

## A SUMMABILITY FACTOR THEOREM FOR ABSOLUTE SUMMABILITY INVOLVING QUASI POWER INCREASING SEQUENCES\*

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**Abstract** – We obtain sufficient conditions for the series  $\sum a_n \lambda_n$  to be absolutely summable of order  $k$  by a triangular matrix.

**Keywords** – absolute summability, weighted mean matrix, cesaro matrix, summability factor

### 1. INTRODUCTION

Quite recently, Bor and Ozarslan [1] obtained sufficient conditions for  $\sum a_n \lambda_n$  to be summable  $|\bar{N}, p_n|_k, k \geq 1$ . Unfortunately, they used an incorrect definition of absolute summability (see, e.g. [2]). In this paper we obtain the corresponding result for triangular matrix using the correct definition of absolute summability. We obtain the correct form of [1] as a corollary.

Let  $T$  be a lower triangular matrix,  $\{s_n\}$  a sequence. Then

$$T_n := \sum_{v=0}^n t_{nv} s_v.$$

A series  $\sum a_n$  is said to be summable  $|T|_k, k \geq 1$  if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty. \quad (1)$$

We may associate with  $T$  two lower triangular matrices,  $\bar{T}$  and  $\hat{T}$ , defined as follows:

$$\bar{t}_{nv} = \sum_{r=v}^n t_{nr} \quad n, v = 0, 1, 2, \dots,$$

and

$$\hat{t}_{nv} = \bar{t}_{nv} - \bar{t}_{n-1,v}, \quad n = 1, 2, 3, \dots$$

We may write

$$T_n = \sum_{v=0}^n a_{nv} \sum_{i=0}^v a_i \lambda_i = \sum_{i=0}^n a_i \lambda_i \sum_{v=i}^n a_{nv} = \sum_{i=0}^n \bar{a}_{ni} a_i \lambda_i.$$

Thus

$$T_n - T_{n-1} = \sum_{i=0}^n \bar{a}_{ni} a_i \lambda_i - \sum_{i=0}^{n-1} \bar{a}_{n-1,i} a_i \lambda_i$$

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\*Received by the editor January 25, 2005 and in final revised form September 17, 2006

$$\begin{aligned}
&= \sum_{i=0}^n \bar{a}_{ni} a_i \lambda_i - \sum_{i=0}^n \bar{a}_{n-1,i} a_i \lambda_i \\
&= \sum_{i=0}^n (\bar{a}_{ni} - \bar{a}_{n-1,i}) a_i \lambda_i \\
&= \sum_{i=0}^n \hat{a}_{ni} a_i \lambda_i = \sum_{i=1}^n \hat{a}_{ni} \lambda_i (s_i - s_{i-1}) \\
&= \sum_{i=1}^n \hat{a}_{ni} \lambda_i s_i - \sum_{i=1}^n \hat{a}_{ni} \lambda_i s_{i-1} \\
&= \sum_{i=1}^{n-1} \hat{a}_{ni} \lambda_i s_i + a_{nn} \lambda_n s_n - \sum_{i=1}^n \hat{a}_{ni} \lambda_i s_{i-1} \\
&= \sum_{i=1}^{n-1} \hat{a}_{ni} \lambda_i s_i + a_{nn} \lambda_n s_n - \sum_{i=0}^{n-1} \hat{a}_{n,i+1} \lambda_{i+1} s_i \\
&= \sum_{i=1}^{n-1} (\hat{a}_{ni} \lambda_i - \hat{a}_{n,i+1} \lambda_{i+1}) s_i + a_{nn} \lambda_n s_n .
\end{aligned}$$

We may write

$$\begin{aligned}
(\hat{a}_{ni} \lambda_i - \hat{a}_{n,i+1} \lambda_{i+1}) &= \hat{a}_{ni} \lambda_i - \hat{a}_{n,i+1} \lambda_{i+1} - \hat{a}_{n,i+1} \lambda_i + \hat{a}_{n,i+1} \lambda_i \\
&= (\hat{a}_{ni} - \hat{a}_{n,i+1}) \lambda_i + \hat{a}_{n,i+1} (\lambda_i - \lambda_{i+1}) \\
&= \lambda_i \Delta_i \hat{a}_{ni} + \hat{a}_{n,i+1} \Delta \lambda_i .
\end{aligned}$$

