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Expression and dynamics of the solutions of some rational recursive sequences

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Abstract

In this paper we obtain the expression of the solutions of the following recursive sequences

$$x_{n+1}=\frac{x_{n-2}x_{n-3}x_{n-4}}{x_nx_{n-1}(\pm 1\pm x_{n-2}x_{n-3}x_{n-4})},\qquad n=0,1,\ldots,$$

where the initial conditions are arbitrary real numbers. Also, we study the behavior of the solution of these equations.

***Keywords**:* Difference equations; recursive sequences; stability; periodic solution

1. Introduction

In this paper, the form of the solutions of the following recursive sequences is obtained

$$x_{n+1}=\frac{x_{n-2}x_{n-3}x_{n-4}}{x_nx_{n-1}(\pm 1\pm x_{n-2}x_{n-3}x_{n-4})},n=0,1,\ldots,\qquad (1)$$

where the initial conditions are arbitrary real numbers. Also, we study the behavior of the solution of these equations.

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, physics, etc. (Elabbasy, Elsadany et al. 2007; Kocic and Ladas, 1993). The study of nonlinear difference equations is of paramount importance not only in their own right but in understanding the behavior of their differential counterparts. Also, difference equations are appropriate models for describing situations where population growth is not continuous but seasonal with overlapping generations.

El-Metwally et al., 2003, investigated the asymptotic behavior of the population model:

$$x_{n+1}=\alpha+\beta x_{n-1}e^{-x_n},$$

where α is the immigration rate and β is the population growth rate.

Grove, et al. (2000); studied the stability and the semicycles of solutions of the biological

$$x_{n+1}=ax_n+bx_{n-1}e^{-x_n}.$$

The generalized Beverton-Holt stock recruitment model was investigated by (Beverton and Holt, 1957; DeVault et al. 1998): $x_{n+1}=ax_n+\frac{bx_{n-1}}{1+cx_{n-1}+dx_n}$.

See also (Elettreby and El-Metwally, 2007; El-Metwally, 2007; El-Metwally and El-Afifi, 2008; Kulenovic and Ladas, 2001; Ladas, 1989; Mackey and Glass, 1977; Mickens, 1987).

There has been a great deal of work concerning the global asymptotic behavior of solutions of rational difference equations (Aloqeili, 2006; Atalay et al., 2005; Elabbasy et al., 2011; Elabbasy et al., 2012; Elsayed, 2011; Elsayed, 2012; Elsayed, 2013; Elsayed, 2014; Wang et al, 2009; Zayed and El-Moneam, 2005; Zayed and El-Moneam, 2011). In particular, (Elabbasy et al., 2006; Elabbasy et al., 2007) investigated the global stability, periodicity character of the solution for the following recursive sequences

$$x_{n+1}=ax_n-\frac{bx_n}{cx_n-dx_{n-1}},$$

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$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Agarwal and Elsayed (2008) investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = a + \frac{d x_{n-l} x_{n-k}}{b - c x_{n-s}}.$$

Aloqeili (2006) has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar (2004) investigated the solutions of the following difference equation

$$x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Karatas et al. (2006) obtained the form of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}.$$

Yalçinkaya and Cinar (2009) studied the dynamics of the difference equation

$$x_{n+1} = \frac{a x_{n-k}}{b + c x_n^p}.$$

Other related results on rational difference equations can be found in (Elsayed and El-Metwally, 2013; Hamza and Barbary, 2008; Rafiq, 2006; Saleh and Abu-Baha, 2006; Touafek and Elsayed, 2012(a,b); Wang and Wang, 2009; Zayed and El-Moneam, 2012(a,b); Zayed and El-Moneam, 2013).

Here, we recall some notations and results which will be useful in our investigation.

Let I be an interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0,1,\ldots, \qquad (1)$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. (Equilibrium Point)
A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2. (Stability)
(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \varepsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).
(v) The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, \ldots, \overline{x})}{\partial x_{n-i}} y_{n-i}.$$

Theorem A (Grove and Ladas, 2005): Assume that $p_i \in R, i = 1,2,\ldots,k$ and $k \in \{0,1,2,\ldots\}$. Then

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, \quad n = 0,1,\ldots .$$

Definition 3. (Periodicity)
A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

2. On the Difference Equation $x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}(1+x_{n-2}x_{n-3}x_{n-4})}$

In this section we give a specific form of the solution of the first difference equation in the form

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}(1+x_{n-2}x_{n-3}x_{n-4})}, \quad (3)$$

where the initial values are arbitrary non zero real numbers.

