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ON EINSTEIN (α,β)-METRICS*

B. REZAEI[1**], A. RAZAVI[2] AND N. SADEGHZADEH[3]

Department of Mathematics and Computer Science of Amir-Kabir University, Tehran, I. R. of Iran
Emails: [1]b_rezaei@cic.aut.ac.ir, [2]arazavi@cic.aut.ac.ir
[3]nasrin_sadeghi@cic.aut.ac.ir

Abstract – In this paper we consider some (α,β)-metrics such as generalized Kropina, Matsumoto and $F=\frac{(\alpha+\beta)^2}{\alpha}$ metrics, and obtain necessary and sufficient conditions for them to be Einstein metrics when β is a constant Killing form. Then we prove with this assumption that the mentioned Einstein metrics must be Riemannian or Ricci flat.

Keywords – Einstein Finsler metrics, (α,β) - metrics, Schur lemma

1. INTRODUCTION

A Finsler space is a manifold M equipped with a family of smoothly varying Minkowsky norms, one on each tangent space. Riemannian metrics are examples of Finsler norms that arise from an inner-product. A Finsler metric function $L(x,y)$ is called an (α,β)-metric if L is a positively homogeneous function of a Riemannian metric $\alpha(x,y)=\sqrt{a_{ij}(x)y^i y^j}$ and a differential 1-form $\beta(x,y)=b_i(x)y^i$ of degree one. The especially interesting examples of (α,β)-metrics are Randers and Kropina metrics. Randers metric and its Ricci tensor are related by their histories in physics. The well-known Ricci tensor was introduced in 1904 by G. Ricci. Nine years later Ricci's work was used to formulate Einstein's gravitation theory [1].

Einstein metrics are defined in the next section but, loosely, we will say a Finsler metric F is Einstein if the average of its flag curvatures at a flag pole y is a function of position x alone, rather than the priori position x and flag pole y. C. Robles investigated Randers Einstein metrics in her Ph.D. Thesis in 2003. She obtained necessary and sufficient conditions for Randers metric to be Einstein and by using Einstein navigation description, she proved the second Schur lemma [2].

The classification of projectively related Einstein Finsler metrics on compact manifold is investigated in [3, 4]. In this paper we consider the famous (α,β)-metrics such as generalized Kropina, Matsumoto and $F=\frac{(\alpha+\beta)^2}{\alpha}$ metrics, and obtain the necessary and sufficient conditions for them to be Einstein metrics when β is a constant Killing form. Also, we prove under these conditions that the above metrics must be Riemannian or Ricci flat. We also use Einstein convention in the following.

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**Corresponding author

2. PRELIMINARIES

Let M be an n-dimensional C^{∞} manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ the tangent space of M. Each element of TM has the form (x,y), where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM \to M$ is given by $\pi(x,y) = x$. The *pull-back tangent bundle* $\pi^* TM$ is a vector bundle over TM_0 whose fiber $\pi_v^* TM$ at $v \in TM_0$ is just T_xM, where $\pi(v) = x$. Then

$$\pi^* TM = \{(x,y,v) \mid y \in T_xM_0, v \in T_xM\}.$$

A *Finsler metric* on a manifold M is a function $F : TM \to [0,\infty)$, having the following properties:
(i) F is C^{∞} on TM_0;
(ii) $F(x,\lambda y) = \lambda F(x,y) \quad \lambda > 0$;
(iii) For any tangent vector $y \in T_xM$, the vertical Hessian of $\frac{F^2}{2}$ given by

$$g_{ij}(x,y) = \left[\frac{1}{2}F^2\right]_{y^i y^j},$$

is positive definite [5].