Therefore

$$\begin{aligned}
T_n - T_{n-1} &= \sum_{i=1}^{n-1} \Delta_i \hat{a}_{ni} \lambda_i s_i + \sum_{i=1}^{n-1} \hat{a}_{n,i+1} \Delta \lambda_i s_i + a_{nn} \lambda_n s_n . \\
&= T_{n1} + T_{n2} + T_{n3} , \text{ say.}
\end{aligned}$$

A triangle is a lower triangular matrix with all nonzero main diagonal entries.

A positive sequence  $\{\gamma_n\}$  is said to be a quasi  $\beta$ -power increasing sequence if there exists a constant  $K = K(\beta, \gamma) \geq 1$  such that  $K n^\beta \gamma_n \geq m^\beta \gamma_m$  holds for all  $n \geq m \geq 1$ .

It should be noted that every almost increasing sequence is a quasi  $\beta$ -power increasing sequence for any non-negative  $\beta$ , but the converse need not be true as can be seen by taking the example, say  $\gamma_n = n^{-\beta}$  for  $\beta > 0$ .

## 2. MAIN RESULT

**Theorem 1.** Let A be a lower triangular matrix with non-negative entries satisfying

- (i)  $\bar{a}_{n0} = 1, \quad n = 0, 1, \dots,$
- (ii)  $a_{n-1,v} \geq a_{nv}$  for  $n \geq v + 1$ , and
- (iii)  $na_{nn} = O(1)$ .

Let  $\{X_n\}$  be a quasi  $\beta$  – power increasing sequence for some  $0 < \beta < 1$ , and let  $\{\beta_n\}$  and  $\{\lambda_n\}$  be sequences such that

- (iv)  $|\Delta\lambda_n| \leq \beta_n$ ,
- (v)  $\lim \beta_n = 0$ ,
- (vi)  $\sum_{n=1}^{\infty} n |\Delta\beta_n| X_n < \infty$ , and
- (vii)  $|\lambda_n| X_n = O(1)$

if

$$(viii) \sum_{n=1}^m \frac{1}{n} |s_k|^k = O(X_m),$$

then the series  $\sum a_n \lambda_n$  is summable  $|A|_k, k \geq 1$ .

We need the following lemma for the proof of our theorem.

**Lemma ([3]).** Under the conditions on  $\{X_n\}$ ,  $\{\beta_n\}$  and  $\{\lambda_n\}$  as taken in the statement of the theorem, the following conditions hold when (vi) is satisfied :

$$(a) n\beta_n X_n = O(1) \text{ and}$$

$$(b) \sum_{n=1}^{\infty} \beta_n X_n < \infty.$$

**Proof of Theorem 1.** The complete proof is sufficient, by Minkowski’s inequality, to show that

$$\sum_{n=1}^{\infty} n^{k-1} |T_{nr}|^k < \infty, \text{ for } r=1, 2, 3.$$

From the definition of  $\hat{A}$  and using (i) and (ii);

$$\begin{aligned} \hat{a}_{n,i+1} &= \bar{a}_{n,i+1} - \bar{a}_{n-1,i+1} \\ &= \sum_{v=i+1}^n a_{nv} - \sum_{v=i+1}^{n-1} a_{n-1,v} \\ &= 1 - \sum_{v=0}^i a_{nv} - 1 + \sum_{v=0}^i a_{n-1,v} \\ &= \sum_{v=0}^i (a_{n-1,v} - a_{nv}) \geq 0. \end{aligned} \tag{2}$$

From (vii), it follows that  $\lambda_n = O(1)$ . Using Hölder’s inequality and (iii)

$$\begin{aligned} I_1 : &= \sum_{n=1}^{m+1} n^{k-1} |T_{n1}|^k \leq \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{i=1}^{n-1} |\Delta_i \hat{a}_{ni}| |\lambda_i| |s_i| \right)^k \\ &\leq \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{i=1}^{n-1} |\Delta_i \hat{a}_{ni}| |\lambda_i|^k |s_i|^k \right) x \left( \sum_{i=1}^{n-1} |\Delta_i \hat{a}_{ni}| \right)^{k-1}. \end{aligned}$$

$$\begin{aligned}
\Delta_i \widehat{a}_{ni} &= \widehat{a}_{n,i} - \widehat{a}_{n,i+1} \\
&= \overline{a}_{ni} - \overline{a}_{n-1,i} - \overline{a}_{n,i+1} + \overline{a}_{n-1,i+1} \\
&= a_{ni} - a_{n-1,i} \leq 0.
\end{aligned}$$

