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"Research Note"

CONFORMAL VECTOR FIELDS ON TANGENT BUNDLE OF A RIEMANNIAN MANIFOLD*

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Abstract – Let M be an *n*-dimensional Riemannian manifold and TM its tangent bundle. The conformal and fiber preserving vector fields on TM have well-known physical interpretations and have been studied by physicists and geometricians. Here we define a Riemannian or pseudo-Riemannian lift metric $\tilde{g}$ on TM, which is in some senses more general than other lift metrics previously defined on TM, and seems to complete these works. Next we study the lift conformal vector fields on $(TM,\tilde{g})$ and prove among the others that, every complete lift conformal vector field on TM is homothetic, and moreover, every horizontal or vertical lift conformal vector field on TM is a Killing vector.

Keywords – Complete lift metric, Conformal, Homothetic, Killing and Fiber-preserving vector fields.

1. INTRODUCTION

Let M be an *n*-dimensional differential manifold with a Riemannian metric g and ϕ be a transformation on M. Then ϕ is called a *conformal* (resp. *projective*) transformation if it preserves the angles (resp. geodesics). Let V be a vector field on M and $\{\varphi_t\}$ be the local one-parameter group of local transformations on M generated by V. Then V is called an *infinitesimal conformal* (resp. *projective) transformation* on M if each φ_t is a local conformal (resp. projective) transformation of M. It is well known that V is an infinitesimal conformal transformation or *conformal vector field* on M if and only if there is a scalar function ρ on M such that $\pounds_V g = 2\rho g$, where $\pounds_V$ denotes Lie derivation with respect to the vector field V. V is called *homothetic* if ρ is constant and is called an *isometry* or *Killing vector field* when ρ vanishes.

Let TM be the tangent bundle over M, and Φ be a transformation on TM. Then Φ is called a *fiber preserving* transformation if it preserves the fibers. Fiber preserving transformations have well known applications in Physics. Let X be a vector field on TM and $\{\Phi_t\}$ the local one parameter group of local transformation on TM generated by X. Then X is called an *infinitesimal fiber preserving transformation* or *fiber preserving vector field* on TM if each Φ_t is a local fiber preserving transformation of TM.

Let $\tilde{g}$ be a Riemannian or pseudo-Riemannian metric on TM. The conformal vector field X on TM is said to be *essential* if the scalar function Ω on TM in $\pounds_X \tilde{g} = 2\Omega\tilde{g}$ depends only on (y^h) (with

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respect to the induced coordinates (x^i, y^i) on TM), and is said to be *inessential* if Ω depends only on (x^h). In other words, Ω is a function on M.

There are some lift metrics on TM as follows:

complete lift metric or g_2, *diagonal* lift metric or $g_1 + g_3$, lift metric $g_2 + g_3$ and lift metric $g_1 + g_2$.

In this area the following results are well known:

Let (M, g) be a Riemannian manifold. If we consider TM with metrics $g_1 + g_3$ or $g_2 + g_3$, then every infinitesimal fiber preserving conformal transformation on TM is homothetic, and induces a homothetic vector field on M [1].

Let (M, g) be a complete, simply connected Riemannian manifold. If we consider TM with metric $g_1 + g_2$, and TM admits an essential infinitesimal conformal transformation, then M is isometric to the standard sphere [2].

Let (M, g) be a Riemannian manifold and V a vector field on M and let X^C, X^V, X^H be complete, vertical and horizontal lifts of V to TM respectively. If we consider TM with metric g_2, then X^C is a conformal vector field on TM if and only if V is homothetic on M. Moreover, if V is a Killing vector on M, then X^C and X^V are Killing vectors on TM [3].

Let (M, g) be a Riemannian manifold. If we consider TM with metric $g_1 + g_3$, then

I) X^C is a conformal vector field if and only if X is homothetic.

II) X^V is a conformal vector field if and only if X is Killing vector field with vanishing second covariant derivative in M.

III) X^H is a conformal vector field if and only if X is parallel [3], [4].

In this paper we are going to replace the cited lift Riemannian or pseudo-Riemannian metrics on TM by $\tilde{g} = ag_1 + bg_2 + cg_3$, that is a combination of diagonal lift and complete lift metrics, where a, b and c are certain positive real numbers. More precisely, we prove the following Theorems.

