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ON THE SYNCHRONIZATION OF COUPLED CHAOTIC SYSTEMS WITH ZERO LYAPUNOV EXPONENTS[*]

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Abstract – The threshold of weak and strong synchronization in a coupled chaotic Gaussian map is demonstrated by using the *Pyraga*s' terminology. However, Vieira and Lichtenberg have shown that the *Pyraga*s' criteria for weak and strong synchronization cannot be used for all chaotic systems. In this article we demonstrated some interesting synchronization behavior of two coupled chaotic systems with some zero and negative Lyapunov exponents. In such cases the various synchronization behaviors may also depend upon the eigenvalues of a system obtained by subtracting two chaotic systems.

Keywords – Chaos synchronization, Lyapunov exponents

1. INTRODUCTION

It is well known that, one important tool to measure the sensitivity of chaotic systems to initial values is *Lyapunov Exponents* (LEs) [1-5]. So, LEs play an important role in describing the qualitative behaviour of synchronization between two chaotic systems. Pyragas [1, 6] discussed the various synchronization stages of coupled chaotic systems by using LEs. He claimed that *Weak Synchronization* (WS) and *Strong Synchronization* (SS) are dependent upon the signs of the LEs of the coupled chaotic systems. However, Vieira and Lichtenberg have shown that this dependency is not generally a distinct property of all coupled chaotic systems [2]. They have shown that the Pyragas transition terminology from WS to SS in one chaotic system to the other is different. In particular, they analytically found that, for the tent map the transition to WS and SS coincide. On the other hand, as we will see, in the Gaussian map WS and SS occur in different thresholds.

In these and similar studies, researchers used LEs to describe the synchronization behaviour of coupled chaotic systems. For example, Shuaiet *et al*. [7] have shown that synchronization between two chaotic systems occurs even in the presence of some positive LEs. In this study we demonstrate that in the case of zero and negative LEs, we may have some interesting synchronization behaviour in coupled chaotic continuous time dependent systems. In these cases the various synchronization behaviors may depend on the eigenvalues of a difference system of two coupled systems. Using Pyragas' terminology, we start by showing the occurrence of WS and SS in coupled Gaussian chaotic maps. Then, following Vieira and Lichtenberg [2], we will see that there can be no universal characterization of WS and SS. In section 4 we will demonstrate some interesting examples for synchronization of two chaotic continuous time dependent systems and describe how this synchronization may depend upon the eigenvalues of the difference system between two coupled chaotic systems. We conclude with final remarks and consequences in section 5.

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2. SYNCHRONIZATION OF TWO CHAOTIC GAUSSIAN MAPS

Consider the one dimensional Gaussian map

$$x_{n+1} = e^{[-a^2(x_n - b)^2]}. \tag{1}$$

Various dynamic behaviours of this map may be observed for different real values of *a* and *b*. For $a = 1.0$, $b = 0.5$ and $x_0 \in (0, 1)$, this map has the stable fixed point $x^* = 0.87125$. Fixing $b = 0.5$ and varying *a* among 1.5, 2.1 and 3.5 with initial value $x_0 \in (0, 1)$, the map is periodic, periodic two or chaotic, respectively. In the case of chaos, for $a = 3.5$ and $b = 0.5$ with two different initial values x_0, we observed different chaotic behaviours.

To review the synchronization of chaotic systems and related terminology, consider the three dimensional Gaussian map defined as follows:

