum_{k=1}^{n} E(X_k) / f(n) \right] = C < \infty,$$

(ii) there is a double sequence $\{\rho_{ij}\}$ of non-negative real numbers such that

$$Var(S(n)) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \quad \text{for each } n \geq 1,$$

(iii) $$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho_{ij} / (f(i \vee j))^2 < \infty, \quad i \vee j = \max(i, j).$$

Then

$$[S(n) - E(S(n))] / f(n) \to 0 \text{ a.s.} \qquad \text{as } n \to \infty.$$

2. MAIN RESULTS

In this paper, C stands for a generic constant not necessarily the same in each appearance. Also, $\{f(n)\}$ will stand for an increasing sequence such that $f(n) > 0$ for each n and $f(n) \to \infty$.
In this section, we extend some limited theorems for pairwise *ND* random variables with finite variances and certain conditions.

Theorem 2: Let $\{X_n, n \geq 1\}$ be a sequence of pairwise *ND* r.v.'s with finite $Var(X_n)$. If

$$\text{(a)} \qquad \sup_{n\geq 1}\left[\sum_{k=1}^{n} E(|X_k - E(X_k)|)/f(n)\right] < \infty,$$

and

$$\text{(b)} \qquad \sum_{n=1}^{\infty} (f(n))^{-2} Var(X_n) < \infty.$$

Then

$$[S(n) - E(S(n))]/f(n) \to 0 \text{ a.s.} \qquad \text{as } n \to \infty.$$

Proof: We put $Y_n = (X_n - E(X_n))^+$ and $Z_n = (X_n - E(X_n))^-$, $n \geq 1$. It is sufficient to show that as $n \to \infty$,

$$(f(n))^{-1} \sum_{i=1}^{n} (Y_i - E(Y_i)) \to 0 \text{ a.s, and } (f(n))^{-1} \sum_{i=1}^{n} (Z_i - E(Z_i)) \to 0 \text{ a.s.} \tag{4}$$

Since $E(Y_n) \leq E|X_n - E(X_n)|$ $(n \geq 1)$, it follows that condition (i) of Theorem 1 is valid for $\{Y_n\}$. Similarly, it is valid for $\{Z_n\}$. Under the pairwise *ND* condition we have

$$Var(\sum_{i=1}^{n} Y_i) \leq \sum_{i=1}^{n} Var(Y_i) \leq \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij} \qquad n \geq 1,$$

where $\rho_{ii} = Var(X_i)$, $i = j$ and $\rho_{ij} = 0$ for $i \neq j$. It follows from Theorem 1 that

$$\frac{1}{f(n)} \sum_{i=1}^{n} (Y_i - E(Y_i)) \to 0 \qquad \text{a.s.}$$

Replacing X_n by $W_n = -X_n$ and $Z_n = (X_n - E(X_n))^-$ by $Z_n = (W_n - E(W_n))^+$ one gets the second part of (4). Since

$$\frac{S(n) - E(S(n))}{f(n)} = \frac{\sum_{i=1}^{n}(Y_i - E(Y_i)) - \sum_{i=1}^{n}(Z_i - E(Z_i))}{f(n)} + \frac{(\sum_{i=1}^{n} E(Y_i) - \sum_{i=1}^{n} E(Z_i))}{f(n)},$$

we have $\dfrac{S(n) - E(S(n))}{f(n)} \to 0$ a.s.

Example 1: Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables and $f(n) = \alpha^n$, $\alpha > 1$. It is obvious that conditions of Theorem 2 hold and we have $\frac{S(n) - E(S(n))}{f(n)} \to 0$ a.s.