Theorem 1. Let M be a connected n-dimensional Riemannian manifold and let TM be its tangent bundle with metric $\tilde{g}$. Then every complete lift conformal vector field on TM is homothetic, and moreover, every horizontal or vertical lift conformal vector field on TM is a Killing vector.

Theorem 2. Let M be a connected n-dimensional Riemannian manifold and TM be its tangent bundle with metric $\tilde{g}$. Then every inessential fiber preserving conformal vector field on TM is homothetic.

2. PRELIMINARIES

Let (M, g) be a real n-dimensional Riemannian manifold and (U, x) a local chart on M, where the induced coordinates of the point $p \in U$ are denoted by its image on IR^n, $x(p)$ or briefly (x^i). Using the induced coordinates (x^i) on M, we have the local field of frames $\{\frac{\partial}{\partial x_i}\}$ on T_pM. Let ∇ be a Riemannian connection on M with coefficients Γ^k_{ij}, where the indices $a, b, c, h, i, j, k, m, \ldots$ run over the range $1, 2, \ldots n$. The Riemannian curvature tensor is defined by

$$K(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \forall X, Y, Z \in X(M).$$

Locally we have

$$K_{ijk}{}^{m} = \partial_i \Gamma^m_{jk} - \partial_j \Gamma^m_{ik} + \Gamma^m_{ia} \Gamma^a_{jk} - \Gamma^m_{ja} \Gamma^a_{ik},$$

where $\partial_i = \frac{\partial}{\partial x^i}$ and $K(\partial_i, \partial_j, \partial_k) = K_{ijk}{}^{m} \partial_m$.

3. NON-LINEAR CONNECTION

Let TM be the tangent bundle of M and π the natural projection from TM to M. Consider $\pi_* : TTM \mapsto TM$ and let us put

$$ker\pi_*^v = \{z \in TTM \mid \pi_*^v(z) = 0\}, \forall v \in TM.$$

Then the vertical vector bundle on M is defined by $VTM = \bigcup_{v \in TM} ker\pi_*^v$. A *non-linear connection* or a *horizontal distribution* on TM is a complementary distribution HTM for VTM on TTM. The non-linear nomination arise from the fact that HTM is spanned by a basis which is completely determined by non-linear functions. These functions are called coefficients of non-linear connection and will be noted in the sequel by N_i^j. It is clear that HTM is a horizontal vector bundle. By definition, we have decomposition $TTM = VTM \oplus HTM$ [5].

Using the induced coordinates (x^i, y^i) on TM, where x^i and y^i are called respectively *position* and *direction* of a point on TM, we have the local field of frames $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\}$ on TTM. Let $\{dx^i, dy^i\}$ be the dual basis of $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$. It is well known that we can choose a local field of frames $\{X_i, \frac{\partial}{\partial y_i}\}$ adapted to the above decomposition, i.e. $X_i \in X(HTM)$ and $\frac{\partial}{\partial y_i} \in X(VTM)$ are sections of horizontal and vertical sub-bundle on HTM and VTM, defined by $X_i = \frac{\partial}{\partial x_i} - N_i^j \frac{\partial}{\partial y_j}$, where $N_i^j(x,y)$ are functions on TM and have the following coordinate transformation rule in local coordinates (x^i, y^i) and $(x^{i'}, y^{i'})$ on TM.

$$N_{i'}^{h'} = \frac{\partial x^{h'}}{\partial x^h}(\frac{\partial x^i}{\partial x^{i'}} N_i^h + \frac{\partial^2 x^h}{\partial x^{i'} \partial x^{a'}} y^{a'}).$$

To see a relation between linear and non-linear connections let $\Gamma_{j\ i}^{\ k}$ be the coefficients of the Riemannian connection of (M, g). Then it is easy to check that $y^a \Gamma_{a\ i}^{\ k}$ satisfies the above relation and thus can be regarded as coefficients of the non-linear connection on TM in the sequel.

