IJST (2014) 38A3 (Special issue-Mathematics): 311-320
Iranian Journal of Science & Technology

http://ijsts.shirazu.ac.ir

Primary decomposition in a soft ring and a soft module

T. Shah and S. Medhit*

Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
E-mail: sadia.midhat@yahoo.com

Abstract

The main objective of this study is to swing Krull intersection theorem in primary decomposition of rings and modules to the primary decomposition of soft rings and soft modules. To fulfill this aim several notions like soft prime ideals, soft maximal ideals, soft primary ideals, and soft radical ideals are introduced for a soft ring over a given unitary commutative ring. Consequently, the primary decomposition of soft rings and soft modules is established. In addition, the ascending and descending chain conditions on soft ideals and soft sub modules of soft rings and soft modules are introduced, respectively, enabling us to develop the notions of soft Noetherian rings and soft Noetherian modules.

Keywords: Primary decomposition; soft Noetherian ring (module); soft primary module; minimal soft prime ideal; soft irreducible ideal

1. Introduction

The certainty of information has always been a major challenge; it is quite difficult to obtain the required level of precision information. In order to deal with problem precision in information, several models and theories were developed and employed but none of them served the purpose of exact precision. It was concluded that the fundamental cause of uncertainty lies in the set theory based on classical logic. Russell's paradox is one of the examples that points out the limitation of classical set theory. Molodtsov (Molodtsov, 1999) initiated soft theory as a mathematical tool to solve the problem of precision in information. Maji et al. (Maji. et. al, 2002 and 2003) showed the significance of the soft set theory by applying it in the decision making problems. Moreover, he introduced new functions on soft set. Chen (Chen, 2005) established the notion of soft set parameterization reduction, which made the soft theory more applicable. Later, the concept of normal parameter reduction of soft sets was introduced by Kong (Kong, 2008). Whereas Zuo et al. (Zou, 2008) introduced the soft set data analysis method. Ali et al. (Ali et al. 2009) contributed some new algebraic operations on the soft sets. In (Li 2011) Fu Li obtained some results of soft set theory based on his newly defined algebraic operations and proved that the distributive law holds involving two new operations. Aktas and Çağman (Aktas and Çağman 2007) applied notions of group theory on soft sets. Jun (Jun, 2008) and Jun and Park (Jun and Park, 2008) have explored BCK/BCI-algebras and there application in the soft sense. However, Feng (Feng, 2008) investigated a soft semi-rings, idealistic soft semi-rings and soft ideals. Moreover, Shabir and irfan (Shabir et al. 2009) restricted one binary operation and investigated soft ideals over a semigroup. In continuation in a general setting Shah et al. (Shah et al. 2011) have a soft treatment of ordered Abel-Grassman's Groupoids (AG-Groupoids).

Recently, Soft set theory has incredible growth in the algebraic structures. Aktaş and Çağman (Aktas et al. 2007) extended soft set for soft group. However, in (Acar et al. 2010) the basic idea of a soft ring was introduced, which is in fact a parameterized family of subrings and ideals of a ring. Atagun and Sezgin (Atagun and Sezgin, 2011) presented soft subring and soft ideal, soft subfield over a field and soft sub-module over a left R-module. Celik et al. (Celik et al. 2011) described a new concept of soft rings, soft ideals, and introduced some new operations in soft set theory. The notion of soft modules and its properties are defined in (Sun et al. 2008).

In this study, initially we extended the concepts of soft ideals in a soft ring to soft irreducible ideals, soft prime ideals, soft maximal ideals, soft primary ideals and soft radical ideals. Ultimately the primary decomposition of soft rings and soft modules is proven and consequentially, a shift in Krull intersection theorem in primary decomposition of rings and modules is obtained.

*Corresponding author
Received: 6 March 2013 / Accepted: 6 April 2014

Furthermore, the ascending and descending chain conditions on soft ideals and soft sub modules of soft rings and soft modules are presented. Accordingly, we are able to develop the notions of soft Noetherian rings and soft Noetherian modules.

The paper is organized as: In section 2, some relevant definitions and results on soft sets and soft rings are described. While in section 3, we delimit the primary decomposition of a soft ring. Also, ascending and descending chain conditions on a soft ring are also described in detail. In the same section the notions of a soft Noetherian ring and a soft Laskerian ring are presented. Section 4 contains the discussion on primary decomposition of a soft module. Moreover ascending and descending chain conditions on soft modules are also given with soft Noetherian and soft Laskerian modules. The major objective is obtained as a theorem, analogous to Krull intersection theorem for a soft ring and a soft module.

2. Preliminaries

Molodtsov (Molodtsov, 1999) defined a soft set as: Let U be an initial universe and E be a set of parameters. The power set of U is denoted P(U) and A be a non-empty subset of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $A \to P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A, F(\varepsilon)$ is the set of ε -approximate elements of (F, A).

Here, we recall some basic facts concerning soft sets from (Maji et al. 2002), for two soft sets (F, A) and (G, B) over a common universe U, (F, A) is a soft subset of (G, B) (i.e., $(F, A) \cong (G, B)$). If $A \subset B$ and for all $e \in A$, F(e) and G(e) are identical approximations. (F, A) is said to be a soft superset of (G, B), if (G, B) is a soft subset of (F, A) and is denoted by $(F, A) \supseteq (G, B)$. Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A). Moreover, a soft set (F, A) over U is said to be a null soft set denoted by φ if $F(\varepsilon) = \varphi$ (null set) for all $\varepsilon \in A$, and a soft set (F, A) over U is said to be absolute soft set denoted by $\tilde{A}$ if $F(\varepsilon) = U$ for all $\varepsilon \in A$. Also $\tilde{A}^c = \varphi$ and $\varphi^c = \tilde{A}$. If (F, A) and (G, B) are two soft sets, then "(F, A) OR (G, B)" denoted by $(F, A) \vee (G, B)$ is defined as $(F, A) \vee (G, B) = (O, A \times B)$ where $O(\alpha, \beta) = F(\alpha) \cup G(\beta)$ for all $(\alpha, \beta) \in A \times B$.

The union of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cup B$ for all $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A/B \\ G(e) & \text{if } e \in B/A \\ F(e) \cup G(e) & \text{if } e \in A \cap B \end{cases}$$

and $(F, A) \tilde{\cup} (G, B) = (H, C)$.

The bi-intersection of two soft sets (F, A) and (G, B) over the common universe U can be defined by Feng (Feng 2008) as follow: $(F, A) \tilde{\sqcap} (G, B) = (H, C)$ is the soft set (H, C), where $C = A \cap B$ and $H(e) = F(e) \cap G(e)$ for all $e \in C$. Celik et al. (Celik et al., 2011) gives restricted sum of two soft sets (F, A) and (G, B) over a ring R is defined as $(F, A) \oplus_{\cap} (G, B) = (H, C)$ where $C = A \cap B$ and $H(e) = F(e) + G(e)$ for all $e \in C$. Whereas the extended product of two soft sets (F, A) and (G, B) over a ring R defined by $(F, A) \odot_{\cup} (G, B) = (H, C)$, where $C = A \cup B$ and

$$H(e) = \begin{cases} F(e) & \text{if } e \in A/B \\ G(e) & \text{if } e \in B/A \\ F(e) \cdot G(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$. The restricted product of two soft sets defined by $(F, A) \odot_{\cap} (G, B) = (H, C)$ where $C = A \cap B$ and $H(e) = F(e) \cdot G(e)$ for all $e \in C$.

From now on, we assume that R is a unitary commutative ring and all the soft sets are considered over R. The support of the soft set (F, A) is defined by Acar et al. (Acar et al., 2011).

$$\text{Supp}(F, A) = \{x \in A : F(x) \neq \varphi\}$$

A soft set is said to be non-null if its support is not equal to the empty set.

Definition 2.1. (Acar et al, 2011, Definition 3.1) A soft set (F, A) over a ring R is called a soft ring over R if F(x) is a subring of R for all $x \in A$. A soft subring is defined as R, if (F, A) and (G, B) are soft rings over R. Then (G, B) is called a soft subring of (F, A) if $B \subset A$ and G(x) is a subring of F(x), for all $x \in \text{Supp}(G, B)$.