Example 2: Let $\{X_n, n \geq 1\}$ and $f(n)$ be as above, $Y_n = -a_n X_n, a_n > 0$ and $a_n = O(n^\beta)$, $\beta > 0$. Put $Z_{2n} = X_n$, $Z_{2n-1} = Y_n$ and $S(n) = \sum_{i=1}^{n} Z_i$. It is obvious that $\{Z_n\}$ is a sequence of pairwise *ND* r.v.'s.

$$\sup_{n\geq 1}\left[\sum_{k=1}^{2n} E(|Z_k - E(Z_k)|)/f(2n)\right] = \sup_{n\geq 1}\left[\sum_{k=1}^{n} E(|X_k - E(X_k)|)/f(2n)\right.$$

$$\left. + \sum_{k=1}^{2n} E(|a_k X_k - E(a_k X_k)|)/f(2n)\right]$$

$$= \sup_{n\geq 1}\left[E(|X_1 - E(Z_1)|)(n + \sum_{k=1}^{n} a_k)/f(2n)\right] < \infty.$$

It is easy to show that Condition (a) of Theorem 2 holds. Also

$$\sum_{n=1}^{\infty}(f(n))^{-2} Var(Z_n) = \sum_{n=1}^{\infty}(f(2n))^{-2} Var(Z_{2n}) + \sum_{n=1}^{\infty}(f(2n-1))^{-2} Var(Z_{2n-1})$$

$$= \sum_{n=1}^{\infty}(f(2n))^{-2} Var(X_1) + \sum_{n=1}^{\infty}(f(2n-1))^{-2} a_n^2 Var(X_1) < \infty.$$

Then, by Theorem 2, $\frac{S(n) - E(S(n))}{f(n)} \to 0$ a.s.

Theorem 3: Let $\{X_n, n \geq 1\}$ be a sequence of pairwise *ND* integrable r.v.'s and $\{B_n, n \geq 1\}$ be a sequence of semi intervals $(-\infty, x_n)$ ($(-\infty, x_n]$, $[x_n, \infty)$ or (x_n, ∞)) satisfying the following conditions:

(a) $\sum_{n=1}^{\infty} C_n P(X_n \in B_n^c) < \infty$ where $C_n = 1 \vee (\frac{x_n}{f(n)})^2$,

(b) $\sum_{i=1}^{n} E(X_i I(X_i \in B_i^c)) = o(f(n))$,

(c) $\sum_{n=1}^{\infty}(f(n))^{-2} Var(X_n I(X_n \in B_n)) < \infty$,

and

(d) $\sup_{n\geq 1}[\sum_{k=1}^{n} E(|X_k| I(X_k \in B_k)) / f(n)] < \infty$,

here B_n^c is the complement of B_n. Then

$$[S(n) - E(S(n))]/f(n) \to 0 \text{ a.s.} \qquad \text{as } n \to \infty.$$

Proof: Let $Y_n = X_n I(X_n \in B_n) + x_n I(X_n \in B_n^c)$, $n \geq 1$. By Lemma 1, $\{Y_n\}$ is a sequence of pairwise *ND* r.v.'s. By (a), (c) and (d), Theorem 2, applied to $\{Y_n\}$, yields $(f(n))^{-1}\sum_{i=1}^{n}(Y_i - E(Y_i)) \to 0$ a.s. as $n \to \infty$. By (a) and (b), we get $(f(n))^{-1}\sum_{i=1}^{n}(Y_i - E(X_i)) \to 0$ a.s. as $n \to \infty$. Since, by condition (a) the r.v.'s $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are equivalent, hence by (a) and the first Borel-Cantelli lemma, the desired result follows.

The next theorem is an analogue to Kolmogrov's classical SLLN for independent and identically distributed r.v.'s. Our intention is to replace the conditions of independent and identical distribution by suitable weaker conditions of simple nature.

Theorem 4: Let $\{X_n, n \geq 1\}$ be a sequence of pairwise *ND* r.v.'s and set $G(x) = \sup_{n\geq 1} P(|X_n| \geq x)$ for $x \geq 0$. If

$$\int_0^\infty G(x)dx < \infty, \tag{5}$$

then $n^{-1}\sum_{i=1}^{n}\gamma_i(X_i - E(X_i)) \to 0$ a.s. as $n \to \infty$ for each bounded non-negative (or non-positive) sequence $\{\gamma_n\}$.