Thus using (ii),

$$\sum_{i=0}^{n-1} |\Delta_i \widehat{a}_{ni}| = \sum_{i=0}^{n-1} (a_{n-1,i} - a_{ni}) = 1 - 1 + a_{nm} = a_{nm}. \quad (3)$$

Using (iv), (vii), (viii) and the condition (b) of Lemma.

$$\begin{aligned}
I_1 &:= O(1) \sum_{n=1}^{m+1} (na_{nm})^{k-1} \sum_{i=1}^{n-1} |\Delta_i \widehat{a}_{ni}| |\lambda_i|^k |s_i|^k \\
&:= O(1) \sum_{n=1}^{m+1} (na_{nm})^{k-1} \sum_{i=1}^{n-1} |\Delta_i \widehat{a}_{ni}| |\lambda_i| |\lambda_i|^{k-1} |s_i|^k \\
&:= O(1) \sum_{i=1}^m |\lambda_i| |s_i|^k \sum_{n=1+i}^{m+1} (na_{nm})^{k-1} |\Delta_i \widehat{a}_{ni}| \\
&:= O(1) \sum_{i=1}^m |\lambda_i| |s_i|^k a_{ii} \\
&:= O(1) \sum_{i=1}^m |\lambda_i| \left( \sum_{r=1}^i |s_r| a_{rr} - \sum_{r=1}^{i-1} |s_r| a_{rr} \right) \\
&:= O(1) \left[ \sum_{i=1}^m |\lambda_i| \sum_{r=1}^i |s_r| a_{rr} - \sum_{j=0}^{m-1} |\lambda_{j+1}| \sum_{r=1}^j |s_r| a_{rr} \right] \\
&:= O(1) \left[ \sum_{i=1}^{m-1} \Delta |\lambda_i| \sum_{r=1}^i |s_r| a_{rr} + |\lambda_m| \sum_{r=1}^m |s_r| a_{rr} \right] \\
&:= O(1) \left[ \sum_{i=1}^{m-1} \Delta |\lambda_i| \sum_{r=1}^i \frac{1}{r} |s_r|^k + |\lambda_m| \sum_{r=1}^m \frac{1}{r} |s_r|^k \right] \\
&:= O(1) \left[ \sum_{i=1}^{m-1} \beta_i X_i + |\lambda_m| X_m \right] \\
&:= O(1).
\end{aligned}$$

Using Hölder's inequality,

$$\begin{aligned}
I_2 &= \sum_{n=1}^{m+1} n^{k-1} |T_{n2}| \leq \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{i=1}^{n-1} \widehat{a}_{n,i+1} s_i \Delta \lambda_i \right|^k \\
&\leq \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{i=1}^{n-1} \widehat{a}_{n,i+1} |s_i| |\Delta \lambda_i| \right)^k \\
&\leq \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{i=1}^{n-1} \widehat{a}_{n,i+1} |s_i| \beta_i \right)^k
\end{aligned}$$

$$\leq \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{i=1}^{n-1} \widehat{a}_{n,i+1} |s_i|^k \beta_i \right) x \left( \sum_{i=1}^{n-1} \widehat{a}_{n,i+1} \beta_i \right)^{k-1}.$$

It is easy to see that

$$\sum_{i=1}^{n-1} \widehat{a}_{n,i+1} \beta_i \leq M a_{nn} \tag{4}$$

as in [4].