Theorem 1. Suppose that $\{x_n\}_{n=-4}^{\infty}$ is a solution of Eq.(3). Then for $n = 0,1,\ldots$

$$x_{3n-1} = \frac{b^{n+1}}{e^n}\prod_{i=1}^{n}\left(\frac{1+icde}{1+ibcd}\right),$$

$$x_{3n} = \frac{a^{n+1}}{d^n}\prod_{i=1}^{n}\left(\frac{1+ibcd}{1+iabc}\right),$$

$$x_{3n+1} = \frac{c(de)^{n+1}}{(ab)^{n+1}(1+cde)}\prod_{i=1}^{n}\left(\frac{1+iabc}{1+(i+1)cde}\right),$$

where $x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a.$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. That is;

$$x_{3n-6} = \frac{a^{n-1}}{d^{n-2}}\prod_{i=1}^{n-2}\left(\frac{1+ibcd}{1+iabc}\right),$$

$$x_{3n-5} = \frac{c(de)^{n-1}}{(ab)^{n-1}(1+cde)}\prod_{i=1}^{n-2}\left(\frac{1+iabc}{1+(i+1)cde}\right),$$

$$x_{3n-4} = \frac{b^n}{e^{n-1}}\prod_{i=1}^{n-1}\left(\frac{1+icde}{1+ibcd}\right),$$

$$x_{3n-3} = \frac{a^n}{d^{n-1}}\prod_{i=1}^{n-1}\left(\frac{1+ibcd}{1+iabc}\right),$$

$$x_{3n-2} = \frac{c(de)^n}{(ab)^n(1+cde)}\prod_{i=1}^{n-1}\left(\frac{1+iabc}{1+(i+1)cde}\right),$$

Now, it follows from Eq.(3) that

$$x_{3n-1} = \frac{x_{3n-4}x_{3n-5}x_{3n-6}}{x_{3n-2}x_{3n-3}(1+x_{3n-4}x_{3n-5}x_{3n-6})}$$

$$= \frac{\dfrac{\frac{b^n}{e^{n-1}}\prod_{i=1}^{n-1}\left(\frac{1+icde}{1+ibcd}\right)\frac{c(de)^{n-1}}{(ab)^{n-1}(1+cde)}}{\prod_{i=1}^{n-2}\left(\frac{1+iabc}{1+(i+1)cde}\right)\frac{a^{n-1}}{d^{n-2}}\prod_{i=1}^{n-2}\left(\frac{1+ibcd}{1+iabc}\right)}}{\frac{a^n}{d^{n-1}}\prod_{i=1}^{n-1}\left(\frac{1+ibcd}{1+iabc}\right)\frac{c(de)^n}{(ab)^n(1+cde)}\prod_{i=1}^{n-1}\left(\frac{1+iabc}{1+(i+1)cde}\right)\left(1+\dfrac{\frac{b^n}{e^{n-1}}\prod_{i=1}^{n-1}\left(\frac{1+icde}{1+ibcd}\right)\frac{c(de)^{n-1}}{(ab)^{n-1}(1+cde)}}{\prod_{i=1}^{n-2}\left(\frac{1+iabc}{1+(i+1)cde}\right)\frac{a^{n-1}}{d^{n-2}}\prod_{i=1}^{n-2}\left(\frac{1+ibcd}{1+iabc}\right)}\right)}$$

$$= \frac{\left(\frac{bcd}{1+(n-1)bcd}\right)}{\frac{cde^n}{b^n(1+cde)}\prod_{i=1}^{n-1}\left(\frac{1+ibcd}{1+(i+1)cde}\right)\left(1+\left(\frac{bcd}{1+(n-1)bcd}\right)\right)}$$

$$= \frac{\text{bd}}{\frac{de^n}{b^n(1+cde)}\prod_{i=1}^{n-1}\left(\frac{1+ibcd}{1+(i+1)cde}\right)(1+(n-1)bcd+\text{bcd})}$$

$$= \frac{b^n(1+cde)bd\prod_{i=1}^{n-1}\left(\frac{1+(i+1)cde}{1+ibcd}\right)}{de^n(1+(n-1)bcd+bcd)}$$

$$= \frac{b^{n+1}(1+cde)\prod_{i=1}^{n-1}\left(\frac{1+(i+1)cde}{1+ibcd}\right)}{e^n(1+nbcd)}.$$

Hence, we have

$$x_{3n-1} = \frac{b^{n+1}}{e^n}\prod_{i=1}^{n}\left(\frac{1+icde}{1+ibcd}\right).$$

Similarly

$$x_{3n} = \frac{x_{3n-3}x_{3n-4}x_{3n-5}}{x_{3n-1}x_{3n-2}(1+x_{3n-3}x_{3n-4}x_{3n-5})}$$

$$= \frac{\dfrac{\frac{a^n}{d^{n-1}}\prod_{i=1}^{n-1}\left(\frac{1+ibcd}{1+iabc}\right)\frac{b^n}{e^{n-1}}}{\prod_{i=1}^{n-1}\left(\frac{1+icde}{1+ibcd}\right)\frac{c(de)^{n-1}}{(ab)^{n-1}(1+cde)}\prod_{i=1}^{n-2}\left(\frac{1+iabc}{1+(i+1)cde}\right)}}{\frac{b^{n+1}}{e^n}\prod_{i=1}^{n}\left(\frac{1+icde}{1+ibcd}\right)\frac{c(de)^n}{(ab)^n(1+cde)}\prod_{i=1}^{n-1}\left(\frac{1+iabc}{1+(i+1)cde}\right)\left(1+\dfrac{\frac{a^n}{d^{n-1}}\prod_{i=1}^{n-1}\left(\frac{1+ibcd}{1+iabc}\right)\frac{b^n}{e^{n-1}}}{\prod_{i=1}^{n-1}\left(\frac{1+icde}{1+ibcd}\right)\frac{c(de)^{n-1}}{(ab)^{n-1}(1+cde)}\prod_{i=1}^{n-2}\left(\frac{1+iabc}{1+(i+1)cde}\right)}\right)}$$