Proof: Put $Y_n = X_n^{+}$ and $Z_n = X_n^{-}$ ($n \geq 1$). It is sufficient to show that as $n \to \infty$,

$$(n)^{-1}\sum_{i=1}^{n}\gamma_i(Y_i - E(Y_i)) \to 0 \text{ a.s, and } (n)^{-1}\sum_{i=1}^{n}\gamma_i(Z_i - E(Z_i)) \to 0 \text{ a.s.} \tag{6}$$

Also, it is sufficient to prove the first part of (6) for $\gamma_n = 1$. To this end, we use Theorem 3 with $B_n = (-\infty, n]$ for all $n \geq 1$. It is obvious that $C_n = 1$ for all $n \geq 1$. Since

$$\sum_{n=1}^{\infty} P(Y_n \in B_n^c) = \sum_{n=1}^{\infty} P(Y_n > n) \leq \sum_{n=1}^{\infty} P(|X_n| > n) \leq \sum_{n=1}^{\infty} G(n) < \infty,$$

it follows that condition (a) of Theorem 3 is valid for $\{Y_n, n \geq 1\}$. To verify condition (b), note that for any non-negative random variable Z and $\alpha \geq 0$,

$$E(ZI(Z \geq \alpha)) = \alpha P(Z \geq \alpha) + \int_\alpha^\infty P(Z \geq x)dx.$$

Hence

$$E(Y_n I(Y_n > n)) \leq E(|X_n| I(|X_n| > n)) \leq n P(|X_n| > n) + \int_n^\infty G(x)dx \to 0,$$

so that condition (b) holds for $\{Y_n, n \geq 1\}$. Obviously, condition (d) holds for $\{Y_n, n \geq 1\}$. To obtain the first part in (6), it remains to verify condition (c).

$$\sum_{n=1}^{\infty} n^{-2} E(Y_n^2 I(Y_n \le n)) = \sum_{n=1}^{\infty} n^{-2} \int_0^{\infty} P(Y_n^2 I(X_n \le n) > x)dx$$

$$= \sum_{n=1}^{\infty} n^{-2} (\int_0^{n^2} P(\sqrt{x} < X_n \le n))dx \le \sum_{n=1}^{\infty} n^{-2} \int_0^{n} 2y(P(X_n > y))dy$$

$$\le \sum_{n=1}^{\infty} 2n^{-2} \int_0^{n} yG(y)dy = 2\sum_{i=1}^{\infty}\sum_{n=i}^{\infty} n^{-2} \int_{i-1}^{i} yG(y)dy$$

$$\le 2C \sum_{i=1}^{\infty} \frac{1}{i} \int_{i-1}^{i} yG(y)dy < \infty .$$

The next theorem is an analogue of SLLN of Chung [9]; for other related results, it may be interesting to review Chung's paper [9].

Theorem 5: Let $\{X_n, n \ge 1\}$ be a sequence of pairwise *ND* r.v.'s, $\{a_n\}$ be a sequence of positive constants such that $\{\frac{a_n}{f(n)}\}$ is a bounded sequence and

$$\sup_{n \ge 1} \left[\frac{1}{f(n)} \sum_{k=1}^{n} E(|X_k| I(|X_k| \le a_k)) \right] < \infty.$$