Let us put $X_h = \frac{\partial}{\partial x^h} - y^a \Gamma_{a\ h}^{\ m} \frac{\partial}{\partial y^m}$ and $X_{\bar{h}} = \frac{\partial}{\partial y^h}$. Then $\{X_h, X_{\bar{h}}\}$ is the adapted local field of frames of TM and let $\{dx^h, \delta y^h\}$ be the dual basis of $\{X_h, X_{\bar{h}}\}$, where $\delta y^h = dy^h + y^a \Gamma_{a\ i}^{\ h} dx^i$ and the indices $i, j, h, ...$ and $\bar{i}, \bar{j}, \bar{h}$ ... run over the range 1, 2, ...n.

4. THE RIEMANNIAN OR PSEUDO-RIEMANNIAN METRIC $\tilde{g}$ ON TANGENT BUNDLE

Let (M, g) be a Riemannian manifold. The Riemannian metric g has components g_{ij}, which are functions of variables x^i on M, and by means of the above dual basis it is well known that [3]; $g_1 := g_{ij} dx^i dx^j$, $g_2 := 2g_{ij} dx^i \delta y^j$ and $g_3 := g_{ij} \delta y^i \delta y^j$ are all bilinear differential forms defined globally on TM.

The tensor field:

$$\tilde{g} = ag_1 + bg_2 + cg_3,$$

on TM where a, b and c are certain positive real numbers, has components

$$\begin{pmatrix} ag_{ij} & bg_{ij} \\ bg_{ij} & cg_{ij} \end{pmatrix},$$

with respect to the dual basis of the adapted frame of TM. From linear algebra we have $det\tilde{g} = (ac - b^2)^n detg^2$. Therefore $\tilde{g}$ is nonsingular if $ac - b^2 \neq 0$ and positive definite if $ac - b^2 > 0$ and define, respectively, pseudo-Riemannian or Riemannian lift metrics on $T(M)$.

5. LIE DERIVATIVE

Let M be an *n*-dimensional Riemannian manifold, V a vector field on M, and $\{\phi_t\}$ any local group of local transformations of M generated by V. Take any tensor field S on M, and denote by $\phi_t{}^*(S)$ the pull-back of S by ϕ_t. Then Lie derivation of S with respect to V is a tensor field $\pounds_V S$ on M defined by

$$\pounds_V S = \frac{\partial}{\partial t}\phi_t{}^*(S) \mid_{t=0} = \lim_{t \longrightarrow 0} \frac{\phi_t{}^*(S) - (S)}{t},$$

on the domain of ϕ_t. The mapping $\pounds_V$ which maps S to $\pounds_V(S)$ is called the Lie derivative with respect to V.

Suppose that S is a tensor field of type (n, m). Then the components $(\pounds_V S)^{j_1,\dots,j_n}_{i_1,\dots,i_m}$ of $\pounds_V S$ may be expressed as [6]

$$(\pounds_V S)^{j_1,\dots,j_n}_{i_1,\dots,i_m} = V^a \partial_a S^{j_1,\dots,j_n}_{i_1,\dots,i_m} + \sum_{k=1}^{m} \partial_{i_k} V^a S^{j_1,\dots,j_n}_{i_1,\dots,a,\dots,i_m} - \sum_{k=1}^{n} \partial_a V^{j_k} S^{j_1,\dots,a,\dots,j_n}_{i_1,\dots,i_m},$$

where $S^{j_1,\dots,j_n}_{i_1,\dots,i_m}$ and V^a denote the components of S and V.

The local expression of the Lie derivative $\pounds_V(S)$ in terms of covariant derivatives on a Riemannian manifold for a tensor field of type (1, 2) is given by:

$$\pounds_V S_j{}^h{}_i = v^a \nabla_a S_j{}^h{}_i - S_j{}^a{}_i \nabla_a v^h + S_a{}^h{}_i \nabla_j v^a + S_j{}^h{}_a \nabla_i v^a, \tag{1}$$

where, $S_j{}^h{}_i$ and v^h are components of S and V, and $\nabla_a S_j{}^h{}_i$, $\nabla_a v^h$ are components of covariant derivatives of S and V, respectively [1, 3, 6].