Every Finsler metric F induces a spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ which is defined by

$$G^i(x,y) := \frac{1}{4} g^{il}(x,y)\{2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\} y^j y^k,$$

where the matrix (g^{ij}) means the inverse of matrix (g_{ij}) [6]. The coefficients G^i_j, G^i_{jk} of the Berwald connection can be derived from the spray G^i as follows:

$$G^i_j = \frac{\partial G^i}{\partial y^j}, G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}.$$

The Riemann curvature $\mathbf{K_y} = K^i_k dx^k \otimes \frac{\partial}{\partial x^i}|_p : T_pM \to T_pM$ is defined by

$$K^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{1}$$

When $F = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric, $K^i_k = R^i_{jkl}(x) y^j y^l$, where $R^i_{jkl}(x)$ denote the coefficients of the usual Riemannian curvature tensor. Thus the quantity $\mathbf{K_y}$ in Finsler geometry is still called the Riemann curvature [7].

The Ricci scalar function of F is given by

$$\rho := \frac{1}{F^2} K^i_i.$$

Therefore, the Ricci scalar function is positive homogeneous of degree 0 in y [8]. This means $\rho(x,y)$ depends on the direction of the flag pole y, but not its length. The Ricci tensor of a Finsler metric F is defined by

$$Ric_{ij} := \{\frac{1}{2} K^m_m\}_{y^i y^j}$$

Ricci-flat manifolds are manifolds whose Ricci tensor vanishes. In physics, Riemannain Ricci-flat manifolds are important, because they represent vacuum solutions to Einstein's equations.

A Finsler metric is said to be an Einstein metric if the Ricci scalar function is a function of x alone, equivalently [8]

$$Ric_{ij}=\rho(x)g_{ij}.$$

Ricci-flat manifolds are special cases of Einstein manifolds.

Let (M,F) be an n-dimensional Finsler space equipped with an (α,β)-metric F, where

$$\alpha(y)=\sqrt{a_{ij}(x)y^iy^j},\beta(y)=b_i(x)y^i,$$

M. Matsumoto [9] showed that G^i of (α,β)-metric space are given by

$$2G^i=\gamma^i_{00}+2B^i, \tag{2}$$

where

$$B^i=(E/\alpha)y^i+(\alpha F_\beta/F_\alpha)s^i_0-(\alpha F_{\alpha\alpha}/F_\alpha)\,C\{(y^i/\alpha)-(\alpha/\beta)b^i\},$$
$$E=(\beta F_\beta/F)\,C,C=\alpha\beta(r_{00}F_\alpha-2\alpha s_0F_\beta)/2(\beta^2F_\alpha+\alpha\gamma^2F_{\alpha\alpha}),$$
$$b^i=a^{ir}b_r,b^2=b^rb_r,\gamma^2=b^2\alpha^2-\beta^2,$$
$$r_{ij}:=\frac{1}{2}(b_{i|j}+b_{j|i}),s_{ij}:=\frac{1}{2}(b_{i|j}-b_{j|i}),$$
$$s^i_j:=a^{ih}s_{hj},s_j:=b_is^i_j.$$

The symbol $"|"$ in the above formula stands for the h-covariant derivation with respect to the Riemannian connection in the space (M,α), and the matrix (a^{ij}) means the inverse of matrix (a_{ij}). The functions γ^i_{jk} stand for the Christoffel symbols in the space (M,α), and the suffix 0 means transvecting by y^i.

Putting

$$\tilde{p}=\beta(\beta F_\alpha F_\beta-\alpha FF_{\alpha\alpha})/2F(\beta^2F_\alpha+\alpha\gamma^2F_{\alpha\alpha}), \tag{3}$$

$$\tilde{q}=-\alpha\beta F_\beta(\beta F_\alpha F_\beta-\alpha FF_{\alpha\alpha})/FF_\alpha(\beta^2F_\alpha+\alpha\gamma^2F_{\alpha\alpha}), \tag{4}$$

$$\tilde{r}_1=\alpha F_\beta/F_\alpha \tag{5}$$

$$\tilde{s}_0=\alpha^3F_{\alpha\alpha}/2(\beta^2F_\alpha+\alpha\gamma^2F_{\alpha\alpha}), \tag{6}$$

$$\tilde{t}_1=-\alpha^4F_{\alpha\alpha}F_\beta/F_\alpha(\beta^2F_\alpha+\alpha\gamma^2F_{\alpha\alpha}). \tag{7}$$