Some results for the quick reference are mentioned below.

Theorem 2.2. (Acar et al., 2011, Theorem 3.6) Let (F, A) and (G, B) be soft rings over R. Then

1. If $G(x) \subset F(x)$, for all $x \in B \subset A$, then (G, B) is a soft subring of (F, A).
2. $(F, A) \tilde{\sqcap} (G, B)$ is a soft subring of both (F, A) and (G, B) if it is non-null.

Definition 2.3. (Acar et al., 2011, Definition 4.2) If $I \subset A$ and $\gamma(x)$ is an ideal of F(x), for all $x \in \text{Supp}(\gamma, I)$ and $I \subset A$ and $\gamma(x)$ is an ideal of F(x), for all $x \in \text{Supp}(\gamma, I)$, then (γ, I) is a soft ideal of (F, A).

Theorem 2.4. (Acar et al., 2011, Theorem 4.3) Let (γ_1, I_1) and (γ_2, I_2) be soft ideals of a soft ring (F, A) over R. Then $(\gamma_1, I_1) \widetilde{\sqcap} (\gamma_2, I_2)$ is a soft ideal of (F, A) if it is non-null.

Theorem 2.5. (Acar et al., 2011, Theorem 4.4) Let (γ_1, I_1) and (γ_2, I_2) be soft ideals of soft rings (F, A) and (G, B) over R, respectively. Then $(\gamma_1, I_1) \widetilde{\sqcap} (\gamma_2, I_2)$ is a soft ideal of $(F, A) \widetilde{\sqcap} (G, B)$ if it is non-null.

Theorem 2.6. (Acar et al., 2011, Theorem 4.6) Let (F, A) be a soft ring over R. (γ_1, I_1) and (γ_2, I_2) be soft ideals of over R. If I_1 and I_2 are disjoint, then $(\gamma_1, I_1) \widetilde{\sqcup} (\gamma_2, I_2)$ is a soft ideal of (F, A).

Definition 2.7. (Acar et al., 2011, Definition 5.1) A non-null soft set (F, A) over R is said to be an idealistic soft ring over R if $F(x)$ is an ideal of R for all $x \in Supp(F, A)$.
In this subsection, the idealistic approach is used. The notion of soft prime ideal is defined below.

Definition 2.8. Let (F, A) be a soft ring over the ring R. A non-null soft set (γ, I) over R is called soft prime ideal of (F, A). This is denoted by (γ, I) $(\gamma, I) \triangleright^{p} (F, A)$ if it satisfies the following conditions:
a) $I \subset A$.
b) $\gamma(x)$ is an ideal of $F(x)$, for all $x \in Supp(\gamma, I)$.
c) For $F(a), F(b) \in (F, A)$, (γ, I) $F(a) \cdot F(b) \in (\gamma, I) \Rightarrow$ either $F(a) \in (\gamma, I)$ or $F(b) \in (\gamma, I)$.
Next, we give the definition of soft maximal ideal.

Definition 2.9. Let (F, A) be a soft ring over a ring R. A soft ideal (γ, I) over the ring R is called soft maximal ideal of (F, A) denoted by $(\gamma, I) \triangleright^{m} (F, A)$ if $\gamma(x)$ is soft maximal ideal of $F(x)$, for all $x \in Supp(\gamma, I)$.

The soft primary ideal and radical soft ideal is defined as;

Definition 2.10. Let (F, A) be a soft ring over the ring R. A non-null soft set (γ, I) over R is called soft primary ideal of (F, A) (F, A), denoted by $(\gamma, I) \triangleright^{p'} (F, A)$ if it satisfies the following conditions:
a) $\subset A$.
b) $\gamma(x)$ is an ideal of $F(x)$, for all $x \in Supp(\gamma, I)$.
c) For all $F(a), F(b) \in (F, A)$, $F(a) \cdot F(b) \in (\gamma, I) \Rightarrow$ either $F(a) \in (\gamma, I)$ or $F(b)^n \in (\gamma, I)$, for some $n \in Z^+$.

Definition 2.11. Let (γ, I) be a soft ideal of (F, A) over the ring R. Then radical of the soft ideal (γ, I) is denoted by $rad((\gamma, I))$ and defined as: $rad((\gamma, I)) = \{F(a) \in (F, A): (F(a))^n \in (\gamma, I)\}$.

Proposition 2.12. The radical of a soft primary ideal is a soft prime ideal.

Proof: Obvious.

3. Primary decomposition of Soft rings

Recall that in a unitary commutative ring R, an ideal I of R has a primary decomposition if I is a finite intersection of primary ideals, that is, $I = \bigcap_{i=1}^{n} Q_i$ where Q_i are primary ideals of R. The primary decomposition I is said to be reduced if the prime ideals $\sqrt{Q_i} \neq \sqrt{Q_j}$ and $\sqrt{Q_i} \supseteq \bigcap_{j=1}^{n} \sqrt{Q_j}$ for $1 \leq i \leq n$ and $i \neq j$. If each ideal of a ring can be written as an intersection of primary ideals (resp. finite intersection of primary ideals), then ring is said to be ring with primary decomposition (resp. Laskerian ring).

In this section we will develop the results about the primary decomposition of soft rings which plays fundamental roles in commutative algebra. Also, we prove an analogue to the result that: In a Noetherian ring every ideal can be written as a finite intersection of irreducible ideals. Moreover, it is determined that all proper soft ideals in a soft Noetherian ring have primary decomposition.

Definition 3.1. Let (γ, I) be a soft ideal of a soft ring (F, A) over R. We say that (γ, I) has a primary decomposition if there exist a non-empty family$\{(\gamma_i, I_i): i \in \mathbb{N}\}$ of soft primary ideals of (F, A) such that
1. $I = \bigcap_i I_i$, for all $i \in \mathbb{N}$;
2. $\gamma(x) = \bigcap_i \gamma_i(x)$, for all $i \in \mathbb{N}$.

A short notation $(\gamma, I) = \widetilde{\bigcap}_{i=1} (\gamma_i, I_i)$ for primary decomposition of (γ, I) is used.

Definition 3.2. A soft ring (F, A) over R is said

to have a primary decomposition (resp. a Laskerian soft ring) if each soft ideal of (F,A) has a primary decomposition (resp. finite primary decomposition).

Definition 3.3. A soft ring (F,A) is said to be reduced or irredundant if it has a primary decomposition, that is, each soft ideal $(\gamma,I) = \mathbin{\tilde{\cap}}_{i=1}(\gamma_i,I_i)$, where (γ_i,I_i) are soft primary ideals. Then
a) $rad(\gamma_i,I_i) \neq rad(\gamma_j,I_j)$, for all $i,j \in \mathbb{N}, i \neq j$;
b) $(\gamma_i,I_i) \supsetneq \mathbin{\tilde{\cap}}_{j\neq i}(\gamma_j,I_j)$, for all $i,j \in \mathbb{N}$.

Definition 3.4. Let (F,A) be a soft ring over R and (γ,I) be soft ideal of (F,A). Then (γ,I) is soft irreducible if $(\gamma,I) = (\gamma_1,I_1) \mathbin{\tilde{\cap}} (\gamma_2,I_2)$, where (γ_1,I_1) and (γ_2,I_2) be a soft ideals of (F,A), and either $(\gamma,I) = (\gamma_1,I_1)$ or $(\gamma,I) = (\gamma_2,I_2)$.

Definition 3.5. Let (γ,I) and (γ,J) be two soft ideals of a soft ring (F,A). Then (γ,I) is said to are (γ,J)-primary if (γ,I) is soft primary and $rad((\gamma,I)) = (\gamma,J)$.