Lemma 1. [1], [7] The Lie bracket of adapted frame of TM satisfies the following relations

$$[X_i, X_j] = y^r K_{jir}{}^m X_{\overline{m}},$$

$$[X_i, X_{\bar{j}}] = \Gamma_j{}^m{}_i X_{\overline{m}},$$

$$[X_{\bar{i}}, X_{\bar{j}}] = 0,$$

where $K_{jir}{}^m$ denotes the components of a Riemannian curvature tensor of M.

Lemma 2. [1] Let X be a vector field on TM with components $(X^h, X^{\bar{h}})$ with respect to the adapted frame $\{X_h, X_{\bar{h}}\}$. Then X is fiber-preserving vector field on TM if and only if X^h are functions on M.

Therefore, every fiber-preserving vector field X on TM induces a vector field $V = X^h \frac{\partial}{\partial x_h}$ on M.

Definition 1. [1], [3] Let V be a vector field on M with components V^h. We have the following vector fields on TM which are called respectively, **complete, horizontal** and **vertical** lifts of V:

$$X^C := V^h X_h + y^m(\Gamma_m{}^h{}_a V^a + \partial_m V^h)X_{\bar{h}},$$
$$X^H := V^h X_h,$$
$$X^V := V^h X_{\bar{h}}.$$

From Lemma 2 we know that X^C, X^H and X^V are fiber-preserving vector fields on TM.

Lemma 3. [1] Let X be a fiber-preserving vector field on TM. Then the Lie derivative of the adapted frame and its dual basis are given by:

I) $\pounds_X X_h = (-\partial_h X^a)X_a + \{y^b X^c K_{hcb}{}^a - X^{\bar{b}}\Gamma_b{}^a{}_h - X_h(X^{\bar{a}})\}X_{\bar{a}}$, II) $\pounds_X X_{\bar{h}} = \{X^b\Gamma_b{}^a{}_h - X_{\bar{h}}(X^{\bar{a}})\}X_a$,

III) $\pounds_X dx^h = (\partial_m X^h)dx^m$,

IV) $\pounds_X \delta y^h = -\{y^b X^c K_{mcb}{}^h - X^{\bar{b}}\Gamma_b{}^h{}_m - X_m(X^{\bar{h}})\}dx^m - \{X^b\Gamma_b{}^h{}_m - X_{\bar{m}}(X^{\bar{h}})\}\delta y^m$.

Lemma 4. [8] Let X be a fiber-preserving vector field on TM, which induces a vector field V on M. Then Lie derivatives $\pounds_X g_1$, $\pounds_X g_2$ and $\pounds_X g_3$ are given by:

I) $\pounds_X g_1 = (\pounds_V g_{ij})dx^i dx^j$,

$$II)\ \pounds_X g_2 = 2[-g_{jm}\{y^b X^c K_{icb}{}^m - X^{\bar{b}}\Gamma_b{}^m{}_i - X_i(X^{\overline{m}})\}dx^i dx^j + \{\pounds_V g_{ij} - g_{jm}\nabla_i X^m + g_{jm}X_{\bar{i}}(X^{\overline{m}})\}dx^j\delta y^i],$$

$$III)\ \pounds_X g_3 = -2g_{mi}\{y^b X^c K_{jcb}{}^m - X^{\bar{b}}\Gamma_b{}^m{}_j - X_j(X^{\overline{m}})\}dx^j\delta y^i + \{\pounds_V g_{ij} - 2g_{mj}\nabla_i X^m + 2g_{mj}X_{\bar{i}}(X^{\overline{m}})\}\delta y^i\delta y^j,$$

where $\pounds_V g_{ij}$ and $\nabla_i X^m$ denote the components of $\pounds_V g$ and the covariant derivative of V respectively.

6. MAIN RESULTS

Proposition 1. Let X be a complete (resp. horizontal or vertical) lift conformal vector field on TM. Then the scalar function $\Omega(x,y)$ in $\pounds_X \tilde{g} = 2\Omega\tilde{g}$ is a function of position alone (resp. $\Omega = 0$).