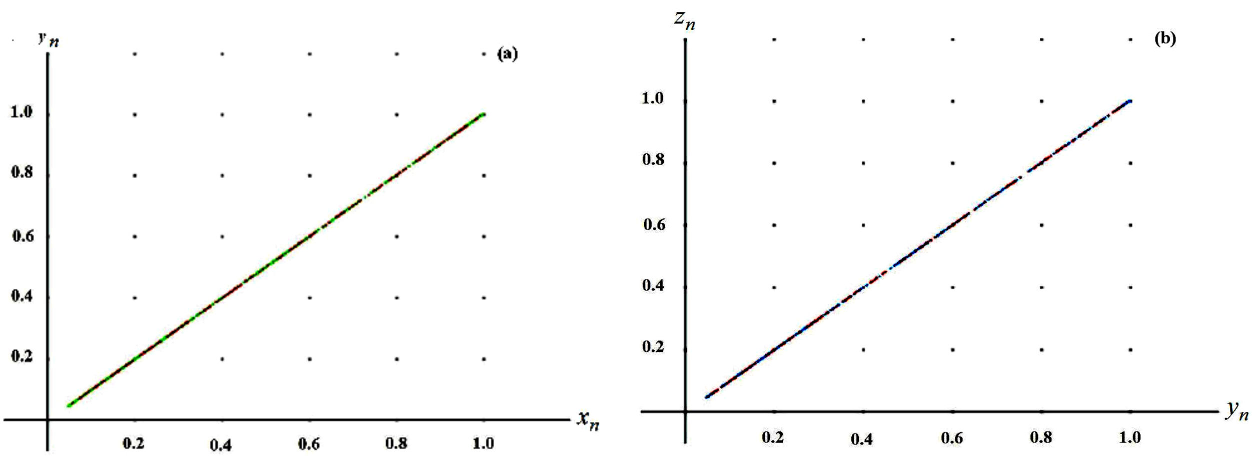
$$x_{n+1} = f(x_n) \tag{2}$$

$$y_{n+1} = f(y_n) - k[f(y_n) - f(x_n)] \tag{3}$$

$$z_{n+1} = f(z_n) - k[f(z_n) - f(x_n)]. \tag{4}$$

In this system $f(x_n) = e^{[-a^2(x_n - b)^2]}$, and the maps (2), (3) and (4) are the *drive, response and auxiliary systems*, respectively. All three of these maps are chaotic for parameter values $a = 3.5$, $b = 0.5$ and $0 < k < 1$ and for initial values between zero and one. Similar to the Pyragas terminology [1, 6], we can see the various synchronization behaviours of these three maps for different values of $k \in (0, 1)$. For values $0.662 < k < 1$, the maps (2) and (3) have transversely stable invariant manifold $x = y$. In other words, for these values of *k* and with any initial values x_0 and y_0, the terms x_n and y_n are synchronized after some iterations. As defined elsewhere [1] this synchronization is called *Identical Synchronization* (IS) (Fig. 1-a). The same synchronization occurs between the maps (3) and (4). The manifold stabilities of $x = y$ and $y = z$ are called SS (Fig. 1-a and 1-b). In general, if there is a function $h : R \to R$ such that $|h(x_n) - y_n| \to 0$ as $n \to \infty$, then there will be a *General Synchronization* (GS) between (2) and (3). Now if *h* is a smooth function, then the synchronization is called SS and if *h* is non-smooth it is called WS.

We can see the GS in the form of WS in the Gaussian map for $0.42 \le k \le 0.662$ (Fig. 1-c and 1-d). In this case $x = y$ is a transversely unstable invariant manifold but $y = z$ is a stable one. Of course, there are some values of *k* $(0 < k < 0.3)$ for which there is no synchronization between the Gaussian maps (2), (3) and (4). In this case the phases spaces (x, y) and (y, z) are not stable (Fig. 1-e and 1-f).