$$= \frac{\left(\frac{abc}{1+(n-1)abc}\right)}{\frac{bcd^n}{a^n}\prod_{i=1}^{n}\left(\frac{1}{1+ibcd}\right)\prod_{i=1}^{n-1}(1+iabc)\left(1+\left(\frac{abc}{1+(n-1)abc}\right)\right)}$$

$$= \frac{aa^n}{d^n\prod_{i=1}^{n}\left(\frac{1}{1+ibcd}\right)\prod_{i=1}^{n-1}(1+iabc)(1+(n-1)abc+abc)}$$

$$= \frac{a^{n+1}\prod_{i=1}^{n}(1+ibcd)}{d^n\prod_{i=1}^{n-1}(1+iabc)(1+nabc)}.$$

Hence, we have

$$x_{3n} = \frac{a^{n+1}}{d^n}\prod_{i=1}^{n}\left(\frac{1+ibcd}{1+iabc}\right).$$

Also, from Eq. (3) we see that

$$x_{3n+1} = \frac{x_{3n-2}x_{3n-3}x_{3n-4}}{x_{3n}x_{3n-1}(1+x_{3n-2}x_{3n-3}x_{3n-4})}$$

$$= \frac{\frac{c(de)^n}{(ab)^n(1+cde)}\prod_{i=1}^{n-1}\left(\frac{1+iabc}{1+(i+1)cde}\right)\frac{a^n}{d^{n-1}}\prod_{i=1}^{n-1}\left(\frac{1+ibcd}{1+iabc}\right)\frac{b^n}{e^{n-1}}\prod_{i=1}^{n-1}\left(\frac{1+icde}{1+ibcd}\right)}{\frac{a^{n+1}}{d^n}\prod_{i=1}^{n}\left(\frac{1+ibcd}{1+iabc}\right)\frac{b^{n+1}}{e^n}\prod_{i=1}^{n}\left(\frac{1+icde}{1+ibcd}\right)\left(1+\frac{c(de)^n}{(ab)^n(1+cde)}\prod_{i=1}^{n-1}\left(\frac{1+iabc}{1+(i+1)cde}\right)\frac{a^n}{d^{n-1}}\prod_{i=1}^{n-1}\left(\frac{1+ibcd}{1+iabc}\right)\frac{b^n}{e^{n-1}}\prod_{i=1}^{n-1}\left(\frac{1+icde}{1+ibcd}\right)\right)}$$

$$= \frac{(de)^n\left(\frac{cde}{1+ncde}\right)}{(ab)^{n+1}\prod_{i=1}^{n}\left(\frac{1+icde}{1+iabc}\right)\left(1+\left(\frac{cde}{1+ncde}\right)\right)}$$

$$= \frac{c(de)^{n+1}}{(ab)^{n+1}(1+ncde+cde)}\prod_{i=1}^{n}\left(\frac{1+iabc}{1+icde}\right)$$

$$= \frac{c(de)^{n+1}}{(ab)^{n+1}(1+(n+1)cde)}\prod_{i=1}^{n}\left(\frac{1+iabc}{1+icde}\right)$$

Hence, we have

$$x_{3n+1} = \frac{c(de)^{n+1}}{(ab)^{n+1}(1+cde)}\prod_{i=1}^{n}\left(\frac{1+iabc}{1+(i+1)cde}\right).$$

Thus, the proof is completed.

Theorem 2. Eq.(3) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.

Proof: For the equilibrium points of Eq.(3), we can write

$$\bar{x} = \frac{\bar{x}^3}{\bar{x}^2\left(1+\bar{x}^3\right)}.$$

Then we have

$$\begin{aligned}\bar{x}^3\left(1+\bar{x}^3\right) &= \bar{x}^3,\\ \bar{x}^3\left(1+\bar{x}^3-1\right) &= 0,\end{aligned}$$

or

$$\bar{x}^6 = 0.$$

Thus the equilibrium point of Eq.(3) is $\bar{x} = 0$. Let $f:(0,\infty)^5 \to (0,\infty)$ be a function defined by

$$f(u,v,w,t,p) = \frac{wtp}{uv(1+wtp)}.$$

Therefore it follows that

$$\begin{aligned}f_u(u,v,w,t,p) &= -\frac{wtp}{u^2v(1+wtp)},\\ f_v(u,v,w,t,p) &= -\frac{wtp}{uv^2(1+wtp)},\\ f_w(u,v,w,t,p) &= \frac{tp}{uv(1+wtp)^2},\\ f_t(u,v,w,t,p) &= \frac{wp}{uv(1+wtp)^2},\\ f_p(u,v,w,t,p) &= \frac{tw}{uv(1+wtp)^2},\end{aligned}$$

we see that

$$\begin{aligned}f_u(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) &= -1,\\ f_v(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) &= -1,\\ f_w(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) &= 1,\\ f_t(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) &= 1,\\ f_p(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) &= 1.\end{aligned}$$

The proof follows by using Theorem A.

