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COMMON FIXED POINTS OF TWO NONEXPANSIVE MAPPINGS BY A NEW ONE-STEP ITERATION PROCESS*

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Abstract – We introduce a new one-step iteration process to approximate common fixed points of two nonexpansive mappings in Banach spaces and prove weak convergence of the iterative sequence using (i) Opial's condition and (ii) Kadec-Klee property. Strong convergence theorems are also established in Banach spaces and uniformly convex Banach spaces under the so-called Condition (A'), which is weaker than compactness.

Keywords – One-step Iteration Process, nonexpansive mapping, common fixed point, condition (A') weak and strong convergence

1. INTRODUCTION

Let E be a real Banach space, C a nonempty convex subset of E and $T: C \to C$ a mapping. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C.$$

A point $x \in C$ is a fixed point of T provided $Tx = x$. Throughout this paper, $\mathbb{N}$ denotes the set of positive integers. Construction of fixed points is an important subject in the theory of nonexpansive mappings and its applications can be found in a number of applied areas; in particular, in image recovery and signal processing (see, for example, [1] and the references therein). Different iteration processes have been used to approximate fixed points of nonexpansive mappings. Among these iteration processes, the Picard iteration process defined by $x_{n+1} = Tx_n$ is the simplest. However, it does not converge even weakly to a fixed point of a nonexpansive mapping T [2]. For any $x_1 \in C$, define a sequence by $x_{n+1} = (1 - a_n)x_n + a_n Tx_n$, where $\{a_n\}$ is a sequence of numbers in $[0,1]$ satisfying $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=0}^{\infty} a_n = \infty$. This iteration process is referred to as the Mann iteration process [3] and has been studied extensively by many authors to approximate fixed points of various mappings including nonexpansive mappings. It is well-known that, if C is a nonempty compact convex subset of a real Banach space and $T: C \to C$ is a nonexpansive mapping, then the Mann iteration process converges strongly to a fixed

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point of T. However, the Mann iteration process fails to converge for Lipschitz pseudocontractive mappings [4]. Ishikawa [5] gave the following two-step iteration process to overcome this drawback:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)x_n + a_n T y_n, \\ y_n = (1-b_n)x_n + b_n T x_n, \qquad \forall n \in \mathbb{N} \end{cases}$$

where $\{a_n\}$ and $\{b_n\}$ are certain sequences in $[0,1]$. To approximate common fixed points of two mappings, many authors have used the following Ishikawa type two-step iteration process [6-8]:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)x_n + a_n S y_n, \\ y_n = (1-b_n)x_n + b_n T x_n, \qquad \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{a_n\}$ and $\{b_n\}$ are certain sequences in $[0,1]$. Note that the notion of approximating fixed points of two mappings has a direct link with the minimization problem [9]. Recently, Liu et al. [10] introduced the following process.

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)Sx_n + a_n T y_n, \\ y_n = (1-b_n)Sx_n + b_n T x_n, \qquad \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{a_n\}$ and $\{b_n\}$ are certain sequences in $[0,1]$. In this paper, we introduce a new one-step iteration process to compute common fixed points of two nonexpansive mappings in Banach spaces. For $S, T : C \to C,$ we define the iteration process as follows:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = a_n S y_n + (1-a_n) T x_n, \qquad \forall n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{a_n\}$ is a sequence in $[0,1]$ satisfying appropriate conditions. From a computational point of view, (1.3) is simpler than both (1.1) and (1.2) to approximate the common fixed points of two mappings. It is worth mentioning that our process is of independent interest. Neither (1.1) nor (1.2) implies (1.3); the converse of this statement does not hold either. However, our process reduces to the Mann iteration process when $S = I$ (: the identity mapping).

Let us recall some definitions.

A Banach space E is said to satisfy Opial's condition [11] if, for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ ($\rightharpoonup$ denotes weak convergence) implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \qquad \forall y \in E (y \neq x).$$

Examples of Banach spaces satisfying this condition are Hilbert spaces and all spaces $l_p (1 < p < \infty)$. On the other hand, the spaces $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition. A mapping $T : C \to E$ is called demiclosed with respect to $y \in E$ if, for each sequence $\{x_n\}$ in C and $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

A Banach space E is said to satisfy the Kadec-Klee property if, for every sequence $\{x_n\}$ in E converging weakly to x together with $\|x_n\|$ converging strongly to $\|x\|$ imply that $\{x_n\}$ converges strongly to a point $x \in E$.