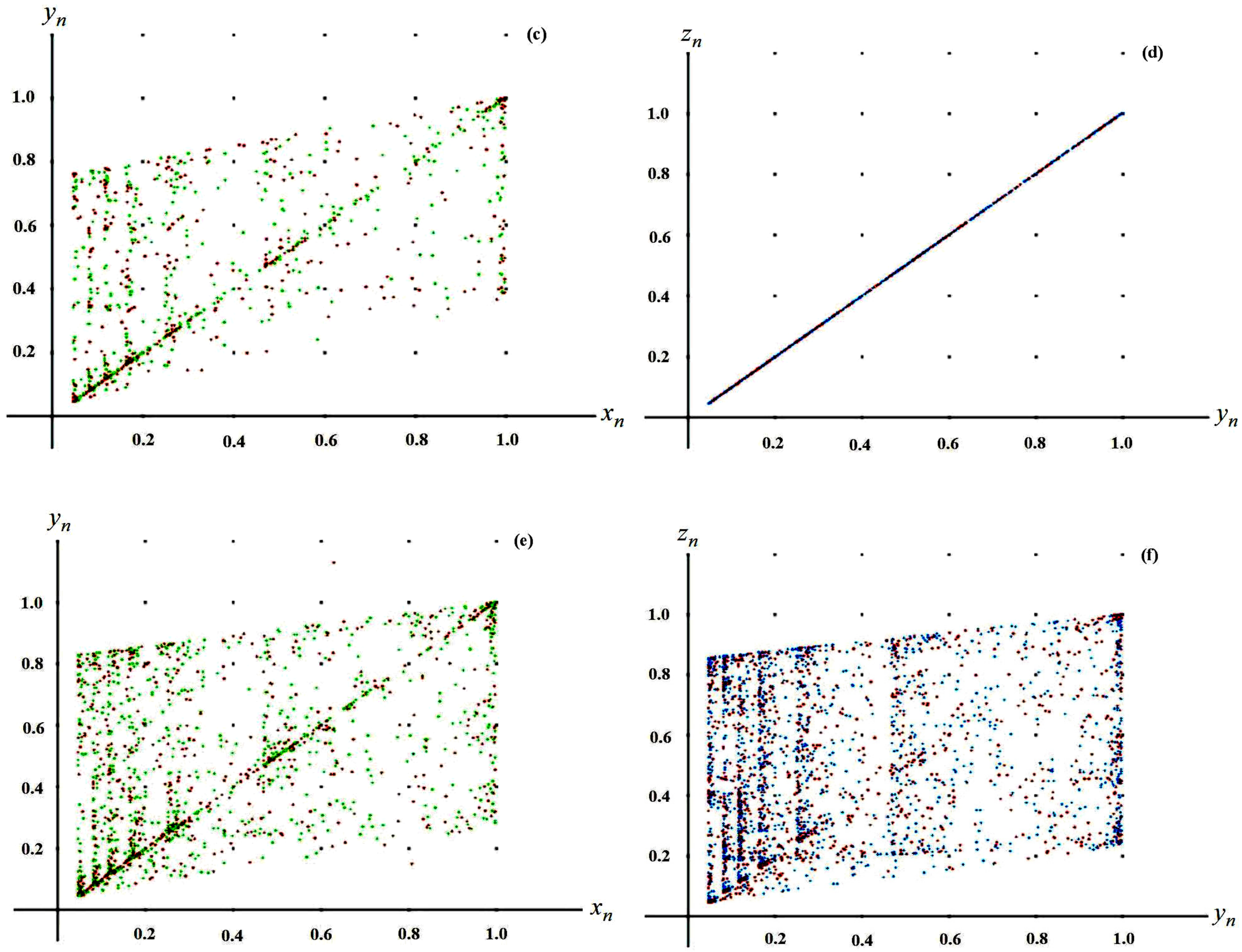


Fig. 1. Synchronization of two coupled chaotic Gaussian maps for different values of k . (a) and (b) strong synchronization for $k = 0.7$. (c) and (d) weak synchronization for $k = 0.45$. (e) and (f) no synchronization between $(x,\ y)$ and $(y,\ z)$ occurs for $k = 0.2$

3. LYAPUNOV EXPONENTS AND SYNCHRONIZATION

Pyragas [1, 6] used LEs to analyze the WS and SS criteria. He defines the *Conditional LE* as

$$\lambda^R = \ln(1-k) + \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \ln|f'(y_n)| \tag{5}$$

and he used this exponent to study the stability of the phase space $(y,\ z)$. The *Transverse LE* is defined as

$$\lambda^I = \ln(1-k) + \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \ln|f'(x_n)| . \tag{6}$$

These LEs can be calculated numerically for higher dimensional spaces [8] and analytically in most one dimensional spaces.

Using different parameter values for which both λ^R and λ^I are negative we have SS, but for $\lambda^R < 0$ and $\lambda^I > 0$ we have WS [1]. Thus if we calculate λ^R and λ^I for the Gaussian map at the fixed point $\bar{x} = \bar{y} = 0.678082$ for different values $0 < k < 1$, then the overall synchronization behaviors can be characterized as follows:

1. If $k = 0.42$, $\lambda^R = 0$, and if $k = 0.662$, $\lambda^I = 0$. Thus, WS occurs for those values of $0.42 < k < 0.662$ for which $\lambda^R < 0$ and $\lambda^I > 0$.
2. If $k > 0.662$, then λ^R and λ^I are both negative, SS occurs.
3. For a value of k less than 0.42, both λ^R and λ^I are positive and in this case there is no synchronization between (2), (3) and (4).