Definition 3.6. Let (γ_1,I_1) and (γ_2,I_2) be two soft ideals of (F,A) over R. Denote soft ideal quotient by the set,
$((\gamma_1,I_1):(\gamma_2,I_2)) = \{F(a): F(a)(\gamma_2,I_2) \subseteq (\gamma_1,I_1)\}$, where the product $F(a)\cdot\gamma_2(b) \in (\gamma_1,I_1)$ for all $\gamma_2(b) \in (\gamma_2,I_2)$, implies that $((\gamma_1,I_1):(\gamma_2,I_2))$ is a soft ideal of (F,A).

Theorem 3.7. Let (F,A) be a soft ring over R and $(\gamma_i,I_i)_{i\in\mathbb{N}}$ be soft ideals of (F,A). The following conditions are equivalent:
1. Every ascending chain of soft ideals is stationary, that is,
a) The set of subsets I_i of a given set A is ordered by inclusion.
b) $\gamma_1(x) \subseteq \gamma_2(x) \subseteq \gamma_3(x) \subseteq \cdots$ such that $\gamma_n(x) \subseteq \gamma_{n+1}(x)$, for all $x \in Supp(\cap_{i\in\mathbb{N}}(\gamma_i,I_i))$ and $(\gamma_1,I_1) \subseteq (\gamma_2,I_2) \subseteq (\gamma_3,I_3) \subseteq \cdots \subseteq (\gamma_n,I_n) \subseteq (F,A)$.
2. Every non empty set of ideals in (F,A) having a maximal element.

Proof: Suppose (1) holds. Let us consider S as a set of all proper soft ideals in a soft ring (F,A). Since the soft ideals are ordered by inclusion, it is implied that every ascending chain of soft ideal in S has an upper bound. By using Zorn's lemma, S contains a soft maximal element. As S contains proper ideals, the maximal element in S is a proper soft ideal of (F,A). Thus, there is a soft maximal ideal for inclusion among all proper soft ideals. Hence every non empty set of ideals in (F,A) has a maximal element.
Conversely, assume that $(\gamma_1,I_1) \subseteq (\gamma_2,I_2) \subseteq (\gamma_3,I_3) \subseteq \cdots$ is being an ascending chain of soft ideals. Suppose $(\gamma,I) = \tilde{\cup}_{i\in\mathbb{N}}(\gamma_i,I_i)$, and S be the set of soft ideals that are properly contained in (γ,I). Therefore, S contains a maximal element. Since every nonempty set of ideals has a maximal element, for some $n \in \mathbb{N}$, each (γ_i,I_i) is contained in (γ_n,I_n). Hence
$(\gamma_n,I_n) = (\gamma_{n+1},I_{n+1}) = (\gamma_{n+2},I_{n+2}) = \cdots = (\gamma,I)$.
Thus, every ascending chain of soft ideals is stationary.

Definition 3.8. Let (F,A) be a soft ring over a ring R and $(\gamma_i,I_i)_{i\in\mathbb{N}}$ be soft ideals of (F,A). (F,A) is said to be soft Noetherian if it holds any one of the following conditions:
1. Every ascending chain condition on soft ideals is stationary, that is,
$a)$ $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ there exists a positive integer n such that $I_n = I_{n+1}$.
$b)$ $\gamma_1(x) \subseteq \gamma_2(x) \subseteq \gamma_3(x) \subseteq \cdots$ such that $\gamma_n(x) = \gamma_{n+1}(x)$, for all $x \in Supp(\cap_{i\in\mathbb{N}}(\gamma_i,I_i))$ and it can be represented by a chain,
$$(\gamma_1,I_1) \subseteq (\gamma_2,I_2) \subseteq (\gamma_3,I_3) \subseteq \cdots \subseteq (\gamma_n,I_n) \subseteq (F,A)$$
2. Every non-empty set of soft ideals of (F,A) is contained in the soft maximal ideal.

Example 3.9. Let (γ_1,I_1) and (γ_2,I_2) be soft ideals of a soft ring (F,A) over a ring R. Consider the ring $R = A = \mathbb{Z}$ and $I_1 = I_2 = I_3 = \mathbb{Z} - \{0\}$. Let us consider the set-valued function $F: A \to P(R)$ given by $F(x) = x\mathbb{Z}$. Then (F,A) is a soft ring over R. Now consider the functions $\gamma_i: I_i \to P(R)$, for $1 \leq i \leq 3$, given by, $\gamma_1(x) = 8x\mathbb{Z}, \gamma_2(x) = 4x\mathbb{Z}, \gamma_3(x) = 2x\mathbb{Z}$, where $x \in Supp(\gamma_i,I_i)$. Thus $(\gamma_1,I_1) \subseteq (\gamma_2,I_2) \subseteq (\gamma_3,I_3) \subseteq (F,A)$ and (γ_3,I_3) is a soft maximal ideal of (F,A).

Remark 3.10. Ascending chain of soft ideals needs not to be stationary. For instance consider the ring $R = \mathbb{Z} + X\mathbb{Q}[X], A = \mathbb{Q}$ *and* $I_i = \frac{1}{2^i}\mathbb{Z}$. Consider the

set valued function $F: A \to P(R)$ such that $F(a) = \{\left(\frac{X}{a}\right), a \in Supp(F, A)\}$. Consider the function $\gamma_i : I_i \to P(R)$ given by $\gamma_i(a) = \{\left(\frac{X}{a}\right) : a \in Supp(\gamma_i, I_i)\}$. This gives $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ and $\gamma_1(x) \subseteq \gamma_2(x) \subseteq \gamma_3(x) \subseteq \cdots$. So,
$(\gamma_1, I_1) \subseteq (\gamma_2, I_2) \subseteq (\gamma_3, I_3) \subseteq (\gamma_4, I_4) \subseteq \cdots$.
Hence, we get a non-terminating ascending chain of soft ideals. This is a non Noetherian ring. Here the soft maximal ideal of the soft ring is $\left(F, A\right)$ itself.

Definition 3.11. Let $\left(F, A\right)$ be a soft ring over a ring R. Then the soft prime ideal (γ, I) is said to be a minimal soft prime ideal if it is minimal in $Spec\left(F, A\right)$ with respect to inclusion.

Proposition 3.12. The following conditions hold for conductor ideals.
1. $(\gamma, I) \subseteq (\gamma, I): (\zeta, J)$
2. $\left((\gamma, I): (\zeta, J)\right) \odot_{\cap} (\zeta, J) \subseteq (\gamma, I)$
3. $\left((\gamma, I): (\zeta, J)\right): (\eta, L) =$
$\left((\gamma, I): (\zeta, J)\right) \odot_{\cup} (\eta, L)$
4. $\left((\gamma, I): (\zeta, J)\right) = \mathbb{m}_{n=1}^{\infty} \left((\gamma_n, \mathrm{I}_n): (\zeta, J)\right)$,
where $(\gamma, I) = \mathbb{m}_{n=1}^{\infty} (\gamma_n, I_n)$.
5. $\left((\gamma, I): (\zeta, J)\right) = \mathbb{m}_{n=1}^{\infty} \left((\gamma, I): (\zeta_n, J_n)\right)$,
where $(\zeta, J) = \oplus_{\cap}(\zeta_n, J_n)$
for all $n \in \mathbb{N}$.