Proof: Let TM be the tangent bundle over M with Riemannian metric $\tilde{g}$ and X be a complete (resp. horizontal or vertical) lift conformal vector field on TM. By definition, there is a scalar function Ω on TM such that

$$\pounds_X \tilde{g} = 2\Omega\tilde{g}.$$

Since the complete horizontal and vertical lift vector fields are fiber preserving, by applying $\pounds_X$ to the definition of $\tilde{g}$, using Lemma 4 and the fact that $dx^i dx^j$, $dx^i\delta y^j$ and $\delta y^i\delta y^j$ are linearly independent, we have following three relations

$$a(\pounds_V g_{ij} - 2\Omega g_{ij}) = bg_{im}(y^b X^c K_{jcb}{}^m - X^{\bar{b}}\Gamma_b{}^m{}_j - X_j(X^{\overline{m}})) + g_{jm}(y^b X^c K_{icb}{}^m - X^{\bar{b}}\Gamma_b{}^m{}_i - X_i(X^{\overline{m}}))], \tag{2}$$

$$b(\pounds_V g_{ij} - 2\Omega g_{ij}) = bg_{im}(\nabla_j X^m - X_{\bar{j}}(X^{\overline{m}})) + cg_{jm}(y^b X^c K_{icb}{}^m - X^{\bar{b}}\Gamma_b{}^m{}_i - X_i(X^{\overline{m}})). \tag{3}$$

Using relation 1, we have $\pounds_V g_{ij} = \nabla_i V_j + \nabla_j V_i$, from which we obtain

$$2\Omega g_{ij} = g_{mj} X_{\bar{i}}(X^{\overline{m}}) + g_{mi} X_{\bar{j}}(X^{\overline{m}}). \tag{4}$$

Applying $X_{\bar{k}}$ to the relation 4 and using the fact that g_{ij} is a function of position alone, we have

$$2g_{ij} X_{\bar{k}}(\Omega) = g_{mj} X_{\bar{k}} X_{\bar{i}}(X^{\overline{m}}) + g_{mi} X_{\bar{k}} X_{\bar{j}}(X^{\overline{m}}). \tag{5}$$

By means of definition 1 for complete lift vector fields, and by replacing the value of $X^{\overline{m}}$ in relation 5, we have

$$2g_{ij} X_{\bar{k}}(\Omega) = g_{mj} X_{\bar{k}} X_{\bar{i}}(y^l(\Gamma_{l\ a}^{\ m} V^a + \partial_l V^m)) + g_{mi} X_{\bar{k}} X_{\bar{j}}(y^l(\Gamma_{l\ a}^{\ m} V^a + \partial_l V^m)).$$

Since the coefficients of the Riemannian connection on M, and components of vector field V are functions of position alone, the right hand side of the above relation becomes zero, from which we have $X_{\bar{k}}(\Omega) = 0$. This means that the scalar function $\Omega(x,y)$ on TM depends only on the variables (x^h).

Similarly, for vertical lift vector fields, by using the fact that the components of V are functions of position alone and from relation 4, we have $\Omega = 0$. Finally, for horizontal lift vector field by means of relation 4, we have $\Omega = 0$.

Proposition 2. Let M be a connected manifold and X be a complete lift conformal vector field on TM. Then the scalar function $\Omega(x,y)$ in $\pounds_X \tilde{g} = 2\Omega\tilde{g}$ is constant.

Proof: Let X be a complete lift conformal vector field on TM with components $(X^h, X^{\bar{h}})$, with respect to the adapted frame $\{X_h, X_{\bar{h}}\}$.

Let us put

$$A_a^m = \Gamma_{a\ h}^{\ m} X^h + \partial_a X^m.$$

The coordinate transformation rule implies that A_a^m are the components of (1, 1) tensor field A. Then its covariant derivative is

$$\nabla_i A^m_{\ a} = \partial_i A^m_{\ a} + \Gamma^m_{i\ k} A^k_{\ a} - \Gamma^k_{i\ a} A^m_{\ k},$$

where $\nabla_i A^m_{\ a}$ is the component of the covariant derivative of tensor field A.