These results are consistent with those of Fig. 1 in the previous section. Nevertheless, Vieira and Lichtenberg have shown [2] that the above criteria for WS and SS in terms of LEs cannot be used for all chaotic systems described by (2-4). Following their studies, we are able to find the LEs of the system (2-4) analytically. That is, the LEs corresponding to the valuables *x*, *y* and *z* are

$$\lambda_1 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \ln|f'(x_n)| \tag{7}$$

$$\lambda_2 = \ln(1-k) + \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \ln|f'(y_n)| \tag{8}$$

$$\lambda_3 = \ln(1-k) + \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \ln|f'(z_n)|. \tag{9}$$

In calculating these LEs we first note that for the initial values of y and z in the area that $y_n \to z_n$, we will have $\lambda_2 = \lambda_3$, because the parameters of the maps are the same. Second, $\lambda^R = \lambda_2$ and this equality occurs even when y_n and z_n are not synchronized. On the other hand, λ^I is not an LE of the global system, particularly in the region where *x* and *y* are not weakly synchronized. However, λ^I and λ_1 are related by $\lambda^I = \ln(1-k) + \lambda_1$. Finally, it is obvious that in the region of SS $\lambda^I = \lambda^R = \lambda_2$.

As they have showed in the tent map the WS does not necessarily imply SS. Since the tent map

$$f(x) = \begin{cases} ax, & \text{if } 0 \le x \le \frac{1}{2}, \\ a(1-x), & \text{if } \frac{1}{2} \le x \le 1 \end{cases} \tag{10}$$

has period one for $0 \le a < 1$, with fixed point $x^* = 0$ and for $1 < a < 2$ the map is chaotic [9]. If $a > 2$, then the generic orbit of this map diverges. Since $|f'(x_n)| = a$, it follows from equations (7)-(9) that the LEs of this system are $\lambda_1 = \ln a$ and $\lambda_2 = \lambda_3 = \ln(1-k) + \ln a$. For fixed $a \in (0, 2)$, λ_2 decreases monotonically as *k* increases from zero to one. Thus there is no region of WS for this map. In fact, synchronization between *x*, *y* and *z* occurs simultaneously in this map, and we have only SS, which occurs whenever $\lambda_2 < 0$, so $k = 1 - \frac{1}{a}$.

4. SOME INTERESTING SYNCHRONIZATION

Other criteria for the synchronization of two coupled chaotic continuous time dependent systems have been described by Guemez *et al.* [10] and Pecora and Carroll [11]. Following their criteria, suppose $\mathbf{x}' = \mathbf{f(x, s)}$ and $\mathbf{y}' = \mathbf{f(y, s)}$ are two chaotic drive and response systems that are coupled by some vector-valued function of time $\mathbf{s} = \mathbf{h(x)}$. Synchronization of this pair of (identical) systems occurs if the dynamical system describing the evolution of the difference,

$$\mathbf{e} = \mathbf{y} - \mathbf{x},\ \mathbf{e}' = \mathbf{f(y, s)} - \mathbf{f(x, s)} = \mathbf{f(x + e, s)} - \mathbf{f(x, s)}$$

has a stable fixed point at $\mathbf{e} = \mathbf{0}$. Note that here we assume that **x** and **y** are drive and response systems, respectively. This stability can be checked analytically by using the linearized system

$$\mathbf{e}' = \mathbf{Df_x(x, s).e} \tag{11}$$

for some small value $\mathbf{e}$. Because analytical stability is impossible in some cases, the stability can also be checked numerically by computing the LEs of this linearized system [8]. If all LEs are negative or have negative real parts, then synchronization occurs between the two coupled chaotic systems. But what happens if there are some zero LEs for this linearized system? Is there any synchronization between these two chaotic systems? Recently, Shuaiet *et al*. [7] have observed that synchronization can be achieved even with positive LEs, and we will show that synchronization may occur with zero LEs too. To this end, consider the following examples.