We get,

$$B^i=(\tilde{p}r_{00}+\tilde{q}_0s_0)y^i+\tilde{r}_1s^i_0+(\tilde{s}_0r_{00}+\tilde{t}_1s_0)b^i. \tag{8}$$

Substituting (8) in (2) and (1), we obtain **Berwald**'s formula in split and covariantized form:

$$K^i_k(y)=\overline{K}^i_k+\{2B^i_{|k}-y^j(B^i_{|j})_{y^k}-(B^i)_{y^j}(B^j)_{y^k}+2B^j(B^i)_{y^jy^k}\}.$$

The one form β is said to be Killing (closed) one form if $r_{ij}=0$ ($s_{ij}=0$ respectively). β is said to be a constant Killing form if it is Killing and has constant length with respect to α, equivalently

$r_{ij}=0,s_i=0.$

Example 2.1. Let $F=\alpha+\beta$ be the family of Randers metrics on S^3 constructed in [10]. β satisfies that $r_{ij}=0$ and $s_i=0$. Moreover, the authors have found a special family of these Randers metrics with constant flag curvature $K=1$.

3. SOME SPECIAL (α,β)-METRIC SPACES

3.1. generalized Kropina spaces

In this section we consider the case of generalized Kropina spaces. A generalized Kropina metric is given by $F=\alpha^{1-p}\beta^p$, where $p\neq 0,1$. We get

$$F_\alpha=(1-p)\alpha^{-p}\beta^p, F_\beta=p\alpha^{1-p}\beta^{p-1}. \tag{9}$$

Substituting (9) in (5), we get

$$\tilde{r}_1=\frac{p\alpha^2}{(1-p)\beta}. \tag{10}$$

Now we suppose that β is a constant Killing form, then by substituting (10) in (8), we have

$$B^i=\frac{p\alpha^2}{(1-p)\beta}s^i_0. \tag{11}$$

From (11), we obtain

$$B^i_{|j}=-\frac{p\alpha^2 b_{0|j}}{(1-p)\beta^2}s^i_0+\frac{p\alpha^2}{(1-p)\beta}s^i_{0|j}, \tag{12}$$

$$B^i_{.j}=\left(\frac{2py_j}{(1-p)\beta}-\frac{pb_j\alpha^2}{(1-p)\beta^2}\right)s^i_0+\frac{p\alpha^2}{(1-p)\beta}s^i_j, \tag{13}$$

where $B^i_{.j}=B^i_{y^j}$. Using (13) we have

$$B^jB^i_{.j.i}=0, \tag{14}$$

$$B^j_{.i}B^i_{.j}=\frac{p^2\alpha^4}{(1-p)^2\beta^2}s^{ij}s_{ij}-\frac{4p^2\alpha^2}{(1-p)^2\beta^2}s^i_0s_{0i}. \tag{15}$$

Derivating (12) by y^i and transvecting by y^j we get

$$y^j(B^i_{|j})_{.i}=0. \tag{16}$$

Substituting (12)-(16) in Berwald's formula, this is re-written as:

$$K^i_i=\overline{K}^i_i+\frac{2p(p+1)\alpha^2}{(1-p)^2\beta^2}s^i_0s_{i0}-\frac{p^2\alpha^4}{(1-p)^2\beta^2}s^{ij}s_{ij}+\frac{2p\alpha^2}{(1-p)\beta}s^i_{0|i}. \tag{17}$$

Where $\overline{K}^i_i$ is the spray curvature of the Riemannian metric a_{ij}.
As a result of (17), we can derive the spray curvature of the Kropina metric with constant Killing, if set p=-1,

$$K^i_i = \overline{K}^i_i - \frac{\alpha^4}{4\beta^2} s^{ij} s_{ij} - \frac{\alpha^2}{\beta} s^i_{0|i}. \tag{18}$$