Uniformly convex Banach spaces, Banach spaces of finite dimension and reflexive locally uniform convex Banach spaces are some of the examples of reflexive Banach spaces which satisfy the Kadec-Klee property.

Next, we state some useful lemmas.

Lemma 1. [12] Suppose that E is a uniformly convex Banach space and $0 < p \le t_n \le q < 1$ for all $n \in \mathbb{N}$. Also, suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in E such that $\limsup_{n\to\infty} \|x_n\| \le r$, $\limsup_{n\to\infty} \|y_n\| \le r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$ hold for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2. [13] Let E be a uniformly convex Banach space and C be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of C into itself. Then $I-T$ is demiclosed with respect to zero.

Lemma 3. [14] Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space and $T: C \to C$ be a nonexpansive mapping. Then there is a strictly increasing and continuous convex function $g:[0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$g(\|T(tx + (1-t)y) - (tTx + (1-t)Ty)\| \le \|x - y\| - \|Tx - Ty\|$$

for all $x, y \in C$ and $t \in [0, 1]$.

Let $\omega_w(\{x_n\})$ denote the set of all weak subsequential limits of a bounded sequence $\{x_n\}$ in E. The following result is Lemma 3.2 of Falset et al. [15].

Lemma 4. Let E be a uniformly convex Banach space such that its dual E^* satisfies the Kadec-Klee property. Assume that $\{x_n\}$ is a bounded sequence in E such that $\lim_{n\to\infty} \|t_n x_n + (1-t) p_1 - p_2\|$ exists for all $t \in [0,1]$ and $p_1, p_2 \in \omega_w(\{x_n\})$, then $\omega_w(\{x_n\})$ is singleton.

2. PREPARATORY LEMMAS

In this section, we prove a pair of lemmas for the development of our convergence results. In the sequel, we will write $F = F(S) \cap F(T)$ for the set of all common fixed points of the mappings S and T.

Lemma 5. Let E be a normed space and C be a nonempty closed convex subset of E. Let $S, T: C \to C$ be nonexpansive mappings. Let $\{x_n\}$ be as in (1.3). If $F \ne \phi$ then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F$.

Proof: Assume that $x^* \in F$. Then we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|a_n S x_n + (1-a_n) T x_n - x^*\| \\
&= \|a_n (Sx_n - x^*) + (1-a_n)(Tx_n - x^*)\| \\
&\le a_n \|Sx_n - x^*\| + (1-a_n)\|Tx_n - x^*\| \\
&\le a_n \|x_n - x^*\| + (1-a_n)\|x_n - x^*\| \\
&= \|x_n - x^*\|.
\end{aligned}$$

Thus $\lim_{n\to\infty} \|x_n - x^*\|$ exists for each $x^* \in F$.

Liu et al. [10] used the condition $\|x - Ty\| \le \|Sx - Ty\|$ for all $x, y \in C$. Call it Condition (L). We propose the condition: $\|x - Sx\| \le \|Sx - Tx\|$ for all $x \in C$ and call it Condition (S). Note that Condition

(S) is weaker than Condition (L). Here is an example of two nonexpansive mappings which satisfy Condition (S), whereas Condition (L) is not satisfied by these mappings. This illustrates the fact that our results are applicable to a larger class of mappings than the corresponding results of Liu et al. [10].

Example 1. Let us define $S,T:\ \mathbb{R}\to\mathbb{R}$ as follows:

$$Sx=\frac{2x+1}{4},\quad Tx=1-x,\qquad \forall x\in\mathbb{R}.$$

Clearly, both S and T are nonexpansive with the common fixed point $\frac{1}{2}$.
Moreover, $\|x-Sx\|\le\|Sx-Tx\|$ for all $x\in\ \mathbb{R}$. In fact,

$$\|x-Sx\|=\left\|x-\frac{2x+1}{4}\right\|=\left\|\frac{2x-1}{4}\right\|$$

And

$$\begin{aligned}\|Sx-Tx\|&=\left\|\frac{2x+1}{4}-(1-x)\right\|\\&=\left\|\frac{2x+1-4+4x}{4}\right\|\\&=3\left\|\frac{2x-1}{4}\right\|.\end{aligned}$$

Thus there are nonexpansive mappings with common fixed points and satisfying the Condition (S). However, if we choose $x=0,\ y=\frac{1}{2}$, then the Condition (L) is not satisfied because

$$\|x-Ty\|=\left\|0-T(\frac{1}{2})\right\|=\frac{1}{2}$$

and

$$\|Sx-Ty\|=\left\|S0-T(\frac{1}{2})\right\|=\frac{1}{4}.$$

We now prove the following result based on the Condition (S).