Example 1. It is well known that both of the following coupled Lorenz systems [12] are chaotic for the parameter values $\sigma = 10$, $b = 8/3$ and $r = 28$.

$$\begin{aligned} x_1' &= -\sigma x_1 + \sigma y_1 \\ y_1' &= -x_1 z_1 + r x_1 - y_1 \\ z_1' &= x_1 y_1 - b z_1 \end{aligned} \qquad \begin{aligned} x_2' &= -\sigma x_2 + \sigma y_2 \\ y_2' &= -x_2 z_1 + r x_2 - y_2 \\ z_2' &= x_2 y_2 - b z_2 \end{aligned}$$

In computing the LEs of the corresponding system (11) by Wolf's method [8] we find LEs of -2.67, -11.0, and 0. Thus the above-mentioned criteria cannot detect synchronization. However, we have determined that these two systems are synchronized in that region where the eigenvalue $\mathbf{Df_x(x, s)}$ in the linearized deference system (11) vanishes. Hence, consider the Jacobian matrix from system (11):

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ r - z_1 & -1 & 0 \\ y_2 & x_2 & -b \end{pmatrix}.$$

The characteristic equation of this yields the eigenvalues $-b$ and $1/2\{-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4\sigma[z_1(t) - (r-1)}\}$. Obviously, in the case in which z_1 becomes equal to the constant $r-1$, we will have a vanishing eigenvalue. In Fig. 2 we display the time series of the trajectories in both the (x_1, t) and (x_2, t) planes together. As we can see in this figure, the response follows a trajectory which is an amplification of the driven attractor by a factor of approximately 10. This factor may change by varying the initial conditions.

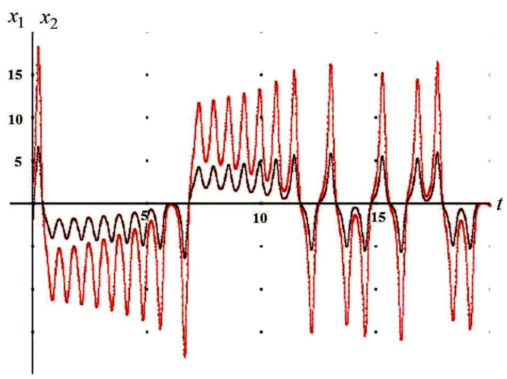


Fig. 2. Time series solutions x_1 and x_2 of coupled Lorenz chaotic systems. The response follows a trajectory which is an amplification of the driven attractor by a factor of approximately 10

Example 2. As another case with zero LEs, consider the following coupled Sprott-C systems [13]. These two systems are coupled using the Pecora-Carroll method [11].

$$\begin{aligned} x_1' &= y_1 z_1 & & \\ y_1' &= x_1 - y_1 & \qquad y_2' &= x_1 - y_2 \\ z_1' &= 1 - x_1^2 & \qquad z_2' &= 1 - x_1^2. \end{aligned}$$

Here again we have a zero LE and a vanishing eigenvalue for the corresponding Jacobian matrix in the system (11), and two other eigenvalues are 1. In this case, as we can see in Fig. 3, the drive and response systems in the (y_1, t) and (y_2, t) planes eventually behave similarly and synchronize.