Proof:
1. Hence obvious.
2. Let $F(a) \in \left((\gamma, I): (\zeta, J)\right)$, where $\zeta(b) = Supp(\zeta, J)$. Then $F(a)\zeta(b_i) \in (\gamma, I)$. Now for any $\zeta\left(b_j\right) \in Supp(\zeta, J)$, it follows $F(a)\zeta(b_i)\zeta\left(b_j\right) \in (\gamma, I)$. Hence $\left((\gamma, I): (\zeta, J)\right) \odot_{\cap} (\zeta, J) \subseteq (\gamma, I)$.
3. Let $F(a)\eta(b) \in \left((\gamma, I): (\zeta, J)\right)$. Then $F(a)\eta(b)\zeta(c) \in \left((\gamma, I): (\zeta, J)\right)$, where $b \in Supp(\eta, L)$ and $c \in Supp(\zeta, J)$ and $\eta(b)\zeta(c) \in (\eta, L) \odot_{\cup} (\zeta, J)$.
Hence $F(a) \in \left((\gamma, I): (\zeta, J)\right) \odot_{\cup} (\eta, L)$.
Similarly, the existence of the converse can be proved.
4. Let $F(a)\zeta(b) \in (\gamma, I)$. Then $F(a)\zeta(b) \in (\gamma_n, I_n)$, for $b \in Supp(\zeta, J)$. Thus $F(a) \in \left((\gamma_n, I_n): (\zeta, J)\right)$, for $n \in \mathbb{N}$, and $F(a) \in \mathbb{m}_{n=1}^{\infty} \left((\gamma_n, I_n): (\zeta, J)\right)$. Hence $(\gamma, I): (\zeta, J) \subseteq \mathbb{m}_{n=1}^{\infty} \left((\gamma_n, I_n): (\zeta, J)\right)$ and vice versa.
5. Let $F(a)\zeta(b) \in (\gamma, I)$ where $(b) = \zeta_1(b)\zeta_2(b)\zeta_3(b) \ldots$ for $b \in Supp(\zeta, J)$. Hence $F(a) \, \mathbb{m}_{n=1}^{\infty} \left((\gamma, I): (\zeta_n, J_n)\right)$.
Thus $\left((\gamma, I): (\zeta, J)\right) \subseteq \mathbb{m}_{n=1}^{\infty} \left((\gamma, I): (\zeta_n, J_n)\right)$
and vice versa.

Theorem 3.13. Let (F, A) be a soft Noetherian ring over R. Each soft ideal (γ, I) of (F, A) over R is a finite intersection of soft irreducible ideals.

Proof: Suppose on the contrary, that the soft ideal (γ, I) can't be written as a finite intersection of soft irreducible ideals. Set $\widetilde{N} = \{(\gamma, I) | (\gamma, I)\ cannot\ be\ written\ as\ finite\ product\ of\ soft\ irreducible\ ideals\}$. Since $\left(F, A\right)$ is a soft Noetherian, there exist a maximal ideal $(\gamma', I') \in \widetilde{N}$, such that (γ', I') can't be written as a finite product of soft irreducible ideals.
Also (γ', I') is not a soft irreducible ideal, therefore there exists (γ_1, I_1) and (γ_2, I_2) such that the restricted intersection of $(\gamma_1, I_1) \, \mathbb{m} \, (\gamma_2, I_2) = (\gamma', I')$ implies either $(\gamma', I') \subseteq (\gamma_1, I_1)$ or $(\gamma', I') \subseteq (\gamma_2, I_2)$. The maximality of (γ', I') implies that $(\gamma_1, I_1) \notin \widetilde{N}$ and $(\gamma_2, I_2) \notin \widetilde{N}$. This implies (γ_1, I_1) and (γ_2, I_2) can be written as the finite intersection of soft irreducible ideals, that is, (γ', I') can be written as the finite intersection of soft irreducible ideals which is a contradiction. Hence, it is proved.

Theorem 3.14. Let (F, A) be a soft Noetherian ring over a ring. Every soft irreducible ideal of (F, A) is a soft primary ideal of (F, A).

Proof: Let (γ, I) be a soft irreducible ideal over (F, A). Then for any $F(a), F(b) \in (F, A)$, such that $F(a)F(b) \in (\gamma, I)$ and $F(b) \notin (\gamma, I)$. This gives, $\left((\gamma, I): F(a)\right) \subseteq \left((\gamma, I): F(a)^2\right) \subseteq \left((\gamma, I): F(a)^3\right) \subseteq \cdots$ is an ascending chain of soft ideals of (F, A) over R. Since (F, A) is Noetherian, $\left((\gamma, I): F(a)^{\mathrm{N}}\right) = \left((\gamma, I): F(a)^{N+1}\right)$, for $N \geq n$. Now, we will see that whether this soft ideal (γ, I) can be written as the restricted intersection of two soft ideals or not.
$(\gamma, I) =$
$\left((\gamma, I) \oplus_{\cap} F(a)^n \cdot (F, A)\right) \, \mathbb{m} \, \left((\gamma, I) \oplus_{\cap} F(b) \cdot (F, A)\right)$, for all $n \in \mathbb{N}$.
If $F(a) \in (\gamma, I), then\ \gamma(a) \subseteq \gamma(a) + F(a)^n F(c)\ and\ \gamma(a) \subseteq \gamma(a) + F(b)F(d)\ \ where\ F(c), F(d) \in (F, A)$. This implies $(\gamma, I) \subseteq \left((\gamma, I) \oplus_{\cap} F(a)^n \cdot (F, A)\right) \, \mathbb{m} \, \left((\gamma, I) \oplus_{\cap} F(b) \cdot (F, A)\right)$.
Conversely, assume that $\gamma(c) \in \left((\gamma, I) \oplus_{\cap} F(a)^n \cdot (F, A)\right) \, \mathbb{m} \, \left((\gamma, I) \oplus_{\cap} F(b) \cdot (F, A)\right)$
and $\gamma(c) = \gamma(b_i) + F(a)^n F(c) = \gamma(b_i) + F(b)F(d)$.
For $c \in Supp(\gamma, I)$
$$\gamma(c) \cdot F(a) = \gamma(b_i) \cdot F(a) + F(a)^{n+1} F(c)$$
$$= \gamma(b_i) \cdot F(a) + F(a)F(b)F(d)$$
Since $F(a)F(b) \in (\gamma, I)$, therefore $F(a)^{n+1} F(c) \in (\gamma, I)$. Also, $F(c) \in \left((\gamma, I) \oplus_{\cap} F(a)^n \cdot (F, A)\right)$ and $\gamma(c) \in (\gamma, I)$, therefore $\left((\gamma, I) \oplus_{\cap} F(a)^n \cdot (F, A)\right) \, \mathbb{m} \, \left((\gamma, I) \oplus_{\cap} F(b) \cdot (F, A)\right) \subseteq (\gamma, I)$
As (γ, I) is irreducible, hence

$(\gamma, I) \subset \big((\gamma, I) \oplus_{\cap} F(a)^n \cdot (F, A)\big)$, due to the fact that $F(b) \notin (\gamma, I)$. So,
$(\gamma, I) = \big((\gamma, I) \oplus_{\cap} F(a)^n \cdot (F, A)\big)$.

This proves that $F(a)^n \in (\gamma, I)$ is soft primary. Thus, in a Noetherian ring every soft irreducible ideal is soft primary ideal.

Theorem 3.15. Every soft Noetherian ring is a soft Laskerian ring.

Proof: Follows directly from (Theorem 3.13 and Theorem 3.14).

Theorem 3.16. Let (γ, I) be (ξ, P) soft primary ideal of a soft ring (F, A) over a ring R and $F(a) \in (F, A)$. The following holds,

1. If $F(a) \in (\gamma, I)$, then $\big((\gamma, I) : F(a)\big) = (\gamma, I)$.
2. If $F(b) \notin (\gamma, I)$, then $\big((\gamma, I) : F(a)\big) \subseteq (\xi, P)$ is a soft primary ideal.

Proof:
1. It is given that $F(a) \in (\gamma, I)$. Since (γ, I) is a soft ideal, therefore $F(a)\gamma(\beta) \subseteq (\gamma, I)$. Hence $\big((\gamma, I) : F(a)\big) \subseteq (\gamma, I)$. The converse is directly from (proposition 2.12), that is, $\big((\gamma, I) : F(a)\big) \supseteq (\gamma, I)$. Thus, equality holds.
2. Suppose $F(b) \in \big((\gamma, I) : F(a)\big)$, then $F(b)F(a) \in (\gamma, I)$. Since $F(a) \notin (\gamma, I)$, therefore $F(b)^n \in (\gamma, I)$ which gives $F(b) \in rad(\gamma, I) = (\xi, P)$. Thus, from (proposition 6)$((\gamma, I) : F(a)) \subseteq (\xi, P)$ and $(\gamma, I) \subseteq ((\gamma, I) : F(a)) \subseteq (\xi, P)$. Now taking the radical of above inclusion; $rad(\gamma, I) \subseteq rad((\gamma, I) : F(a)) \subseteq rad(\xi, P)$. Since $rad(\gamma, I) = (\xi, P)$, $rad((\gamma, I) : F(a)) = (\xi, P)$. Now to verify that $rad((\gamma, I) : F(a))$ is soft primary ideal, let us consider that $F(x), F(y) \in (F, A)$ with $F(x), F(y) \in ((\gamma, I) : F(a))$ and $F(y) \notin rad((\gamma, I) : F(a)) = (\xi, P) = rad(\gamma, I)$. This implies $F(x) \in \big((\gamma, I) : F(a)\big)$. Hence $\big((\gamma, I) : F(a)\big)$ is (ξ, P) soft primary.