Lemma 6. Let E be a uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $S,T:C\to C$ be nonexpansive mappings and $\{x_n\}$ be as in (1.3), where $\{a_n\}$ is a sequence in $[\delta,\ 1-\delta]$ for some $\delta\in(0,\ 1)$. Suppose that the Condition (S) holds. If $F\ne\phi$, then

$$\lim_{n\to\infty}\|Sx_n-x_n\|=0=\lim_{n\to\infty}\|Tx_n-x_n\|.$$

Proof: Let $x^*\in F$. By Lemma 5, $\lim_{n\to\infty}\|x_n-x^*\|$ exists. Suppose that there exist $c\ge 0$ such that $\lim_{n\to\infty}\|x_n-x^*\|=c$. Then $\|Sx_n-x^*\|\le\|x_n-x^*\|$ implies that

$$\limsup_{n\to\infty}\|Sx_n-x^*\|\le c$$

Similarly,

$$\limsup_{n\to\infty}\|Tx_n - x^*\| \le c.$$

Further, $\lim_{n\to\infty}\|x_{n+1} - x^*\| = c$ gives that

$$\lim_{n\to\infty}\|a_n(Sx_n - x^*) + (1-a_n)(Tx_n - x^*)\| = c.$$

Applying Lemma 1, we obtain

$$\lim_{n\to\infty}\|Sx_n - Tx_n\| = 0.$$

Using the Condition (S), we get

$$\limsup_{n\to\infty}\|x_n - Sx_n\| \le 0.$$

That is,

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0.$$

Also, $\|x_n - Tx_n\| \le \|x_n - Sx_n\| + \|Sx_n - Tx_n\|$

implies that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

Lemma 7. Let E be a uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $S,T : C \to C$ be nonexpansive mappings and $\{x_n\}$ be as in (1.3). Then, for any $p_1, p_2 \in F$,

$$\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\| = 0 \text{ exists for all } t \in [0,1].$$

Proof: By Lemma 5, $\lim_{n\to\infty}\|x_n - p_n\|$ exists for all $p \in F$ and so $\{x_n\}$ is bounded. Thus there exists a real number $r > 0$ such that $\{x_n\} \subseteq D \equiv \overline{B_r(0)} \cap C$, so that D is a closed convex bounded nonempty subset of C. Put $\alpha_n(t) = \|tx_n + (1-t)p_1 - p_2\|$. Notice that $\alpha_n(0) = \|p_1 - p_2\|$ and $\alpha_n(1) = \|x_n - p_2\|$ exist as proved in Lemma 5. For each $n \in \mathbb{N}$, define $W_n : D \to D$ by

$$W_n x = a_n Sx + (1-a_n)Tx, \qquad \forall x, y \in D.$$

It is easy to verify that

$$\|W_n x - W_n y\| \le \|x - y\|, \ \forall x, y \in D.$$

Set

$$R_{n,m} = W_{n+m-1}W_{n+m-2}\ldots W_n,$$

and

$$b_{n,m} = \|R_{n,m}(tx_n + (1-t)p_1) - (tR_{n,m}x_n + (1-t)p_1)\|, \quad \forall n, m \in \mathbb{N}.$$

Then it follows that $\|R_{n,m}x - R_{n,m}y\| \le \|x - y\|$ for all $x, y \in D$, $R_{n,m}x_n = x_{n+m}$ and $R_{n,m}p = p$ for all $p \in F$. By Lemma 3, there exists a strictly increasing continuous function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$g(b_{n,m}) \le \|x_n - p_1\| - \|R_{n,m}x_n - R_{n,m}p_1\|$$
$$= \|x_n - p_1\| - \|x_{n+m} - p_1\|.$$

Since $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$, so $g(b_{n,m}) \to 0$ as $n \to \infty$. Hence $b_{n,m} \to 0$ for all $m \in \mathbb{N}$ as $n \to \infty$.