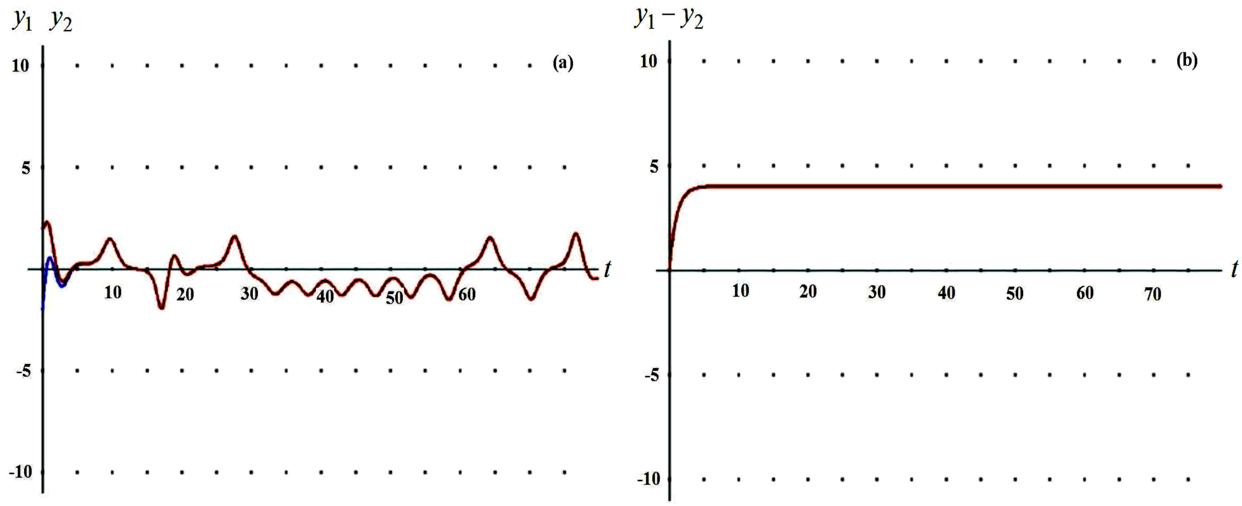


Fig. 3. (a) time series solutions y_1 and y_2 of coupled chaotic Sprott-C systems.
(b) difference between y_1 and y_2, which shows eventual synchronization

Example 3. An interesting situation of zero LEs occurs whenever the Jacobian matrix has complex conjugate eigenvalues with vanishing real part. Consider, for example, the following pair of Sprott-L systems coupled by the second method of Pecora and Carroll.

$$\begin{aligned} x_1' &= y_1 + 3.9 z_1 & & \\ y_1' &= 0.9 x_1^2 - y_1 & \qquad x_2' &= y_1 + 3.9 z_2 \\ z_1' &= 1 - x_1 & \qquad z_2' &= 1 - x_2 \end{aligned}$$

The eigenvalues of the Jacobian matrix are $\pm i\sqrt{3.9}$. Fig. 4 shows the time series solutions for x_1 and x_2. In this figure, the differences between drive and response change in an oscillatory fashion are shown. The frequency of this oscillation depends on the imaginary part of the eigenvalues, with constant amplitude, which is related to the difference at the moment at which the connection starts. This behaviour is called *marginal oscillatory synchronization* [10, 14].

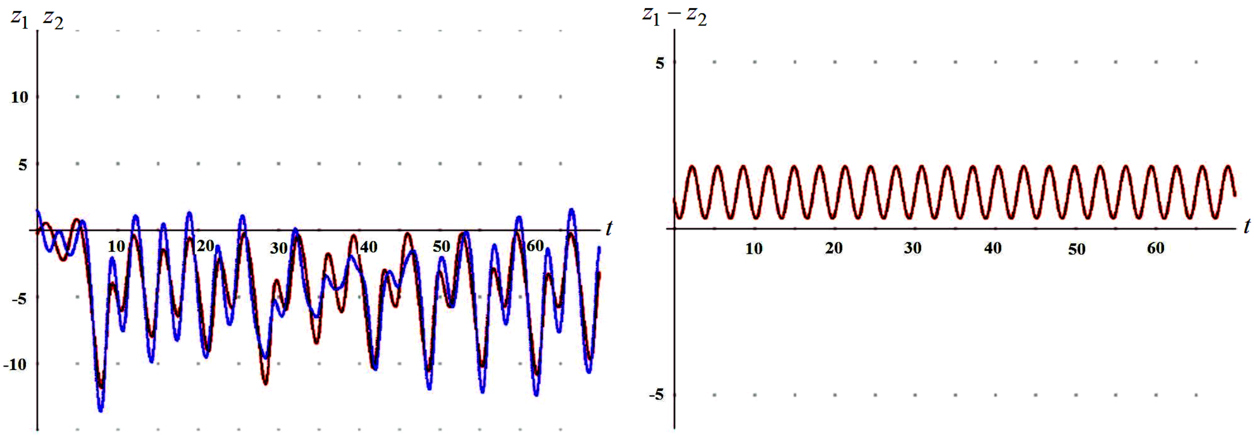


Fig. 4. (a) time series solutions Z_1 and z_2 of coupled chaotic Sprott-L. (b) shows the differences between drive and response change in an oscillatory fashion