Remark 3.17. If the soft ideal (γ, I) is a (ξ, P) soft primary ideal of a soft ring (F, A). Then for $F(a) \in (\gamma, I)$ it is not necessary that $((\gamma, I) : F(a)) \neq (F, A)$. For instance, consider ring $R = \mathbb{Z}_4 = \{0,1,2,3\}$, its subsets $P = \{0,1,2\}$ and $I = \{0,1\}$. The set-valued function $F : A \to P(R)$ given by $F(x) = \{y \in R : x.y \in \{0,2\}\}$. Therefore, $F(0) = F(2) = \mathbb{Z}_4, F(1) = F(3) = \{0,2\}$. We see, all these sets are subrings of R. Hence (F, A) is a soft ring over R. Now consider the function $\gamma : I \to P(R)$ given by $\gamma(x) = \{y \in R : x \cdot y = 0\}$. This implies $\gamma(0) = \mathbb{Z}_4, \gamma(1) = \{0\}$. So $(\gamma, I) = \{\mathbb{Z}_4, \{0\}\}$. The mapping $\xi : P \to P(R)$ is defined as $\xi(x) = \{y \in R : x.y = 0\}$, $\xi(0) = \mathbb{Z}_4, \xi(1) = \{0\}$ and $\xi(2) = \{0,2\}$. So $(\xi, P) = \{\mathbb{Z}_4, \{0\}, \{0,2\}\}$. It is observe that (ξ, P) is a soft prime ideal and (γ, I) is (ξ, P) soft primary ideal. For $F(0) \in (\gamma, I)$, clearly $\big((\gamma, I) : F(0)\big) = \{F(0), F(2)\} \subseteq (F, A)$ but $\big((\gamma, I) : F(a)\big) \not\supseteq (F, A)$.

Proposition 3.18. If (γ_1, I_1) and (γ_2, I_2) be primary decomposition ideals of a soft ring (F, A) over a ring R. Then $(\gamma_1, I_1) \oplus_{\cup} (\gamma_2, I_2)$ is a primary decomposition soft ideal of (F, A), where $I_1 \cap I_2 = \varphi$.

Proof: Obvious.

Proposition 3.19: Let (γ_1, I_1) and (γ_2, I_2) (γ_2, I_2) be primary decomposition ideals of a soft ring (F, A) over R. Then $(\gamma_1, I_1) \hat{\wedge} (\gamma_2, I_2)$ need not to be a primary decomposition ideal of (F, A).

Proof: Obvious.

Remark 3.20. Let (γ_1, I_1) and (γ_2, I_2) be primary decomposition of soft ideals of a soft ring (F, A) over R. So
(a) $(\gamma_1, I_1) \oplus_{\cap} (\gamma_2, I_2)$ need not be a primary decomposition soft ideal of (F, A).
(b) $(\gamma_1, I_1) \odot_{\cap} (\gamma_2, I_2)$ need not be a primary decomposition soft ideal of (F, A).
(c) $(\gamma_1, I_1) \Cap (\gamma_2, I_2)$ need not be a primary decomposition soft ideal of (F, A).

Theorem 3.21. Every soft ideal of a soft Noetherian ring contains the power of its soft radical.

Proof: Let (γ, I) be a soft ideal of a soft ring (F, A). Take $F(a_i) \in rad(\gamma, I)$, where $a_i \in Supp(F, A)$. Then $F(a_i)^{n_i} \in (\gamma, I)$, for some $n_i \in \mathbb{N}$. Put $n = 1 + \sum(n_i - 1)$, then $(\gamma, I)^n$ is generated by $F(a_1)^{m_1} \cdot F(a_2)^{m_2} \cdot F(a_3)^{m_3} \cdots F(a_i)^{m_i}$, where $\sum m_i = n$. At least one $m_i \geq n_i$ making each $F(a_i)^{m_i}$, an element of (γ, I). Hence $(\gamma, I)^n \subseteq (\gamma, I)$.

Theorem 3.22. Restricted product of two soft ideals is contained in their restricted intersection.

Proof: Let (γ_1, I_1) and (γ_2, I_2) be two soft ideals of soft ring (F, A) over a ring R. Take $\gamma_1(a)\gamma_2(a) \in (\gamma_1, I_1) \odot_{\cap} (\gamma_2, I_2)$ *where* $\gamma_1(a) \in (\gamma_1, I_1)$, $\gamma_2(a) \in$

(γ_2, I_2) and $a \in Supp((\gamma_1, I_1) \odot_\cap (\gamma_2, I_2))$. Since $a \in I_1 \cap I_2$, hence $\gamma_1(a)\gamma_2(a) \in (\gamma_1, I_1)$ and $\gamma_1(a)\gamma_2(a) \in (\gamma_2, I_2)$. Thus $\gamma_1(a)\gamma_2(a) \in (\gamma_1, I_1) \Cap (\gamma_2, I_2)$. Hence,
$(\gamma_1, I_1) \odot_\cap (\gamma_2, I_2) \subseteq (\gamma_1, I_1) \Cap (\gamma_2, I_2)$.

Proposition 3.23. Let (γ, I) be a soft prime ideal and $(\gamma_1, I_1), (\gamma_2, I_2), \cdots, (\gamma_n, I_n)$ any $n \geq 0$ soft ideals of (F, A). The following statements are equivalent:
1. (γ, I) contains (γ_j, I_j), for some j.
2. $\Cap_{i=1}^{n} (\gamma_i, I_i) \subseteq (\gamma, I)$ *and* $\odot_\cap (\gamma_i, I_i) \subseteq (\gamma, I)$ *for* $1 \leq i \leq n$.

Proof: Obvious.

Theorem 3.24. Let $(\gamma, I), (\gamma_1, I_1), (\gamma_2, I_2), \cdots, (\gamma_n, I_n)$ be a set of $n+1$ soft ideals of (F, A) over R such that $(\gamma, I) = \Cap_{i=1}^{n} (\gamma_i, I_i)$, is a reduced primary decomposition of soft ideal (γ, I). Let (ξ, P) be a soft prime ideal of (F, A) of R. Then the following statements are equivalent:
1. $(\xi, P) = (\xi_i, P_i)$, for some i, where $(\xi_i, P_i) = rad(\gamma_i, I_i)$.
2. There exists an element of soft set $F(a) \in (F, A)$ such that $((\gamma, I): F(a))$ is a (ξ, P) soft primary ideal.
3. There exists a $F(a) \in (F, A)$ such that $(\xi, P) = rad((\gamma, I): F(a))$.

Proof: Obvious.

Theorem 3.25. Let (γ, I) and (ξ, P) be soft ideals of soft ring (F, A) over a ring R. Then (γ, I) is a soft primary for (ξ, P) if and only if
(a) $(\gamma, I) \subseteq (\xi, P) \subseteq rad((\gamma, I))$
(b) If $F(a)F(b) \in (\gamma, I)$ and $F(a) \notin (\gamma, I)$, then $F(b) \in (\xi, P)$.