3.2. Matsumoto spaces

In this section we consider the case of Matsumoto spaces. The Matsumoto metric is $F = \frac{\alpha^2}{\alpha-\beta}$, so

$$F_\alpha = (\alpha^2 - 2\alpha\beta)/(\alpha-\beta)^2, F_\beta = \alpha^2/(\alpha-\beta)^2. \tag{19}$$

Substituting (19) in (5), we get

$$\tilde{r}_1 = \frac{\alpha^2}{\alpha - 2\beta}. \tag{20}$$

Now we suppose that β is a constant Killing form, so by substituting (20) in (8), we have

$$B^i = \frac{\alpha^2}{\alpha-2\beta} s^i_0, \tag{21}$$

from (21), we get

$$B^i_{|j} = \frac{2\alpha^2 b_{0|j}}{(\alpha-2\beta)^2} s^i_0 + \frac{\alpha^2}{\alpha-2\beta} s^i_{0|j}, \tag{22}$$

$$B^i_{.j} = \frac{2y_j}{\alpha-2\beta} s^i_0 - \frac{\alpha y_j}{(\alpha-2\beta)^2} s^i_0 + \frac{2b_j\alpha^2}{(\alpha-2\beta)^2} s^i_0 + \frac{\alpha^2}{\alpha-2\beta} s^i_j. \tag{23}$$

Using (23), we get

$$B^j B^i_{.j.i} = 0, \tag{24}$$

$$B^j_{.i} B^i_{.j} = \frac{\alpha^4}{(\alpha-2\beta)^2} s^{ij} s_{ij} - \frac{2\alpha^3}{(\alpha-2\beta)^3} s^i_0 s_{0i} + \frac{4\alpha^2}{(\alpha-2\beta)^2} s^i_0 s_{0i}. \tag{25}$$

Derivating (22) by y^i and transvecting by y^j

$$y^j (B^i_{|j})_{.i} = 0. \tag{26}$$

Substituting (22)-(26) in Berwald's formula, this is re-written as:

$$K^i_i = \overline{K}^i_i + \frac{-2\alpha^3}{(\alpha-2\beta)^3} s^i_0 s_{i0} + \frac{2\alpha^2}{\alpha-2\beta} s^i_{0|i} - \frac{\alpha^4}{(\alpha-2\beta)^2} s^{ij} s_{ij}. \tag{27}$$

Recollect $\overline{K}^i_i$ is the spray curvature of the Riemannian metric a_{ij}.

3.3. $F = \frac{(\alpha+\beta)2}{\alpha}$ ***metrics***

In this section we consider the space of $F = \frac{(\alpha+\beta)^2}{\alpha}$ metric. From (5) we have

$$F_\alpha = (\alpha^2 - \beta^2)/\alpha^2, F_\beta = 2(\alpha+\beta)/\alpha. \tag{28}$$

Substituting (28) in (5), we get

$$r_1 = \frac{2\alpha^2}{\alpha - \beta}. \tag{29}$$

Now we suppose that β is a constant Killing form, so by substituting (29) in (8), we have

$$B^i = \frac{2\alpha^2}{\alpha - \beta} s^i_0. \tag{30}$$

From (30), we get

$$B^i_{|j} = \frac{2\alpha^2 b_{0|j}}{(\alpha-\beta)^2} s^i_0 + \frac{2\alpha^2}{\alpha-\beta} s^i_{0|j}, \tag{31}$$

$$B^i_{.j} = \frac{4y_j}{(\alpha-\beta)} s^i_0 - \frac{2\alpha y_j}{(\alpha-\beta)^2} s^i_0 + \frac{2b_j\alpha^2}{(\alpha-\beta)^2} s^i_0 + \frac{2\alpha^2}{\alpha-\beta} s^i_j. \tag{32}$$

