Iranian Journal of Science & Technology, Transaction A, Vol. 32, No. A1
Printed in The Islamic Republic of Iran, 2008

THE STRUCTURE OF DERIVATIONS FROM A FULL MATRIX ALGEBRA INTO ITS DUAL*

R. ALIZADEH[1] AND G. H. ESSLAMZADEH[2**]

[1]Department of Mathematics, Shahed University, P. O. Box 18151-159, Tehran, I. R. of Iran
Email: alizadeh@shahed.ac.ir
[2]Department of Mathematics, Shiraz University, Shiraz 71454, I. R. of Iran
Email: esslamz@shirazu.ac.ir

Abstract – Let A be a unital algebra over a field of characteristic zero. We show that every derivation from $M_n(A)$ into its dual $M_n(A)^*$ is the sum of an inner derivation and a derivation induced by a derivation from A into A^*.

Keywords – Derivation, full matrix algebra, dual space

1. INTRODUCTION

Throughout A is a unital algebra over a field of characteristic zero and M is an A-bimodule. We denote the full matrix algebra of $n \times n$ matrices over A with the usual operations by $M_n(A)$. E_{ij} s, $1 \le i, j \le n$ are also usual matrix units in $M_n(A)$. For all $x \in A$ we display matrix whose (i, j) th entry is x and zero elsewhere, by $x \otimes E_{ij}$. The dual A^* of A is the set of all linear maps from A into its field. We denote the action of an element $g \in A^*$ on an element $a \in A$ with $< g, a >$. Also, A^* is an A-bimodule with the following module operations:

$$< a.f, b > = < f, ba > \text{ and } < f.a, b > = < f, ab >, \forall a, b \in A, f \in A^*.$$

A *derivation* $D : A \to M$ is a linear map which satisfies the identity $D(ab) = D(a)b + aD(b)$ $a, b \in A$. We say that D is *inner* if there exists $m \in M$ such that $D(a) = am - ma$ for all $a \in A$. Every *derivation* $D : A \to M$ induces a derivation $\overline{D} : M_n(A) \to M_n(M)$ by $\overline{D}((a_{ij})) = (D(a_{ij}))$.
Benkart and Osborn [1] characterized derivations of $M_n(A)$, where A is a unital nonassociative algebra with char(A) $\neq$ 2,3 and $n > 2$. They showed that every derivation of $M_n(A)$ is a sum of an inner derivation generated by a matrix with entries in the nucleus N of A, and a derivation induced by a derivation of A. A similar result for full matrix rings was proved in [2]. The case of the centers of upper triangular matrix rings over simple algebras which are finite dimensional modulo, was discussed in [3]. Coelho and Milies in [4] proved a similar result for upper triangular matrices over an arbitrary ring with identity. Jondrup gave a new proof of the latter result in [5].
In this article we prove an analog of the above mentioned result for derivations from $M_n(A)$ into its dual $M_n(A)^*$. In other words, we prove the following theorem.

Theorem: Every derivation from $M_n(A)$ into its dual $M_n(A)^*$ is the sum of an inner derivation and a derivation induced by a derivation from A into A^*.

---

*Received by the editor May 27, 2007 and in final revised form August 28, 2007
**Corresponding author

2. MAIN RESULT

Let $g \in M_n(A)^*$ and $1 \le i, j \le n$. Define $g_{ij} \in A^*$ by $< g_{ij}, a >=< g, a \otimes E_{ij} >$. We can identify $M_n(A)^*$ with $M_n(A^*)$ via the map

$$\phi : M_n(A)^* \to M_n(A^*), \quad g \mapsto (g_{ij}).$$

From now on we display $g \in M_n(A)^*$ by (g_{ij}). For all $(a_{ij}) \in M_n(A)$ we have

$$< g, (a_{ij}) >=< (g_{ij}), (a_{ij}) >= \sum_{i,j=1}^{n} < g_{ij}, a_{ij} > .$$

Theorem: Every derivation from $M_n(A)$ into its dual $M_n(A)^*$ is the sum of an inner derivation and a derivation induced by a derivation from A into A^*.

Proof: Let $(f_{ij}) \in M_n(A)^*$, $(a_{ij}) \in M_n(A)$ and $a \in A$. Then we have

$$< (f_{ij})(a_{ij}), a \otimes E_{kl} >=< (f_{ij}), (a_{ij})[a \otimes E_{kl}] >=< (f_{ij}), \sum_{s=1}^{n} [a_{sk} a \otimes E_{sl}] >$$
$$= \sum_{s=1}^{n} < f_{sl}, a_{sk} a >=< \sum_{s=1}^{n} f_{sl} a_{sk}, a > .$$

Thus

$$[(f_{ij})(a_{ij})]_{kl} = \sum_{s=1}^{n} f_{sl} a_{sk} . \tag{1}$$

Similarly, we have:

$$[(a_{ij})(f_{ij})]_{kl} = \sum_{s=1}^{n} a_{ls} f_{ks} . \tag{2}$$

Suppose $D : M_n(A) \to M_n(A)^*$ is a derivation. Define

$$D_{ij}^{kl} : A \to A^*, \; D_{ij}^{kl}(a) = [D(a \otimes E_{ij})]_{kl}, \; 1 \le i, j, k, l \le n. \tag{3}$$