Numerical examples

For confirming the results of this section, we consider numerical examples which represent different types of solutions to Eq. (3).

Example 1. We assume $x_{-4} = 5, x_{-3} = 13, x_{-2} = 7,\ x_{-1} = 3,\ x_0 = 9.$ See Fig. 1.

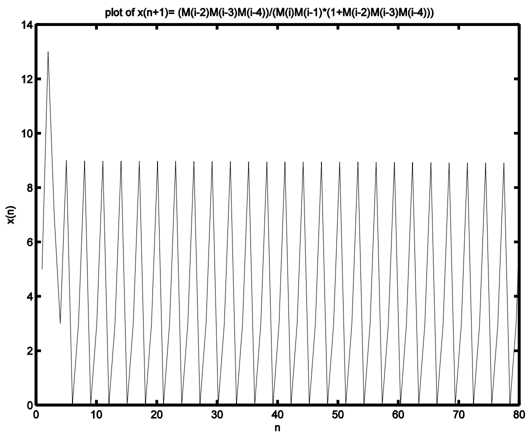

Fig. 1. This Figure shows the solution of the difference equation (3), with initial conditions $x_{-4} = 5, x_{-3} = 13, x_{-2} = 7,\ x_{-1} = 3,\ x_0 = 9$

Example 2. See Fig. 2, since $x_{-4} = 11, x_{-3} = 0.3,$ $x_{-2} = 9,\ x_{-1} = 0.8,\ x_0 = 2.$

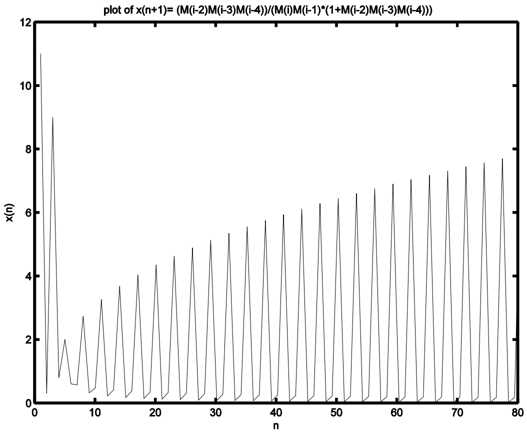

Fig. 2. This Figure shows the solutio of Eq.(3) when $x_{-4}=11, x_{-3}=0.3,\ x_{-2}=9,\ x_{-1}=0.8,\ x_0=2$

3. On the Difference Equation $x_{n+1}=\frac{x_{n-2}x_{n-3}x_{n-4}}{x_nx_{n-1}(-1+x_{n-2}x_{n-3}x_{n-4})}$

In this section we obtain the solution of the second difference equation in the form

$$x_{n+1}=\frac{x_{n-2}x_{n-3}x_{n-4}}{x_nx_{n-1}(-1+x_{n-2}x_{n-3}x_{n-4})}, \qquad (4)$$

where the initial values are arbitrary non zero real numbers with $x_0x_{-1}x_{-2}\neq 1,\ x_{-1}x_{-2}x_{-3}\neq 1, x_{-2}x_{-3}x_{-4}\neq 1$.

Theorem 3. Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(4). Then for $n=0,1,...$

$$x_{6n-2}=\frac{c(de)^{2n}(-1+abc)^n}{(ab)^{2n}(-1+cde)^n},$$
$$x_{6n-1}=\frac{b^{2n+1}(-1+cde)^n}{e^{2n}(-1+bcd)^n},$$
$$x_{6n}=\frac{a^{2n+1}(-1+bcd)^n}{d^{2n}(-1+abc)^n},$$
$$x_{6n+1}=\frac{c(de)^{2n+1}(-1+abc)^n}{(ab)^{2n+1}(-1+cde)^{n+1}},$$
$$x_{6n+2}=\frac{b^{2n+2}(-1+cde)^{n+1}}{e^{2n+1}(-1+bcd)^{n+1}},$$
$$x_{6n+3}=\frac{a^{2n+2}(-1+bcd)^{n+1}}{d^{2n+1}(-1+abc)^{n+1}},$$

where $x_{-4}=e,\ x_{-3}=d,\ x_{-2}=c,\ x_{-1}=b,\ x_0=a.$

Proof: For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. That is;

$$x_{6n-8}=\frac{c(de)^{2n-2}(-1+abc)^{n-1}}{(ab)^{2n-2}(-1+cde)^{n-1}},$$
$$x_{6n-7}=\frac{b^{2n-1}(-1+cde)^{n-1}}{e^{2n-2}(-1+bcd)^{n-1}},$$
$$x_{6n-6}=\frac{a^{2n-1}(-1+bcd)^{n-1}}{d^{2n-2}(-1+abc)^{n-1}},$$
$$x_{6n-5}=\frac{c(de)^{2n-1}(-1+abc)^{n-1}}{(ab)^{2n-1}(-1+cde)^{n}},$$
$$x_{6n-4}=\frac{b^{2n}(-1+cde)^{n}}{e^{2n-1}(-1+bcd)^{n}},$$
$$x_{6n-3}=\frac{a^{2n}(-1+bcd)^{n}}{d^{2n-1}(-1+abc)^{n}}.$$