Using (32) we get

$$B^j B^i_{.j.i} = 0, \tag{33}$$

$$B^j_{.i} B^i_{.j} = \frac{4\alpha^4}{(\alpha-\beta)^2} s^{ij} s_{ij} - \frac{8\alpha^3}{(\alpha-\beta)^3} s^i_0 s_{0i} + \frac{16\alpha^2}{(\alpha-\beta)^2} s^i_0 s_{0i}. \tag{34}$$

Derivating (31) by y^i and transvecting by y^j

$$y^j (B^i_{|j})_{.i} = 0. \tag{35}$$

Substituting (31)-(35) in Berwald's formula, this is re-written as:

$$K^i_i = \overline{K}^i_i + \frac{12\alpha^2}{(\alpha-\beta)^2} s^i_0 s_{i0} + \frac{4\alpha^2}{\alpha-\beta} s^i_{0|i} - \frac{4\alpha^4}{(\alpha-\beta)^2} s^{ij} s_{ij} - \frac{8\alpha^3}{(\alpha-\beta)^3} s^i_0 s_{i0}. \tag{36}$$

Where $\overline{K}^i_i$ is the spray curvature of the Riemannian metric a_{ij}. The following example is special case of this metric such that Berwald has investigated it.

Example 3.1. Suppose

$$F = \frac{(\sqrt{|y|^2 - (|x|^2|y|^2 - <x,y>^2)} + <x,y>)^2}{(1-|x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - <x,y>^2)}}.$$

F is the well-known metric defined on the unit ball $B^n \subseteq R^n$ centered at the origin. F is projectively flat and, moreover, has zero flag curvature $K = 0$ [11].

4. EINSTEIN CRITERION

In this section we assume that Ricci scalars of the mentioned (α, β)-metrics are functions of x alone, i.e. F is Einstein. We have $F^2 Ric(x) = K^i_i$, so we can derive necessary and sufficient conditions for them to be Einstein.

4.1. Einstein Matsumoto metrics

From (27),

$$0=\overline{Ric}_{00}-\frac{2\alpha^3}{(\alpha-2\beta)^3}s^i_0 s_{i\,0}+\frac{2\alpha^2}{\alpha-2\beta}s^i_{0|i}-\frac{\alpha^4}{(\alpha-2\beta)^2}s^{ij}s_{ij}$$

$$-\frac{\alpha^4}{(\alpha-\beta)^2}Ric(x). \tag{37}$$

Multiplying (37) by $(\alpha-\beta)^2(\alpha-2\beta)^3$ removes y from the denominators and we can derive the criterion for the Matsumoto metric to be Einstein as follows:

$$Rat+\alpha Irrat=0,$$

where Rat and Irrat are, respectively, degree 7 and degree 6 polynomials in y.

Lemma 4.1. A Matsumoto metric with constant Killing form β is Einstein if and only if both Rat=0 and Irrat=0 hold.

$$Rat=(-8\alpha^4\beta-24\alpha^2\beta^3-8\beta^5)\overline{Ric}_{00}+4\alpha^4\beta s^i_0 s_{i\,0}+(2\alpha^6+26\alpha^4\beta^2+8\alpha^2\beta^4)s^i_{0|i}$$

$$-(4\alpha^6\beta+2\alpha^4\beta^3)s^{ij}s_{ij}+(6\alpha^6\beta+8\alpha^4\beta^3)Ric(x),$$

$$Irrat=(\alpha^4+19\alpha^2\beta^2+22\beta^4)\overline{Ric}_{00}-2(\alpha^4+\alpha^2\beta^2)s^i_0 s_{i\,0}-12(\alpha^4\beta+2\alpha^2\beta^3)s^i_{0|i}$$

$$+(\alpha^6+5\alpha^4\beta^2)s^{ij}s_{ij}-6(\alpha^6+\alpha^4\beta^2)Ric(x). \tag{38}$$

Proof: We know that α can never be polynomial in y. Otherwise, the quadratic $\alpha^2=a_{ij}(x)y^i y^j$ would have been factored into two linear terms. Its zero set would then consist of a hyper-plane, contradicting the positive definiteness of a_{ij}. Now, suppose the polynomial Rat is not zero. The above equation would imply that it is the product of polynomial Irrat with a non-polynomial factor α. This is not possible. So Rat must vanish and, since α is positive at all $y\neq 0$, we see that Irrat must be zero as well.