Using (iii)

$$\begin{aligned} I_2 &= O(1) \sum_{n=1}^{m+1} (na_{nn})^{k-1} \sum_{i=0}^{n-1} \widehat{a}_{n,i+1} |s_i|^k \beta_i \\ &= O(1) \sum_{i=1}^m \beta_i |s_i|^k \sum_{n=1+i}^{m+1} (na_{nn})^{k-1} \widehat{a}_{n,i+1} \\ &= O(1) \sum_{i=1}^m \beta_i |s_i|^k \sum_{n=1+i}^{m+1} \widehat{a}_{n,i+1}. \end{aligned}$$

From (2)

$$\begin{aligned} \sum_{n=i+1}^{m+1} \left( \sum_{v=0}^i (a_{n-1,v} - a_{nv}) \right) &= \sum_{v=0}^i \sum_{n=i+1}^{m+1} (a_{n-1,v} - a_{nv}) \\ &= \sum_{v=0}^i (a_{iv} - a_{m+1,v}) \\ &\leq \sum_{v=0}^i a_{iv} = 1. \end{aligned} \tag{5}$$

Using (v), (vi) and (viii)

$$\begin{aligned} I_2 : &= O(1) \sum_{i=1}^m \beta_i |s_i|^k = O(1) \sum_{i=1}^m i \beta_i \frac{1}{i} |s_i|^k \\ &= O(1) \sum_{i=1}^m i \beta_i \left[ \sum_{r=1}^i \frac{|s_r|^k}{r} - \sum_{r=1}^{i-1} \frac{|s_r|^k}{r} \right] \\ &= O(1) \left[ \sum_{i=1}^m i \beta_i \sum_{r=1}^i \frac{|s_r|^k}{r} - \sum_{j=1}^{m-1} (j+1) \beta_{j+1} \sum_{r=1}^j \frac{|s_r|^k}{r} \right] \\ &= O(1) \left[ \sum_{i=1}^{m-1} \Delta(i \beta_i) \sum_{r=1}^i \frac{|s_r|^k}{r} + O(1) m \beta_m \sum_{i=1}^m \frac{|s_i|^k}{i} \right] \\ &= O(1) \sum_{i=1}^{m-1} |\Delta(i \beta_i)| X_i + O(1) m \beta_m X_m \\ &= O(1) \sum_{i=1}^{m-1} i |\Delta(\beta_i)| X_i + O(1) \sum_{i=1}^{m-1} \beta_{i+1} X_{i+1} + O(1) m \beta_m X_m \\ &= O(1), \end{aligned}$$

again using the conditions of Lemma.

Using (iii) and (vii),

$$\begin{aligned} \sum_{n=1}^{m+1} n^{k-1} |T_{n3}|^k &\leq \sum_{n=1}^{m+1} n^{k-1} |a_{nn} \lambda_n s_n|^k \\ &= O(1) \sum_{n=1}^m (na_{nn})^{k-1} a_{nn} |\lambda_n|^k |s_n|^k \\ &= O(1) \sum_{n=1}^m a_{nn} |\lambda_n|^{k-1} |\lambda_n| |s_n|^k \\ &= O(1), \end{aligned}$$

as in the proof of  $I_L$ .

**Corollary 1.** Let  $\{p_n\}$  be a positive sequence such that  $P_n = \sum_{k=1}^n p_k \rightarrow \infty$ , and satisfies (i)  $np_n = O(P_n)$ .

Let  $\{X_n\}$  be a quasi  $\beta$ -power increasing sequence for some  $0 < \beta < 1$  and let  $\{\beta_n\}$  and  $\{\lambda_n\}$  be sequences such that

$$(ii) |\Delta \lambda_n| \leq \beta_n,$$

$$(iii) \lim \beta_n = 0,$$

$$(iv) \sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty, \text{ and}$$

$$(v) |\lambda_n| X_n = O(1).$$

If

$$(vi) \sum_{n=1}^m \frac{1}{n} |s_k|^k = O(X_m),$$

then the series  $\sum a_n \lambda_n$  is summable  $|\bar{N}, p_n|_k, k \geq 1$ .

**Proof:** Conditions (ii), (iii), (iv), (v) and (vi) of Corollary 1 are, respectively, conditions (iv), (v), (vi), (vii) and (viii) of Theorem 1. Conditions (i) and (ii) of Theorem 1 are automatically satisfied for any weighted mean method. Condition (iii) of Theorem 1 becomes condition (i) of Corollary 1.

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