From definition 1, $X^{\overline{m}} = A^m_{\ a} y^a$. By means of relation 3, we have

$$b[\pounds_V g_{ij} - 2\Omega g_{ij} - g_{im}(\nabla_j X^m - A_j^m)] = cg_{jm}[y^a X^c K_{ica}^{\ \ \ m} - \Gamma_{k\ i}^{\ m} A_a^k y^a - X_i(A_h^m y^h)].$$

Note that the components of A are functions of position alone, from which the right hand side of this relation becomes

$$cg_{jm}[y^a X^c K_{ica}^{\ \ \ m} - \Gamma_{k\ i}^{\ m} A_a^k y^a - (\frac{\partial}{\partial x^i} - y^a \Gamma_{a\ i}^{\ k} \frac{\partial}{\partial y^k})(A_h^m y^h)]$$

$$= cg_{jm}[y^a X^c K_{ica}^{\ \ \ m} - \Gamma_{k\ i}^{\ m} A_a^k y^a - y^a \tfrac{\partial}{\partial x^i} A_a^m + \Gamma_{a\ i}^{\ k} A_k^m y^a]$$

$$= cy^a(X^c K_{icaj} - g_{mj} \nabla_i A^m_{\ a}).$$

Thus we have

$$b[\pounds_V g_{ij} - 2\Omega g_{ij} - g_{mi}(\nabla_j X^m - A^m_j)] = cy^a(X^c K_{icaj} - g_{mj}\nabla_i A^m_a).$$

By means of Proposition 1 the left hand side of the above relation is a function of position alone. Applying $X_{\bar{k}} = \frac{\partial}{\partial y^k}$ to this relation gives

$$X^c K_{icaj} - g_{mj}\nabla_i A^m_a = 0,$$

Or

$$X^c K_{icaj} = \nabla_i A_{ja}.$$

From which

$$\nabla_i A_{ja} + \nabla_i A_{aj} = 0. \tag{6}$$

Now by replacing $X^{\bar{m}}$ in relation 4

$$2\Omega g_{ij} = g_{mj} X_{\bar{i}}\{y^h(\Gamma_{h\ a}^{\ m} X^a + \partial_h X^m)\} + g_{mi} X_{\bar{j}}\{y^h(\Gamma_{h\ a}^{\ m} X^a + \partial_h X^m)\}$$

$$= g_{mj}(\Gamma_{i\ a}^{\ m} X^a + \partial_i X^m) + g_{mi}(\Gamma_{j\ a}^{\ m} X^a + \partial_j X^m)$$

$$= g_{mj} A^m_i + g_{mi} A^m_j.$$

Applying covariant derivation ∇_k to this relation gives

$$2g_{ij}\nabla_k \Omega = \nabla_k A_{ji} + \nabla_k A_{ij}.$$

From relation 6, we get $\nabla_k \Omega = \frac{\partial}{\partial x_h}\Omega = 0$.

Since M is connected, the scalar function Ω is constant.

Theorem 1. Let M be a connected *n*-dimensional Riemannian manifold and TM be its tangent bundle with metric $\tilde{g}$. Then every complete lift conformal vector field on TM is homothetic, moreover, every horizontal or vertical lift conformal vector field on TM is a Killing vector.

Proof: Let M be an *n*-dimensional Riemannian manifold, TM its tangent bundle with the metric $\tilde{g}$ and X a complete (resp. horizontal or vertical) lift conformal vector field on TM. Then by means of Proposition 1 the scalar function $\Omega(x,y)$ in $\pounds_X \tilde{g} = 2\Omega\tilde{g}$ is a function of position alone (resp. $\Omega = 0$), and by means of Proposition 2 it is constant. Thus, every complete lift conformal vector field on TM is homothetic and every horizontal or vertical lift conformal vector field on TM is a Killing vector.

Theorem 2. Let M be a connected n-dimensional Riemannian manifold and TM be its tangent bundle with metric $\tilde{g}$. Then every inessential fiber preserving conformal vector field on TM is homothetic.