Notice that Rat=0 shows that α^2 divides $\beta^5\overline{Ric}_{00}$. Since α^2 is an irreducible degree two polynomials in y, and β^5 factors into five linear terms, it must be the case that α^2 divides Ric_{00}. That is, α is Einstein.
Therefore,

$$\overline{R}ic_{00}=c\alpha^2,$$

where c must be a constant by the Riemannian Schur Lemma. We can get some more information out of the two equations Rat=0 and Irrat=0, in Lemma 4.1. By replacing every instance of $\overline{R}ic_{00}$ with $c\alpha^2$ and so divide through by the common factor of α^2 to Rat=0, we will get an expression of the form

$$0=\alpha^2 P_1+Q_1, \tag{39}$$

where P_1 is a polynomial of degree one, and Q_1 is a polynomial of degree three, as follows:

$$P_1=(-8\alpha^2\beta-24\beta^3)c+4\beta s^i_0 s_{i\,0}+(2\alpha^2+26\beta^2)s^i_{0|i}$$

$$-(4\alpha^2\beta+2\beta^3)s^{ij}s_{ij}+(6\alpha^2\beta+8\beta^3)Ric(x),$$

$$Q_1=-8\beta^3c+8\beta^4 s^i_{0|i}.$$

From (39) we conclude that α^2 divides Q_1 and so $\beta=0$. Then the Matsumoto metric is actually Riemannian. In fact, we proved the following theorem:

Theorem 4.2. Let (M,F) be a Matsumoto Finsler space with constant Killing form β, then F is Einstein if and only if it is Reimannian Einstein metric.

4.2. Einstein $F=\frac{(\alpha+\beta)^2}{\alpha}$ metrics

From (36),

$$0=\overline{Ric}_{00}+\frac{12\alpha^2}{(\alpha-\beta)^2}s^i_0 s_{i0}+\frac{4\alpha^2}{\alpha-\beta}s^i_{0|i}-\frac{4\alpha^4}{(\alpha-\beta)^2}s^{ij}s_{ij}-\frac{8\alpha^3}{(\alpha-\beta)^3}s^i_0 s_{i0}.$$

$$-\frac{(\alpha+\beta)^4}{\alpha^2}Ric(x). \tag{40}$$

Multiplying (40) by $(\alpha-\beta)^3\alpha^2$ removes y from the denominators and we can derive the criterion for the above metric to be Einstein as follows:

$$Rat+\alpha Irrat=0,$$

where Rat and Irrat are, respectively, degree 7 and degree 6 polynomials in y .

Lemma 4.3. A Finsler space of metric $F=\frac{(\alpha+\beta)^2}{\alpha}$ is Einstein if and only if Rat=Irrat=0 hold.
where,

$$Rat=(-3\alpha^4\beta-\alpha^2\beta^3)\overline{Ric}_{00}-12\alpha^4\beta s^i_0 s_{i0}+4(\alpha^6+\alpha^4\beta^2)s^i_{0|i}+4\alpha^6\beta s^{ij}s_{ij}$$

$$-(\alpha^6\beta-3\alpha^4\beta^3+3\alpha^2\beta^5-\beta^7)Ric(x),$$

$$Irrat=(\alpha^4+3\alpha^2\beta^2)\overline{Ric}_{00}-8\alpha^4\beta s^i_{0|i}-4\alpha^6 s^{ij}s_{ij}+4\alpha^4 s^i_0 s_{i0}$$

$$-(\alpha^6-3\alpha^4\beta^2+3\alpha^2\beta^4-\beta^6)Ric(x). \tag{41}$$

By similar arguments such as Matsumoto metrics, we have the following theorem:

Theorem 4.4. Let $(M,F=\frac{(\alpha+\beta)^2}{\alpha})$ be a Finsler space with constant Killing form β. If F is Einstein then it is Reimannian metric.