Let $g_n : (0,\infty) \to (0,\infty)$ be a sequence of functions, $g_n(0)$ being defined arbitrarily, such that for each $n \ge 1$

i) $\quad \frac{g_n(x)}{x} \uparrow \quad$ and $\quad \frac{g_n(x)}{x^2} \downarrow ;$

and

ii) $$\sum_{n=1}^{\infty} \frac{E(g_n(|X_n|))}{g_n(a_n)} < \infty .$$

Then

$$[S(n) - E(S(n))] / f(n) \to 0 \text{ a.s.} \qquad \text{as } n \to \infty .$$

Proof: We use Theorem 3 with $B_n = (-\infty, a_n]$, $a_n > 0$. Put $Y_n = X_n^+$ and $Z_n = X_n^-$, $n \ge 1$. It suffices to show that as $n \to \infty$,

$$(f(n))^{-1} \sum_{i=1}^{n} (Y_i - E(Y_i)) \to 0 \text{ a.s,} \quad \text{and} \quad (f(n))^{-1} \sum_{i=1}^{n} (Z_i - E(Z_i)) \to 0 \text{ a.s .} \qquad (7)$$

It is obvious that $C_n < C$ for all $n \ge 1$. Also it is sufficient to prove the first part of (7). To verify condition (a) note that

$$\sum_{n=1}^{\infty} C_n P(Y_n > a_n) \le C \sum_{n=1}^{\infty} P(|X_n| > a_n) \le C \sum_{n=1}^{\infty} P(g_n |X_n| \ge g_n(a_n)) < \infty .$$

Next, note that

$$\sum_{n=1}^{\infty} E(Y_n I(Y_n > a_n)/f(n)) \le \sum_{n=1}^{\infty} \frac{1}{f(n)} E(\frac{|X_n|}{g_n(|X_n|)} g_n(|X_n|) I(|X_n| \ge a_n))$$

$$\le \sum_{n=1}^{\infty} \frac{a_n}{f(n) g_n(a_n)} E(g_n(|X_n|)) < \infty,$$

so that condition (b) is followed by Kronecker lemma (see Page 123 of Chung [2]). Condition (c) follows, since

$$\sum_{n=1}^{\infty} \frac{1}{(f(n))^2} E(Y_n^2 I(Y_n \le a_n)) \le \sum_{n=1}^{\infty} \frac{1}{(f(n))^2} E(\frac{Y_n^2}{g_n(Y_n)} g_n(Y_n) I(Y_n \le a_n))$$

$$\le \sum_{n=1}^{\infty} \frac{a_n^2}{f^2(n)} \frac{E g_n(|X_n|)}{g_n(a_n)} < \infty.$$

It follows that the first part of (7) holds.

Corollary 2: If $\{X_n, n \ge 1\}$ is a sequence of *NA* r.v.'s, then Theorems (2-5) are valid.

***Acknowledgements*-** The authors wish to thank the referees and the editor for their useful comments and suggestions.

REFERENCES

1. Chandra, T. K. & Goswami, A. (1992). Cesaro uniform integrability and the strong law of large numbers, *Sankhya*: *The Indian Journal of Statistics*, 54, 215-231.
2. Chung, K. L. (1974). *A course in probability theory*, second edition, New York, Academic Press.
3. Landers, D. & Rogge, L. (1987). Laws of large numbers for pairwise independent uniformly integrable random variables, M*ath. Nachr.* 130, 189-192.
4. Matula, P. (1992). A note on the almost sure convergence of sums of negatively dependent random variables*, Stat. Probab. Letters,* 15, 209-213.
5. Bozorgnia, A., Patterson, R. F. & Taylor, R. L. (1996). Limit Theorems for dependent random variables, *World Congress Nonlinear Analysts,* 92, 1639-1650.
6. Amini, M. (2000). Some contribution to limit theorems for negatively dependent random variables, Ph.D. Thesis*,* Mashhad University.
7. Joag-Dav, K. & Proschan, F. (1983). Negative association of random variables with applications, *Ann. of Stat.*, 11, 286-296.
8. Csörgo, S., Tandori, K. & Totik, V. (1983). On the strong law of large numbers for pairwise independent random variables, *Acta Math. Hungarica*, 42, 319-330.
9. Chung, K. L. (1947). Note on some strong laws of large numbers, *Amer. J. Math*., 69, 189-192.