Now, it follows from Eq.(4) that

$$x_{6n-2}=\frac{x_{6n-5}x_{6n-6}x_{6n-7}}{x_{6n-3}x_{6n-4}(-1+x_{6n-5}x_{6n-6}x_{6n-7})}$$

$$=\frac{\frac{c(de)^{2n-1}(-1+abc)^{n-1}}{(ab)^{2n-1}(-1+cde)^{n}}\frac{a^{2n-1}(-1+bcd)^{n-1}}{d^{2n-2}(-1+abc)^{n-1}}\frac{b^{2n-1}(-1+cde)^{n-1}}{e^{2n-2}(-1+bcd)^{n-1}}}{\left(\frac{a^{2n}(-1+bcd)^{n}}{d^{2n-1}(-1+abc)^{n}}\frac{b^{2n}(-1+cde)^{n}}{e^{2n-1}(-1+bcd)^{n}}\right)\left(-1+\frac{c(de)^{2n-1}(-1+abc)^{n-1}}{(ab)^{2n-1}(-1+cde)^{n}}\frac{a^{2n-1}(-1+bcd)^{n-1}}{d^{2n-2}(-1+abc)^{n-1}}\frac{b^{2n-1}(-1+cde)^{n-1}}{e^{2n-2}(-1+bcd)^{n-1}}\right)}$$

$$=\frac{\frac{cde}{(-1+cde)}}{\left(\frac{(ab)^{2n}(-1+cde)^{n}}{(ed)^{2n-1}(-1+abc)^{n}}\right)\left(-1+\frac{cde}{(-1+cde)}\right)}$$

$$=\frac{cde(ed)^{2n-1}(-1+abc)^{n}}{(ab)^{2n}(-1+cde)^{n}(-(-1+cde)+cde)}.$$

Then, we have

$$x_{6n-2}=\frac{c(de)^{2n}(-1+abc)^{n}}{(ab)^{2n}(-1+cde)^{n}}.$$

Similarly

$$x_{6n-1}=\frac{x_{6n-4}x_{6n-5}x_{6n-6}}{x_{6n-2}x_{6n-3}(-1+x_{6n-4}x_{6n-5}x_{6n-6})}$$

$$=\frac{\frac{b^{2n}(-1+cde)^{n}}{e^{2n-1}(-1+bcd)^{n}}\frac{c(de)^{2n-1}(-1+abc)^{n-1}}{(ab)^{2n-1}(-1+cde)^{n}}\frac{a^{2n-1}(-1+bcd)^{n-1}}{d^{2n-2}(-1+abc)^{n-1}}}{\left(\frac{c(de)^{2n}(-1+abc)^{n}}{(ab)^{2n}(-1+cde)^{n}}\frac{a^{2n}(-1+bcd)^{n}}{d^{2n-1}(-1+abc)^{n}}\right)\left(-1+\frac{b^{2n}(-1+cde)^{n}}{e^{2n-1}(-1+bcd)^{n}}\frac{c(de)^{2n-1}(-1+abc)^{n-1}}{(ab)^{2n-1}(-1+cde)^{n}}\frac{a^{2n-1}(-1+bcd)^{n-1}}{d^{2n-2}(-1+abc)^{n-1}}\right)}$$

$$=\frac{\frac{bcd}{(-1+bcd)}}{\left(\frac{cd(e)^{2n}(-1+bcd)^{n}}{b^{2n}(-1+cde)^{n}}\right)\left(-1+\frac{bcd}{(-1+bcd)}\right)}$$

$$=\frac{b^{2n+1}(-1+cde)^{n}}{e^{2n}(-1+bcd)^{n}(-(-1+bcd)+bcd)}.$$

Then we obtain

$$x_{6n}=\frac{a^{2n+1}(-1+bcd)^{n}}{d^{2n}(-1+abc)^{n}}.$$

Similarly, we can easily obtain the other relations. Thus, the proof is completed.

Theorem 4. Eq.(4) has two equilibrium points which are $0, \sqrt[3]{2}$ and these equilibrium points are not locally asymptotically stable.