4.3. Einstein generalized Kropina metrics

From (17),

$$0=\overline{Ric}_{00}+\frac{2p(p+1)\alpha^2}{(1-p)^2\beta^2}s^i_0 s_{i0}-\frac{p^2\alpha^4}{(1-p)^2\beta^2}s^{ij}s_{ij}$$

$$+\frac{2p\alpha^2}{(1-p)\beta}s^i_{0|i}-\alpha^{2(1-p)}\beta^{2p}Ric(x). \tag{42}$$

The relation $\overline{K}^i_i=\overline{Ric}_{00}$ is a consequence of the definition of $\overline{Ric}_{ij}$ and Euler's theorem. Consider the following two possibilities:

Case 1. $1-p<0$. In this case, by multiplying $\alpha^{2(p-1)}$, $(1-p)^2\beta^2$ removes y from the denominators.

Case 2. $1-p>0$. Then p is negative (since it is not equal to 0 or 1), by multiplying $(1-p)^2$, $\beta^{|2p|}$ remove y from the denominators.

In case 1, removing y from the denominator leaves an expression of the form

$$0=\alpha^{(2p-2)}P+Q. \tag{43}$$

where P is a polynomial of degree 4, and Q is a polynomial of degree 2p+2, as follows

$$P=(1-p)^2\beta^2\overline{Ric}_{00}+2p(p+1)\alpha^2s^i_0s_{i0}-p^2\alpha^4s^{ij}s_{ij}+2p(1-p)\alpha^2\beta s^i_{0|i},$$

$$Q=-(1-p)^2\beta^{2(p+1)}Ric(x).$$

Lemma 4.5. A generalized Kropina metric is Einstein if P=Q=0.
The statement holds if P=Q=0. Suppose that F is Einstein metric. We claim $Ric(x)=0$. Otherwise, from (43), we can say that $\alpha^{(2p-2)}$ divides Q. In particular, α^2 divides Q,

$$Q=-(1-p)^2\beta^{(2p+2)}Ric(x). \tag{44}$$

But it forces α^2 to divide $\beta^{(2p+2)}$, (because α^2 and $Ric(x)$ are relatively prime) which is not possible since α^2 is irreducible and $\beta^{(2p+2)}$ factors into linear terms. Hence $\beta=0$, but this is a contradiction and so we have $Ric(x)=0$. Therefore, P=Q=0.

Theorem 4.6. An Einstein generalized Kropina metric with constant Killing form β is Ricci flat and α is Einstein.
This is just case 1. One can derive similar results for case 2.

If set $p=-1$, we derive the condition for a Kropina metric, with a constant Killing β form, to be Einstein as follows:

$$P_1=Q_1=0,$$

where,

$$P_1=4\beta^2\overline{Ric}_{00}$$

$$Q_1=-\alpha^4s^{ij}s_{ij}-4\beta\alpha^2s^i_{0|i}-4\alpha^4Ric(x).$$

In [10], Robles and Bao proved a Schur lemma for Randers metric:

Theorem 4.7. [2] The Ricci scalar Ric(x) of any Einstein Randers metric in a dimension greater than two is necessarily constant.

The Einstein navigation description was used for Randers metric. However, nothing is known about corresponding for the other (α,β)-metrics yet. We have the following theorem in the case of scalar flag curvature metrics:

Theorem 4.8. [9] Let F be a scalar flag curvature Finsler space, then F is Einstein if and only if F is a constant flag curvature.

Suppose that F is one of the (α, β)-metrics in this paper with constant Killing form β. If F is Einstein metric, then from the above theorems, F is constant Ricci scalar. Therefore we can get the following corollary:

Corollary 4.9. Ricci scalar functions of any Einstein metrics of Matsumoto, generalized Kropina and $F = \frac{(\alpha+\beta)^2}{\alpha}$ type, with constant Killing form is necessarily constant.

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