Proof: Let X be an inessential fiber preserving conformal vector field on TM with components $(X^h, X^{\bar{h}})$, with respect to the adapted frame $\{X_h, X_{\bar{h}}\}$. Using the same argument in proof of Proposition 1, it is obvious that we have relations 2, 3 and 4. From relation 4, we have

$$\Omega g_{ii} = g_{mi} X_{\bar{i}}(X^{\bar{m}}).$$

Since $\Omega(x,y)$ in $\pounds_X \tilde{g} = 2\Omega\tilde{g}$ is supposed to be a function of position alone, by applying $X_{\bar{i}}$ to the above relation we have

$$X_{\bar{i}}(X_{\bar{i}}(X^{\overline{m}})) = 0.$$

Applying $X_{\bar{i}}$ to relation 4 again and using above relation gives

$$X_{\bar{i}}(X_{\bar{j}}(X^{\overline{m}})) = 0.$$

Thus we can write

$$X^{\overline{m}} = \alpha^m{}_a y^a + \beta^m, \tag{7}$$

where α^m_a and β^m are certain functions of position alone. Replacing relation 7 in relation 3, we have

$$b(\pounds_V g_{ij} - 2\Omega g_{ij}) = bg_{im}(\nabla_j X^m - \alpha^m{}_j) + cg_{jm}(y^b X^c K_{icb}{}^m - y^a\alpha^b{}_a\Gamma_b{}^m{}_i - \beta^b\Gamma_b{}^m{}_i -$$

$$y^a \tfrac{\partial}{\partial x_i}\alpha^m{}_a - \tfrac{\partial}{\partial x_i}\beta^m + y^a\Gamma_a{}^k{}_i\alpha^m{}_k)$$

$$= bg_{im}(\nabla_j X^m - \alpha^m{}_j) + cg_{jm}(y^b X^c K_{icb}{}^m - y^a\nabla_i\alpha^m{}_a) - cg_{jm}\nabla_i\beta^m.$$

Therefore

$$b(\pounds_V g_{ij} - 2\Omega g_{ij} - g_{im}(\nabla_j X^m - \alpha^m{}_j)) + cg_{jm}\nabla_i\beta^m = cg_{jm}y^a(X^c K_{ica}{}^m - \nabla_i\alpha^m{}_a).$$

The left hand side of this relation is a function of position alone. From which by applying $X_{\bar{k}}$ we have

$$X^c K_{ica}{}^m = \nabla_i\alpha^m{}_a. \tag{8}$$

Replacing relation 7 in relation 4 we find

$$2\Omega g_{ij} = \alpha_{ji} + \alpha_{ij}.$$

The covariant derivative of this relation and using relation 8 gives

$$\nabla_k\Omega = \frac{\partial}{\partial x_h}\Omega = 0.$$

Since M is connected, then the scalar function Ω on M is constant. This completes the proof of Theorem 2.

REFERENCES

1. Yamauchi, K. (1995). On infinitesimal conformal transformations of the tangent bundles over Riemannian manifolds. *Ann Rep. Asahikawa. Med. Coll., 16*, 1-6, and (1996). *Ann. Rep. Asahikawa. Med. Coll. 17*, 1-7, and (1997). *Ann. Rep. Asahikawa. Med. Coll., 18*, 27-32.
2. Hasegawa, I. & Yamauchi, K. (2003). Infinitesimal conformal transformations on tangent bundles with the lift metric $1+2$. *Scientiae Mathematicae Japonicae 57*, (1), 129-137, e7, 437-445.
3. Yano, K. & Ishihara, S. (1973). *Tangent and Cotangent Bundles*. Department of Mathematics Tokyo Institute of Technology, Marcel Dekker, Tokyo, Japan.
4. Yano, K. & Kobayashi, H. (1996). Prolongations of tensor fields and connection to tangent bundle I,

General theory. *Jour. Math. Soc. Japan*, 18194-210.

5. Bejancu, A. (1990). Finsler geometry and applications. *Ellis Horwood Limited publication*.
6. Nakahara, M. (1990). *Geometry Topology and Physics*. Physics institute, Faculty of Liberal Arts Shizuoka, Japan., Bristol and New York, Adam Hilger.
7. Miron, R. (1981). Introduction to the theory of Finsler spaces. *Proc. Nat. Sem. On Finsler spaces, Brasov* (131-183). & (1987). Some connections on tangent bundle And their applications to the general relativity. *Tensor N. S*. 46, 8-22.
8. Yawata, M. (1991). Infinitesimal isometries of frame bundles with natural Riemannian metric. *Tohoku Math. J.* (2), *43*(1), 103-115.