Proof: For the equilibrium points of Eq.(4), we can write

$$\overline{x} = \frac{\overline{x}^3}{\overline{x}^2\left(-1+\overline{x}^3\right)}.$$

Then we see that

$$\overline{x}^3\left(-1+\overline{x}^3\right) = \overline{x}^3,$$

or

$$\overline{x}^3\left(\overline{x}^3-2\right) = 0.$$

Thus the equilibrium points of Eq.(4) are $0, \sqrt[3]{2}$. Let $f: (0,\infty)^5 \to (0,\infty)$ be a function defined by

$$f(u,v,w,t,p) = \frac{wtp}{uv(-1+wtp)}.$$

Therefore it follows that

$$\begin{aligned} f_u(u,v,w,t,p) &= -\frac{wtp}{u^2v(-1+wtp)}, \\ f_v(u,v,w,t,p) &= -\frac{wtp}{uv^2(-1+wtp)}, \\ f_w(u,v,w,t,p) &= -\frac{tp}{uv(-1+wtp)^2} \\ f_t(u,v,w,t,p) &= -\frac{wp}{uv(-1+wtp)^2}, \\ f_p(u,v,w,t,p) &= -\frac{tw}{uv(-1+wtp)^2}, \end{aligned}$$

we see that

$$\begin{aligned} f_u(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) &= \pm 1, \\ f_v(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) &= \pm 1, \\ f_w(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) &= -1, \\ f_t(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) &= -1, \\ f_p(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) &= -1. \end{aligned}$$

The proof follows by using Theorem A.

Theorem 5. The following statements are true:
(a) If $x_0 \neq x_{-3},\ x_{-1} \neq x_{-4},\ x_{-2}x_{-3}x_{-4} \neq 2$, then all the solutions of Eq.(4) are unbounded.
(b) Eq.(4) has a periodic solutions of period six iff $x_0 = x_{-3},\ x_{-1} = x_{-4}$, and will be take the form

$$\left\{\begin{matrix} x_{-2}, x_{-1}, x_0, \dfrac{x_{-2}}{(-1+x_{-2}x_{-3}x_{-4})}, \\ x_{-1}, x_0, x_{-2}, x_{-1}, x_0, \ldots \end{matrix}\right\}.$$

(c) Eq.(4) has a periodic solutions of period three iff $x_0 = x_{-3},\ x_{-1} = x_{-4},\ x_{-2}x_{-3}x_{-4} = 2$ and will take the form $\{x_{-2}, x_{-1}, x_0, x_{-2}, x_{-1}, x_0, \ldots\}$.

Proof:
(a) The proof in this case follows directly from the form of the solution as given in Theorem 3.
(b) First, suppose that there exists a prime period six solution of Eq.(4) of the form

$$\begin{matrix} x_{-2}, x_{-1}, x_0, \dfrac{x_{-2}}{(-1+x_{-2}x_{-3}x_{-4})}, \\ x_{-1}, x_0, x_{-2}, x_{-1}, x_0, \ldots . \end{matrix}$$

Then we see from the form of solution of Eq.(4) that

$$\begin{aligned} c &= \frac{c(de)^{2n}(-1+abc)^n}{(ab)^{2n}(-1+cde)^n}, \\ b &= \frac{b^{2n+1}(-1+cde)^n}{e^{2n}(-1+bcd)^n}, \\ a &= \frac{a^{2n+1}(-1+bcd)^n}{d^{2n}(-1+abc)^n}, \\ \frac{c}{(-1+cde)} &= \frac{c(de)^{2n+1}(-1+abc)^n}{(ab)^{2n+1}(-1+cde)^{n+1}}, \\ b &= \frac{b^{2n+2}(-1+cde)^{n+1}}{e^{2n+1}(-1+bcd)^{n+1}}, \\ a &= \frac{a^{2n+2}(-1+bcd)^{n+1}}{d^{2n+1}(-1+abc)^{n+1}}. \end{aligned}$$

Then

$$a = d, \quad b = e.$$

Second, suppose that

$$x_0 = x_{-3}, \quad x_{-1} = x_{-4}.$$

Then we see from the solution of Eq.(4) that

$$x_{6n-2} = c,\ \ x_{6n-1} = b, \quad x_{6n} = a, \begin{matrix} x_{6n+1} = \dfrac{c}{(-1+cde)}, \\ x_{6n+2} = b, \quad x_{6n+3} = a, \end{matrix}$$

Thus we have a period six solution and the proof is complete.
(c) The proof in this case follows directly from case (b).

Numerical examples

Here different types of solutions of Eq. (4) will be represented

Example 3. We consider $x_{-4} = 7,\ x_{-3} = 3, x_{-2} = 9,\ x_{-1} = 8,\ x_0 = 2$. See Fig. 3.

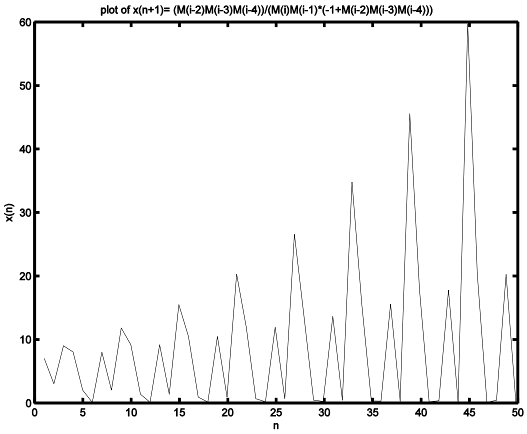

Fig. 3. This Figure shows the behavior of the solution of difference equation $x_{n+1}=\frac{x_{n-2}x_{n-3}x_{n-4}}{x_nx_{n-1}(-1+x_{n-2}x_{n-3}x_{n-4})}$, where $x_{-4}=7,\ x_{-3}=3, x_{-2}=9,\ x_{-1}=8,\ x_0=2$

Example 4. See Fig. 4, since $x_{-4}=3, x_{-3}=7$, $x_{-2}=6,\ x_{-1}=3,\ x_0=7$.