From (1) and (2) we conclude that for every a, $b \in A$ and every positive integer $m \le n$ the following equality holds:

$$[D(a \otimes E_{im})[b \otimes E_{mj}]]_{kl} = \sum_{r=1}^{n} D_{im}^{rl}(a) b \delta_{mr} \delta_{jk} = D_{im}^{ml}(a) b \delta_{jk} \tag{4}$$

Where δ is the Kronecker's delta. Similarly, we have

$$[[a \otimes E_{im}] D(b \otimes E_{mj})]_{kl} = \sum_{r=1}^{n} a \delta_{il} \delta_{mr} D_{mj}^{kr}(b) = a \delta_{il} D_{mj}^{km}(b). \tag{5}$$

From (4) and (5) we conclude that

$$D_{ij}^{kl}(ab) = D_{im}^{ml}(a) b \delta_{jk} + a \delta_{il} D_{mj}^{km}(b). \tag{6}$$

Therefore D_{ii}^{ii} is a derivation from A to A^*, for $1 \le i \le n$. Using (6) we have

$$D_{ij}^{jl}(a)=D_{ii}^{il}(1)a,\ D_{ij}^{ki}(a)=aD_{jj}^{kj}(1) \qquad a\in A,\ i\neq l,\ k\neq j. \tag{7}$$

Again, using (6) for every $0\le i,j,l\le n$, the following equalities hold.

$$D_{jj}^{ji}(a)=D_{ji}^{ij}(1)a+D_{ij}^{ji}(a). \tag{8}$$

$$D_{ij}^{ji}(a)=D_{il}^{li}(1)a+D_{lj}^{jl}(a). \tag{9}$$

$$D_{lj}^{jl}(a)=D_{ll}^{ll}(a)+aD_{lj}^{jl}(1). \tag{10}$$

$$D_{ji}^{ij}(a)=aD_{ji}^{ij}(1)+D_{jj}^{jj}(a). \tag{11}$$

From (8) we have

$$D_{ji}^{ij}(1)=-D_{ij}^{ji}(1). \tag{12}$$

Also, from (9), (10) and (12) we have

$$D_{ij}^{ji}(a)=D_{il}^{li}(1)a-aD_{jl}^{lj}(1)+D_{ll}^{ll}(a). \tag{13}$$

Using (8) and (11) we conclude that

$$D_{ij}^{ji}(a)=D_{ji}^{ij}(a)-D_{ji}^{ij}(1)a-aD_{ji}^{ij}(1) \tag{14}$$

In addition, using (6), (7) and (14) we obtain

$$\begin{aligned}[D((a_{rs}))]_{ij} &= \sum_{k,l=1}^{n} D_{kl}^{ij}(a_{kl})\\ &=\sum_{k=1}^{n} D_{ki}^{ij}(1)a_{ki}+\sum_{l=1}^{n} a_{jl}D_{jl}^{ij}(1)-D_{ji}^{ij}(1)a_{ji}-a_{ji}D_{ji}^{ij}(1)+D_{ji}^{ij}(a_{ji})\\ &=\sum_{k=1}^{n} D_{kk}^{kj}(1)a_{ki}+\sum_{k=1}^{n} a_{jk}D_{kk}^{ik}(1)+D_{ij}^{ji}(a_{ji}).\end{aligned} \tag{15}$$

On the other hand, we have

$$0=[D(E_{kk}E_{ii})]_{ik}=\sum_{s=1}^{n} D_{kk}^{sk}(1)\delta_{si}+\sum_{s=1}^{n}\delta_{ks}D_{ii}^{is}(1)=D_{kk}^{ik}(1)+D_{ii}^{ik}(1). \tag{16}$$

Therefore,

$$D_{kk}^{ik}(1)=-D_{ii}^{ik}(1). \tag{17}$$

Now, for $1\le k,j\le n$ define $D_{kj}=D_{kk}^{kj}$. From (15) and (17) we conclude that

$$\begin{aligned}[(D(a_{rs}))]_{ij} &= \sum_{k=1}^{n} D_{kj}(1)a_{ki}-\sum_{k=1}^{n} a_{jk}D_{ik}(1)+D_{ij}^{ji}(a_{ji})\\ &=[(D_{rs}(1))(a_{rs})-(a_{rs})(D_{rs}(1))]_{ij}+D_{ij}^{ji}(a_{ji}).\end{aligned} \tag{18}$$

Using (13) and (18) for every $1\le l\le n$ we obtain

$$
\begin{aligned}
D((a_{ij})) = {} & [(D_{ij}(1)) + diag(D_{1l}^{l1}(1), \ldots, D_{nl}^{ln}(1))](a_{ij}) \\
& -(a_{ij})[(D_{ij}(1)) + diag(D_{1l}^{l1}(1), \ldots, D_{nl}^{ln}(1))] + \overline{D_{ll}^{ll}}((a_{ij})).
\end{aligned}
$$

REFERENCES

1. Benkart, G. M. & Osborn, J. M. (1981). Derivations and automorphisms of non associative matrix algebras. *Trans. Amer. Math. Soc. 263*, 411-430.
2. Mathis, D. (1982). Differential polynomial rings and Morita equivalence. *Comm. Alg. 10*, 200l-2017.
3. Jöndrup, S. (1987). Automorphisms of upper triangular matrix rings. *Arch. Math. 49*, 497-502.
4. Coelho, S. P. & Milies, C. P. (1993). Derivations of upper triangular matrix rings. *Linear Alg. Appl. 187*, 263-267.
5. Jöndrup, S. (1995). Automorphisms and derivations of upper triangular matrices. *Linear Alg. Appl. 22*, 205-215.