Proof: Suppose (a) and (b) holds. If $F(a)F(b) \in (\gamma, I)$ and $F(a) \notin (\gamma, I)$, then $F(b) \in (\xi, P) \subseteq rad((\gamma, I))$. Thus, the element $F(b)^n \in (\gamma, I)$ for some $n > 0$. Therefore, (γ, I) is a soft primary ideal. To show (γ, I) is soft primary for (ξ, P). We need only show that $(\xi, P) = rad((\gamma, I))$. By (a), $(\xi, P) \subseteq rad((\gamma, I))$. If $F(b) \in (\gamma, I)$, then for some positive integer n such that $F(b)^n \in (\gamma, I)$. If $n = 1$, then $F(b)^n \in (\gamma, I) \subseteq (\xi, P)$. If $n > 1$, then $F(b)^{n-1}F(b) \in (\gamma, I)$ with $F(b)^{n-1} \notin (\gamma, I)$ by the minimality of n, by (b), $F(b) \in (\xi, P)$. Thus, for $F(b) \in rad((\gamma, I))$, it gives $F(b) \in (\xi, P)$ The converse implication is obvious.

Theorem 3.26. Let $(\gamma, I), (\gamma_1, I_1), (\gamma_2, I_2), \cdots, (\gamma_2, I_n)$ are soft ideals of a soft ring (F, A). Then,
1. $rad(rad(\gamma, I)) = rad(\gamma, I)$.
2. $rad\begin{pmatrix}(\gamma, I) \odot_\cap (\gamma_1, I_1) \odot_\cap (\gamma_2, I_2) \\ \odot_\cap \cdots \odot_\cap (\gamma_n, I_n)\end{pmatrix} = \Cap_{i=1}^{n} (\gamma_i, I_i)$.

Proof:
1. Let $F(a) \in rad(rad(\gamma, I))$. Then $F(a)^n \in rad(\gamma, I)$ for $n \in \mathbb{N}$. Hence, by definition there exist a positive integer m such that $(F(a)^n)^m \in (\gamma, I)$. Thus $F(a) \in rad(\gamma, I)$.
Conversely, Let $F(a) \in rad(\gamma, I)$. Then by definition of soft radical ideal, it is implied that $F(a)^1 \in rad(\gamma, I)$. Hence $F(a) \in rad(rad(\gamma, I))$.
2. Let us consider an element $F(a) \in \Cap_{i=1}^{n} rad(\gamma_i, I_i)$. Then there are $m_1, m_2, \cdots, m_n > 0$ such that $F(a)^{m_i} \in (\gamma_i, I_i)$, for each j. If $m = m_1 + m_2 + \cdots + m_n$, then

$$F(a)^m = F(a)^{m_1}F(a)^{m_2} \cdots F(a)^{m_n} \in (\gamma_1, I_1) \odot_\cap (\gamma_2, I_2) \odot_\cap \cdots \odot_\cap (\gamma_n, I_n).$$

Hence

$$\Cap_{i=1}^{n} rad(\gamma_i, I_i) \subseteq rad\big((\gamma_1, I_1) \odot_\cap (\gamma_2, I_2) \odot_\cap \cdots \odot_\cap (\gamma_n, I_n)\big).$$

Since we know that

$$(\gamma_1, I_1) \odot_\cap (\gamma_2, I_2) \odot_\cap \cdots \odot_\cap (\gamma_n, I_n) \subseteq \Cap_{i=1}^{n} (\gamma_i, I_i).$$

We have $rad((\gamma_1, I_1) \odot_\cap (\gamma_2, I_2) \odot_\cap \cdots \odot_\cap (\gamma_n, I_n)) \subseteq \Cap_{i=1}^{n} rad(\gamma_i, I_i)$.

Theorem 3.27. Let (F, A) be a soft ring over a ring R. If soft prime ideal $(\gamma_i, I_i)_{1 \leq i \leq n}$ are soft primary ideals for the soft prime ideal (ξ, P), then $\Cap_{i=1}^{n} (\gamma_i, I_i)$ is also a soft primary ideal belonging to (ξ, P).

Proof: Let $(\gamma, I) = \Cap_{i=1}^{n} (\gamma_i, I_i)$. Then, by (Theorem 3.26),
$rad(\gamma, I) = rad \Cap_{i=1}^{n} (\gamma_i, I_i) = \Cap_{i=1}^{n} rad(\gamma_i, I_i) = (\xi, P)$, and by using (Theorem 3.25), $(\gamma, I) \subseteq (\xi, P) \subseteq rad(\gamma, I)$. If $F(a)F(b) \in (\gamma, I)$ and $F(a) \notin (\gamma, I)$, $F(a)F(b) \in (\gamma_i, I_i)$ for some i. Since (γ_i, I_i) is (ξ, P) soft primary, $F(b) \in (\xi, P)$. Consequently, (γ, I) itself is (ξ, P) soft primary.

Theorem 3.28. Let (γ, I) be a soft ideal of a soft ring (F, A) over a ring R. If (γ, I) has a primary decomposition of soft rings, then (γ, I) has a reduced primary decomposition.

Proof: Let $(\gamma, I) = \Cap_{i=1}^{n} (\gamma_i, I_i)$ be intersection of soft primary ideals and some soft primary ideals (γ_i, I_i) are contained in $(\gamma_1, I_1) \Cap (\gamma_2, I_2) \Cap \cdots \Cap (\gamma_{i-1}, I_{i-1}) \Cap$

$(\gamma_{i+1}, I_{i+1}) \Cap \cdots \Cap (\gamma_n, I_n)$, so $(\gamma, I) = (\gamma_1, I_1) \Cap (\gamma_2, I_2) \Cap \cdots \Cap (\gamma_{i-1}, I_{i-1}) \Cap (\gamma_{i+1}, I_{i+1}) \Cap \cdots \Cap (\gamma_n, I_n)$ is also a primary decomposition. By eliminating the superfluous (γ_i', I_i') and reindexing we have
$(\gamma, I) = (\gamma_1, I_1) \Cap (\gamma_2, I_2) \Cap \cdots \Cap (\gamma_k, I_k)$ with no soft primary ideal (γ_j, I_j) contained in the intersection of others (γ_j, I_j). Let $(\xi_1, P_1), (\xi_2, P_2), \cdots, (\xi_h, P_h)$ be distinct prime ideals in the set $\{rad(\gamma_1, I_1), rad(\gamma_2, I_2), \cdots, rad(\gamma_k, I_k)\}$. Let (γ_i', I_i') where $1 \le i \le h$ be the intersection of all the (γ_i, I_i) that belong to the soft prime ideal (ξ_i, P_i). Each (γ_i', I_i') is soft primary for (ξ_i, P_i). Clearly no (γ_i', I_i') contains the intersection of all soft primary ideals, therefore
$(\gamma, I) = \Cap_{i=1}^{k} (\gamma_i, I_i) = \Cap_{i=1}^{h} (\gamma_i', I_i')$. Hence (γ, I) has a reduced primary decomposition of soft rings.

4. Primary decomposition of Soft modules

Recall that a module M is said to be Noetherian (resp. artinan) if every ascending chain (resp. descending chain) of sub modules of M is stationary. A proper sub module C of a R-module M is said to be a primary sub module if $r \in R, b \notin M$ and $rb \in C$, this gives $r^n M \in C$ for some positive integer n.

Definition 4.1. A soft set (G, B) over a $R-$module M is called a soft module if each $G(b)$ is a sub module of M, for all $b \in Supp(G, B)$ (see Sun et al., 2008 Definition 10).

In this section we introduce the algebraic notions in soft sense, such as soft Noetherian module, soft primary module and primary decomposition of soft modules. Throughout this section all rings are commutative with identity and all modules are unitary.

Definition 4.2. Let (G, B) be a soft module over an R-module M. It is said to be soft Noetherian module, if the following conditions are equivalent,
1. Every ascending chain of soft sub modules is stationary, that is,
a) The set of subsets of B_i of a given set B are ordered by inclusion,
$B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$ such that $B_n = B_N$, for $n \ge N$.
$b)$ $(G_1, B_1) \subseteq (G_2, B_2) \subseteq (G_3, B_3) \subseteq \cdots$, there exist a positive integer n such that $(G_n, B_n) = (G_N, B_N)$, for $n \ge N$ and chain takes form $(G_1, B_1) \subseteq (G_2, B_2) \subseteq (G_3, B_3) \subseteq \cdots \subseteq (G_n, B_n)$.
2. Every non-empty set of soft sub modules of (G, B) is contained in soft maximal sub module.