Example 4. For our final example, we examine a case in which the Jacobian matrix has conjugate eigenvalues with the positive real parts. Consider the following two coupled chaotic Rossler systems

$$\begin{aligned} x_1' &= -(y_1 + z_1) & \qquad x_2' &= -(y_2 + z_2) \\ y_1' &= x_1 + a y_1 & \qquad y_2' &= x_2 + a y_2 \\ z_1' &= b + z_1(x_1 - c) & \qquad z_2' &= b + z_2(x_1 - c), \end{aligned}$$

with $a = 0.2$, $b = 0.2$ and $c = 4.6$. It is easy to see that the eigenvalues of the corresponding Jacobian matrix are $-c$ and $1/2[a \pm \sqrt{a^2 - 4}]$. In this case, the oscillatory difference between drive and response starts with a small amplitude and increases exponentially with time. Time series solutions for x_1, x_2 and the difference between them are shown in Figs. 5-a and 5-b, respectively.

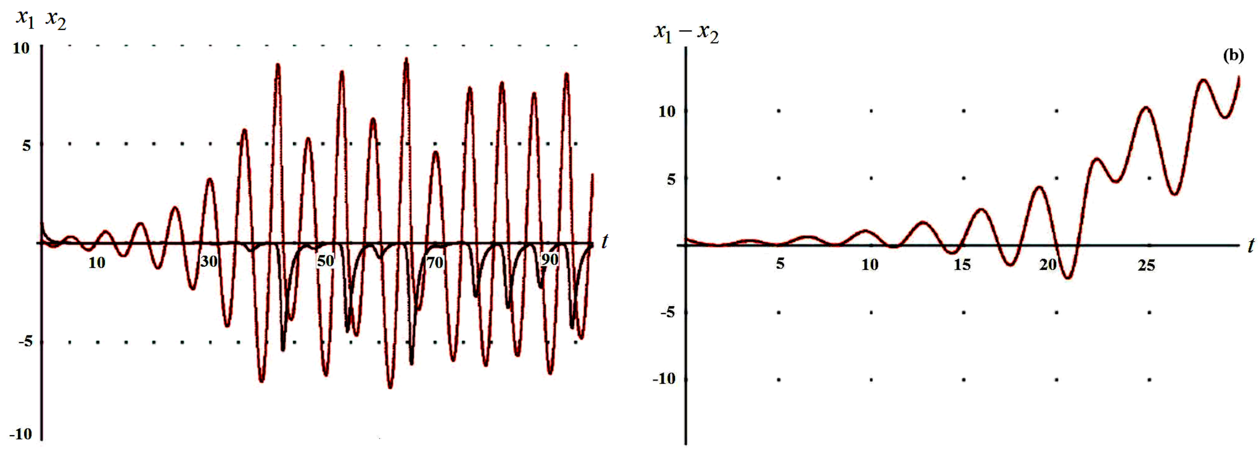


Fig. 5. (a) time series solutions x_1 and x_2 for coupled chaotic Rossler systems. (b) oscillatory difference between x_1 and x_2 start with small amplitude and increases exponentially with time

5. CONCLUSIONS

We have used the Pyragas' terminologies for WS and SS in chaotic Gaussian map. On the other hand, the Tent map is an example by Vieira and Lichtenberg in which WS does not imply SS. Other attempts have been made to determine WS and SS according to Pyragas' terminologies, but there is still no general theorem to describe such synchronization behavior. He and Vaidya [15] tried to find a necessary and sufficient condition for chaotic synchronization of two coupled systems, but their attempt was very specific and left many unanswered questions. In this article we have attempted to classify the types of synchronization which occur with zero LEs in continuous time dependent chaotic systems. In these cases, the sign of other non-zero LEs does not determine the synchronization behavior of two coupled chaotic systems. If there is synchronization, then we analyzed the difference behaviors between the coupled systems, which are related to the eigenvalues of $\mathbf{Df_x(x, s)}$ in system (11). Of the many possible cases, depending on the different types of these eigenvalues, the most interesting are those with zero and positive real parts and in which synchronization between two coupled chaotic systems occurs with oscillatory differences between the drive and response systems.

There are many remaining cases to be classified in terms of the Jacobian matrix eigenvalues in system (11), and there is still no general theorem to characterize the synchronization of coupled chaotic systems with zero LEs in both discrete and continuous time dependent chaotic systems.

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