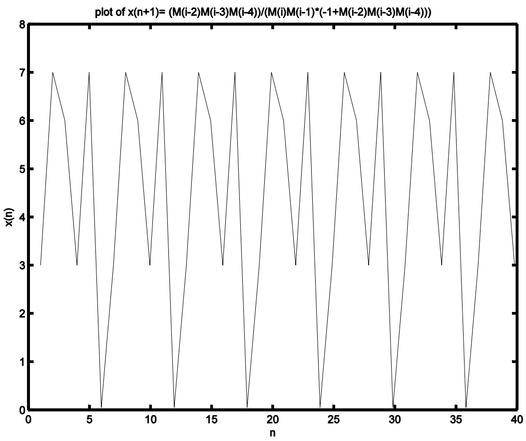

Fig. 4. This Figure shows the periodicity of solution of Eq. (4), when we take the initial conditions are $x_{-4}=3, x_{-3}=7,\ x_{-2}=6,\ x_{-1}=3,\ x_0=7$

The proofs of the theorems in the following section are similar to thoes presented in the previous sections and so will be omitted.

4. On the Difference Equation $x_{n+1}=\frac{x_{n-2}x_{n-3}x_{n-4}}{x_nx_{n-1}(1-x_{n-2}x_{n-3}x_{n-4})}$

In this section we obtain the solution of the following difference equation

$$x_{n+1}=\frac{x_{n-2}x_{n-3}x_{n-4}}{x_nx_{n-1}(1-x_{n-2}x_{n-3}x_{n-4})}, \qquad (5)$$

where the initial values are arbitrary non zero real numbers.

Theorem 6. Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(5). Then for $n=0,1,...$

$$x_{3n-1}=\frac{b^{n+1}}{e^n}\prod_{i=1}^{n}\left(\frac{1-icde}{1-ibcd}\right),$$

$$x_{3n}=\frac{a^{n+1}}{d^n}\prod_{i=1}^{n}\left(\frac{1-ibcd}{1-iabc}\right),$$

$$x_{3n+1}=\frac{c(de)^{n+1}}{(ab)^{n+1}(1-cde)}\prod_{i=1}^{n}\left(\frac{1-iabc}{1-(i+1)cde}\right).$$

Theorem 7. Eq.(5) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.

Example 5. Assume that $x_{-4}=10,\ x_{-3}=4, x_{-2}=9,\ x_{-1}=6,\ x_0=2$ see Fig. 5.

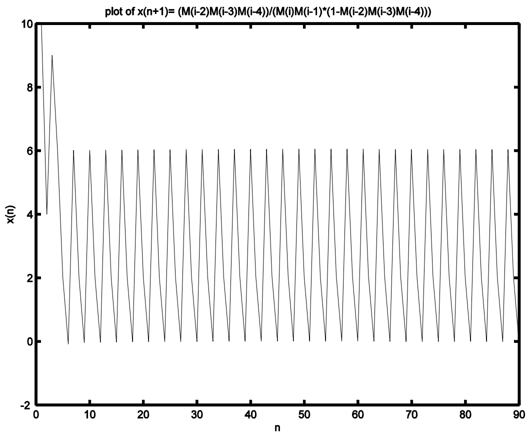

Fig. 5. This Figure shows the solution of the equation (5) when $x_{-4}=10,\ x_{-3}=4, x_{-2}=9,\ x_{-1}=6,\ x_0=2$

Example 6. See Fig. 6 since $x_{-4}=2, x_{-3}=0.7$, $x_{-2}=0.5,\ x_{-1}=0.8,\ x_0=2$.

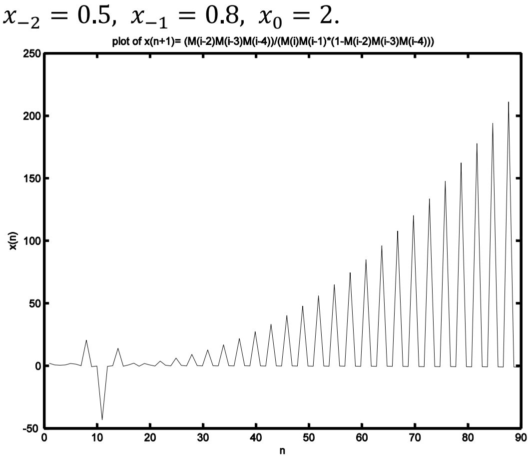

Fig. 6. This Figure shows the solution of the difference equation (5) where $x_{-4}=2, x_{-3}=0.7,\ x_{-2}=0.5$, $x_{-1}=0.8,\ x_0=2$

5. On the Difference Equation $x_{n+1}=\frac{x_{n-2}x_{n-3}x_{n-4}}{x_nx_{n-1}(-1-x_{n-2}x_{n-3}x_{n-4})}$

Here we obtain a form of the solutions of the equation

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}(-1-x_{n-2}x_{n-3}x_{n-4})}, \quad (6)$$

where the initial values are arbitrary non zero real numbers with $x_0x_{-1}x_{-2} \neq -1$, $x_{-1}x_{-2}x_{-3} \neq -1$, $x_{-2}x_{-3}x_{-4} \neq -1$.