Definition 4.3. A soft module (F, A) satisfies the maximal condition (resp. minimum condition) on soft sub modules if every non-empty set of soft sub modules of (F, A) contains a maximal (resp. minimal) element (with respect to set theoretic inclusion).

Definition 4.4. Let (F, A) be a soft ring over a ring R and (G, B) be a soft module over an R-module M. If (γ, I) is a soft prime ideal of (F, A),
$(\gamma, I) \odot (G, B) = \{\gamma(a)G(b): a \in Supp(\gamma, I), b \in Supp(G, B)\}$ is a soft submodule of (G, B).

Example 4.5. For $R = M = \mathbb{Z}$, $A = B = \mathbb{N}$ and $I = 2\mathbb{N}$, let us consider the set value function $F: A \to P(R)$ given by $F(x) = \{x\mathbb{Z}: x \in A\}$. Then (F, A) is a soft ring over R. Also, consider an R-module M and $G: B \to P(M)$ given by $G(b) = M$, for all $b \in B$. Take (G, B) as a soft module over an R-module M. Now again consider $\gamma: I \to P(R)$ given by $\gamma(x) = 3\mathrm{x}\mathbb{Z}$. Then (γ, I) is a soft ideal of (F, A). As $(\gamma, I) \odot (G, B) = 3x\mathbb{Z} \cdot \mathbb{Z} = 3x\mathbb{Z}$, for $x \in Supp(\gamma, I)$ is a soft sub module of (G, B).

Theorem 4.6. A soft module (G, B) satisfies the ascending (resp. descending) chain condition on soft sub modules if and only if (G, B) satisfies the maximal (resp. minimal) condition on soft sub modules.

Proof: Suppose (G, B) satisfies the minimal condition on soft sub modules and $(G_1, B_1) \supseteq (G_2, B_2) \supseteq (G_3, B_3) \supseteq \cdots$ is a chain of soft sub modules. Then the set $\{(G_i, B_i) | i \ge 1\}$ has a minimal element, say (G_n, B_n). Consequently, for $i \ge n$ we have $(G_n, B_n) \supseteq (G_i, B_i)$ by hypothesis and $(G_n, B_n) \tilde{\subseteq} (G_i, B_i)$ by minimality. Hence $(G_n, B_n) = (G_i, B_i)$ for each $i \ge n$, therefore, (F, A) satisfies the descending chain condition. Conversely, suppose (G, B) satisfies the descending chain condition and S is a non-empty set of soft sub modules of (G, B), and a soft sub module $(G_0, B_0) \in S$. If S has no minimal element, then for each soft sub module (G, B) in S there exists at

least one soft submodule (G', B') in S such that $(G, B) \supsetneq (G', B')$. For each (G, B) in S, choose one such (G', B'). This choice then defines a function $f : S \to S$ by $B \mapsto B'$. There is a function: $N \longrightarrow S$ such that $\varphi(0) = (G_0, B_0)$ and $\varphi(n+1) = f(\varphi(n)) = \varphi(n)$. Thus if $(G_n, B_n) \in S$ denotes $\varphi(n)$ then there is a sequence $(G_0, B_0) \supsetneq (G_1, B_1) \supsetneq (G_2, B_2) \supsetneq (G_3, B_3) \supsetneq \cdots$. This contradicts the descending chain condition. Therefore, S must have a minimal element. Hence (G, B) satisfies minimum condition. The proof for ascending chain condition and maximum conditions is analogous.

Definition 4.7. Let (F, A) be a soft ring over a ring R and (G, B) be a soft module over an R-module M A non-null soft subset of (H, C) of soft module (G, B) is said to be soft primary submodule, if it satisfies the following conditions;
$a)\ C \subseteq B$
$b)\ H(c)$ is submodule of $G(c)$ for all $c \in Supp(H, C)$.
$c)\ F(a) \in (F, A)$ such that $F(a)^n G(b) \in (H, C)$ for all $G(b) \in (G, B)$ and $n \in \mathbb{N}$.

Theorem 4.8. Let (F, A) be a soft ring over a ring R and (G, B) be a soft module over an R-module M. Then (H, C) be a soft primary sub module of (G, B) such that, $(\xi, Q) = \{F(a) \in (F, A): F(a)(G, B) \subseteq (H, C)\}$ is soft primary ideal in (F, A)

Proof: Let $F(a_1)F(a_2) \in (\xi, Q)$ and $F(a_2) \notin (\xi, Q)$. Then $F(a_2)(G, B) \notin (H, C)$ for all $b \in Supp(G, B)$. Consequently, there exist $G(b) \in (G, B)$, $F(a_2)G(b) \notin (H, C)$ but $F(a_1)(F(a_2)G(b)) \in (H, C)$. Since (H, C) is a soft primary submodule $F(a_1)(G, B) \subseteq (H, C)$ for some n, that is, $F(a_1)^n \in (\xi, Q)$. Therefore, (ξ, Q) is soft primary.

Example 4.9. For $R = M = \mathbb{Z}$, $A = B = \mathbb{N}$ and $C = 3\mathbb{N}$, let us consider the set value function $F : A \to P(R)$ given by $F(x) = \{x\mathbb{Z}: x \in A\}$. Then (F, A) is a soft ring over R. Also consider a R-module M and $G : B \to P(M)$ given by $G(b) = M$ for all $b \in B$. Then (G, B) is a soft module over a R-module M. Now again consider $C \subseteq B$ and $H : C \to P(M)$ given by $H(m) = 2m\mathbb{Z}$ is soft submodule of (G, B). It is observe that $(\xi, Q) = \{F(2), F(4), \cdots, F(2n): n \in \mathbb{N}\}$ is a soft primary submodule of (G, B).

Definition 4.10. Let (F, A) be a soft ring over a ring R and (G, B) be a soft module over an R-module M. A soft primary submodule (H, C) of a soft module (G, B), is said to be a (σ, P)-soft primary submodule of (G, B) if
$(\sigma, P) = rad(\xi, Q) = \{F(a) \in (F, A): F(a)^n (G, B) \subseteq (H, C)\ for\ n \geq 0\}$, where $(\xi, Q) = \{F(a) \in (F, A): F(a)(G, B) \subseteq (H, C)\}$ is soft primary ideal in (F, A).

Definition 4.11. Let (F, A) be soft ring over a ring R and (G, B) be soft module over an R-module M A soft submodule (H, C) of (G, B) has a primary decomposition if $(H, C) = \mathbin{\Cap}_{i=1}^{n} (H_i, C_i)$ with each (H_i, C_i) is a (σ_i, P_i)-soft primary submodule of (G, B), where (σ_i, P_i) is a soft prime ideal of (F, A).

Definition 4.12. Let (F, A) be soft ring over a ring R and (G, B) be soft module over an R-module M. A soft sub module $(H, C) = \mathbin{\Cap}_{i=1}^{n} (H_i, C_i)$ of (G, B) has a primary decomposition with no $(H, C) \subseteq \mathbin{\Cap}_{j=1}^{n} (H_j, C_j)$ for $i \neq j$ and the soft prime ideals (ξ_i, P_i) are all distinct. Then the soft primary decomposition is said to be reduced primary decomposition.

Theorem 4.13. Let (F, A) be a soft ring over a ring R and (G, B) be a soft module over an R-module M. If a soft submodule (H, C) of (G, B) has a primary decomposition, then (H, C) has a reduced primary decomposition.

Proof: Obvious

Theorem 4.14. Let (F, A) be a soft ring over a ring R and (G, B) be a soft module over an R-module M satisfying ascending chain condition on soft sub modules. Every soft sub module (H, C) of (G, B) has a reduced soft primary decomposition.

Proof: Let S be the set of all soft sub modules of (G, B) that do not have a primary decomposition. Clearly, there is no soft primary sub module in S. We show S is in fact empty. Suppose on the contrary that S is nonempty, then S contains a soft maximal element say (H, C). Since (H, C) is not soft primary, there exist $F(a) \in (F, A)$ and $G(b) \in (G, B) \backslash (H, C)$ such that $F(a)G(b) \in (H, C)$ but $F(a)^n G(b) \notin (H, C)$ for all $n > 0$. Consider $(G_n, B_n) = \{G(b) \in (G, B): F(a)^n G(b) \in (H, C)\}$. Then each (G_n, B_n) is soft sub module of (G, B) and the chain by inclusion is $(G_1, B_1) \subseteq (G_2, B_2) \subseteq (G_3, B_3) \subseteq \cdots$. By hypothesis there exists $k > 0$ such that $(G_i, B_i) \in (G_k, B_k)\ for\ i \geq k$. Let $(K, D) = \{G(b): G(b) = F(a)^k G(b') + H(c)\}$ be soft sub module of (G, B) Clearly, $(H, C) \subseteq (G_k, B_k) \mathbin{\Cap} (K, D)$. To show equilty, if $G(b) \in (G_k, B_k) \mathbin{\Cap} (K, D)$, then $G(b) = G(b') + K(d)$ and

$F(a)^k G(b) = F(a)^k G(b') + F(a)^k K(d) \in (H, C)$. Therefore, $(H, C) = (G_k, B_k) \Cap (K, D)$. Now by maximality of (H, C) in S, (G_k, B_k) and (K, D) must have primary decomposition. Thus S is empty and every soft sub module has a primary decomposition.

Lemma 4.15. Let (F, A) be a soft ring over a ring R and (G, B) be a soft module over an R-module M. Let (γ, I) be a soft prime ideal of (F, A) and (H, C) is (γ, I)-soft primary sub module of soft Noetherian module (G, B), then there exist a smallest integer m such that $(\gamma, I)^m \odot (G, B) \subset (H, C)$.

Proof: Recall that there exist primary ideal (σ, Q) such that $rad(\sigma, Q) = (\gamma, I)$, for some soft primary sub module (H, C) Suppose $\gamma(a) \in (\gamma, I)$ such that $\gamma(a)^{n_i} G(b) \in (H, C)$, for all $b \in Supp(G, B)$ and $n_i \geq 1$. Take $m = max(n_1, n_2, \cdots, n_i)$, hence for all $a \in Supp(\gamma, I)$ we get $\gamma(a)^m G(b) \in (H, C)$, where $b \in Supp(G, B)$. Thus $(\gamma, I)^m \odot (G, B) \subset (H, C)$.

Now we present the Krull intersection theorem in soft sense.

Theorem 4.16. Let (F, A) be a soft ring over a ring R, (γ, I) be a soft ideal of (F, A) and (G, B) be a soft module over an R-module M If $(H, C) = \Cap_{n=1}^{\infty} (\gamma, I)^n \odot (G, B)$, then $(\gamma, I) \odot (H, C) = (H, C)$.

Proof: Let us assume that $(\gamma, I) \odot (H, C) = (G, B)$. Since we know that (H, C) is a soft sub module, therefore $(\gamma, I) \odot (H, C) \subseteq (H, C)$. This implies $(G, B) \subseteq (H, C)$ and hence, $(H, C) = (G, B)$.

If we take $(\gamma, I) \odot (H, C) \neq (G, B)$, then by (lemma 4.15) $(\gamma, I) \odot (H, C)$ has a soft primary decomposition, that is, $(\gamma, I) \odot_{\cap} (H, C) = \Cap_{i=1}^{n} (H_i, C_i)$, where each (H_i, C_i) is (σ_i, P_i) soft primary sub module and (σ_i, P_i) is soft prime ideal of (F, A). Since (H, C) is a soft submodule, therefore $(\gamma, I) \odot (H, C) \subseteq (H, C)$. Now we show the converse inclusion. If $(\gamma, I) \subset (\sigma_i, P_i)$ for some fixed i, then by (lemma 4.15) there is an integer m such that, $(\sigma_i, P_i)^m \odot (G, B) \subset (H_i, C_i)$. Hence, $(H, C) = \Cap_{n=1}^{\infty} (\gamma, I)^n \odot (G, B) \subset (\gamma, I)^m \odot (G, B) \subset (\xi_i, P_i)^m \odot (G, B) \subset (H_i, C_i)$.

Since $(H, C) \subset (H_i, C_i)$, therefore there exists $c \in Supp(H, C)$ and $a \in Supp(\gamma, I)$ such that $\gamma(a) H(c) \in (\gamma, I) \odot (H, C) \subset (H_i, C_i)$, where (H_i, C_i) is soft primary sub module and $\gamma(a)(G, B) \subset (H_i, C_i)$ for some $n > 0$. Thus $(H, C) \subset (\gamma, I) \odot (H, C)$.

5. Conclusion

This paper describes a detailed study of the primary decomposition in soft rings and modules. We analyzed the algebraic structure of soft Noetherian rings and soft Noetherian modules. This work is focused, proving a form of Krull intersection theorem in soft rings and modules. To extend this work, one can study the algebraic properties of soft Noetherian rings and soft Laskerian rings.

References

Acar, U., Koyuncu, F., & Tanay, B. (2010). Soft sets and soft rings. *Comput. Math. Appl.*, 59, 3458–3463.

Aktaş, H., & Çağman, N. (2007). Soft sets and soft groups. *Information Sciences*, 177, 2726–2735.

Ali, M. I., Feng, F., Liu, X., Min, W. K., & Shabir, M. (2009). On some new operations in soft set theory. *Comput. Math. Appl.*, 57, 1547–1553.

Atagun, A. O., & Sezgin, A. (2011). Soft substructures of rings, fields and modules. *Computers and Mathematics with Applications*, 61, 592–601.

Chen, D. (2005). The parameterization reduction of soft sets and its Applications. *Comput. Math. Appl.*, 49, 757–763.

Celik, Y., Ekiz, C., & Yamak, S. (2011). A new view on soft rings. *Hacet. J. Math. Stat.*, 40(2), 273–286.

Feng, F., Jun, Y. B., & Zhao, X. Z. (2008). Soft semirings. *Comput. Math. Appl.*, 56, 2621–2628.

Li, F. (2011). Notes on the soft operations. ARPN j. syst. softw., 1(6), 205–208.

Kong, Z., Gao, L., Wong, L., & Li, S. (2008). The normal parameter reduction of soft sets and its algorithm. *J. Comp. Appl. Math.*, 56, 3029–3037.

Jun, Y. B. (2008). Soft BCK/BCI-algebras. *Comput. Math. Appl.*, 56, 1408–1413.

Jun, Y. B., & Park, C. H. (2008). Applications of soft sets in ideal theory of BCK/BCI-Algebras. *Information Sciences*, 178, 2466–2475.

Maji, P. K., Biswas, R., & Roy, R. (2002). An application of soft sets in a decision making problem. *Comput. Math. Appl.*, 44, 1077–1083.

Maji, P. K., Biswas, R., & Roy, R. (2003). Soft set theory. *Comput. Math. Appl.*, 45, 555–562.

Molodtsov, D. (1999). Soft set theory first results. *Comput. Math. Appl.*, 37, 19–31.

Shabir, M., & Ali, M. I. (2009). Soft ideals and generalized fuzzy ideals in semigroups. *New Math. and Nat. Comp.*, 5, 599–615.

Shah, T., Rehman, I., & Razzaque, A. (2011). Soft ordered Abel-Grassman's Groupoid (AG-Groupoid). *International Journal of Physical Sciences*, 6(25), 6118–6126.

Sun, Q. M., Zhang, Z. L., & Liu, J. (2008). *Soft Sets and Soft Module: Proceedings of the third international conference on rough sets and knowledge technology.* In Wang, G., Li, T., Grzymal-Busse, J.W., Miao, D., Skowrn, A., & Yao, Y. (Ed.), *Lecture Notes in Computer Science*, (Vol. 5009, pp 403–409). Springer.

Zou, Y., & Xiao, Z. (2008). Data analysis approaches of soft sets under incomplete information. *Knowledge-Based Systems*, 21, 941–945.