Theorem 8. Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(6). Then for $n = 0,1, ...$

$$x_{6n-2} = \frac{c(de)^{2n}(-1-abc)^n}{(ab)^{2n}(-1-cde)^n},$$
$$x_{6n-1} = \frac{b^{2n+1}(-1-cde)^n}{e^{2n}(-1-bcd)^n},$$
$$x_{6n} = \frac{a^{2n+1}(-1-bcd)^n}{d^{2n}(-1-abc)^n},$$
$$x_{6n+1} = \frac{c(de)^{2n+1}(-1-abc)^n}{(ab)^{2n+1}(-1-cde)^{n+1}},$$
$$x_{6n+2} = \frac{b^{2n+2}(-1-cde)^{n+1}}{e^{2n+1}(-1-bcd)^{n+1}},$$
$$x_{6n+3} = \frac{a^{2n+2}(-1-bcd)^{n+1}}{d^{2n+1}(-1-abc)^{n+1}}.$$

Theorem 9. Eq. (6) has two equilibrium points which are 0, $\sqrt[3]{-2}$ and these equilibrium points are not locally asymptotically stable.

Theorem 10. The following statements are true:
(a) If $x_0 \neq x_{-3}$, $x_{-1} \neq x_{-4}$, $x_{-2}x_{-3}x_{-4} \neq -2$, then all the solutions of Eq.(6) are unbounded.
(b) Eq.(6) has a periodic solutions of period six iff $x_0 = x_{-3}$, $x_{-1} = x_{-4}$, and will be take the form $\left\{ \begin{matrix} x_{-2}, x_{-1}, x_0, \frac{x_{-2}}{(-1-x_{-2}x_{-3}x_{-4})}, \\ x_{-1}, x_0, x_{-2}, x_{-1}, x_0, \ldots \end{matrix} \right\}$.
(c) Eq.(6) has periodic solutions of period three iff $x_0 = x_{-3}$, $x_{-1} = x_{-4}$, $x_{-2}x_{-3}x_{-4} = -2$ and will be take the form $\{x_{-2}, x_{-1}, x_0, x_{-2}, x_{-1}, x_0, \ldots\}$.

Example 7. Consider $x_{-4} = 0.3$, $x_{-3} = 0.7$, $x_{-2} = -5$, $x_{-1} = 0.8$, $x_0 = 2$ see Fig. 7.

Example 8. Figure 8 shows the solutions when $x_{-4} = -5$, $x_{-3} = 9$, $x_{-2} = 8$, $x_{-1} = -5$, $x_0 = 9$.

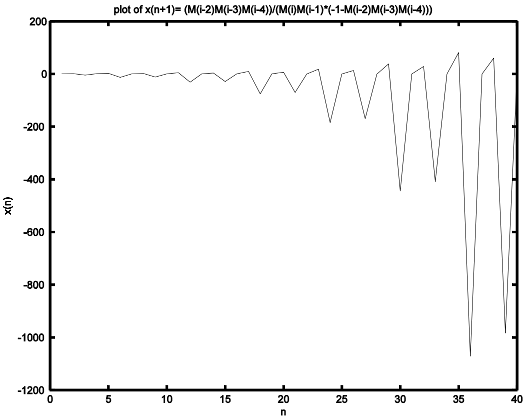

Fig. 7. This Figure shows the solution of the equation $x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}(-1-x_{n-2}x_{n-3}x_{n-4})}$, when $x_{-4} = 0.3$, $x_{-3} = 0.7$, $x_{-2} = -5$, $x_{-1} = 0.8$, $x_0 = 2$

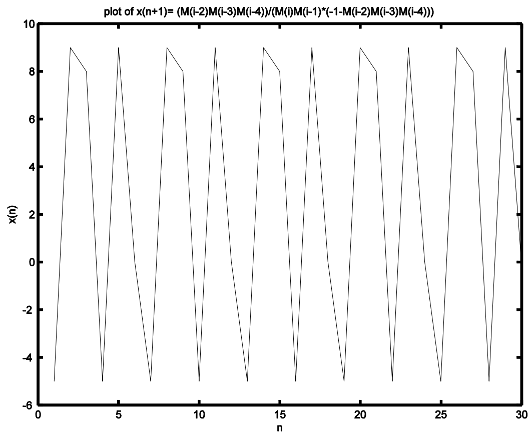

Fig. 8. This Figure shows the solution of the equation (6) with the initial conditions $x_{-4} = -5$, $x_{-3} = 9$, $x_{-2} = 8$, $x_{-1} = -5$, $x_0 = 9$

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