Finally, from the inequality

$$\begin{aligned}\alpha_{n+m}(t) &= \|tx_{n+m} + (1-t)p_1 - p_2\| \\ &\le b_{n,m} + \|R_{n,m}(tx_n + (1-t)p_1) - p_2\| \\ &= b_{n,m} + \|R_{n,m}(tx_n + (1-t)p_1) - R_{n,m}p_2\| \\ &\le b_{n,m} + \|tx_n + (1-t)p_1 - p_2\| \\ &= b_{n,m} + \alpha_n(t),\end{aligned}$$

it follows that

$$\limsup_{m\to\infty} \alpha_{n+m}(t) \le \limsup_{m\to\infty} b_{n,m} + \alpha_n(t).$$

That is,

$$\limsup_{m\to\infty} \alpha_m(t) \le \liminf_{n\to\infty} \alpha_n(t),$$

Hence $\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0, 1]$.

3. WEAK AND STRONG CONVERGENCE

We approximate common fixed points of the mappings S and T through weak convergence of the sequence $\{x_n\}$ in (1.3). In the first result, we use the Opial's condition, whereas the Kadec-Klee property is employed in the second one.

Theorem 1. Let E be a uniformly convex Banach space satisfying the
Opial's condition and C, S, T, $\{x_n\}$ be as in Lemma 6. If $F \neq \phi$, then $\{x_n\}$ converges weakly to a common fixed point of S and T.

Proof: Let $x^* \in F$. Then, as proved in Lemma 5, $\lim_{n\to\infty}\|x_n - x^*\|$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in F.
Since $\{x_n\}$ is a bounded sequence in a uniformly convex Banach space
E, there exist two convergent subsequences $\{x_{ni}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$. Let $z_1 \in C$ and $z_2 \in C$ be weak limits of the subsequences $\{x_{ni}\}$ and $\{x_{n_j}\}$ respectively. By Lemma 6, $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ and $I - S$ is demiclosed with respect to zero by Lemma 2. So we obtain $Sz_1 = z_1$. Similarly, $Tz_1 = z_1$. Again, in the same way, we can prove that $z_2 \in F$.
Next, we prove the uniqueness. For this, suppose that $z_1 \neq z_2$. Then, by the Opial's condition, we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n - z_1\| &= \lim_{n_i\to\infty}\|x_{n_i} - z_1\| \\ &< \lim_{n_i\to\infty}\|x_{n_i} - z_2\| \\ &= \lim_{n\to\infty}\|x_n - z_2\|\end{aligned}$$

$$= \lim_{n_j \to \infty} \left\| x_{n_j} - z_2 \right\|$$
$$< \lim_{n_j \to \infty} \left\| x_{n_j} - z_1 \right\|$$
$$= \lim_{n \to \infty} \left\| x_n - z_1 \right\|,$$

which is a contradiction. Hence $\{x_n\}$ converges weakly to a point in F .

Theorem 2. Let E be a uniformly convex Banach space such that its dual E^* satisfies the Kadec-Klee property. Let C, S, T and $\{x_n\}$ be as in Lemma 6. If $F \neq \phi$, then $\{x_n\}$ converges weakly to a common fixed point of S and T .

Proof: By the boundedness of $\{x_n\}$ and reflexivity of E, we have a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to a point $p \in C$. By Lemma **6**, we have

$$\lim_{i \to \infty} \left\| x_{n_i} - Sx_{n_i} \right\| = 0 = \lim_{i \to \infty} \left\| x_{n_i} - Tx_{n_i} \right\|.$$

This shows that $p \in F$. To prove that $\{x_n\}$ converges weakly to the point p ; suppose that $\{x_{nk}\}$ is another subsequence of $\{x_n\}$ that converges weakly to a point $q \in C$. Then, by Lemmas 6 and 2, $p, q \in W \cap F$, where $W = \omega_w(\{x_n\})$. Since $\lim_{n \to \infty} \left\| tx_n + (1-t)p - q \right\|$ exists for all $t \in [0, 1]$ by Lemma 7, and so $p = q$ by Lemma 4. Consequently, $\{x_n\}$ converges weakly to the point $p \in F$.
The first strong convergence result in an arbitrary real Banach space goes as follows:

Theorem 3. Let E be a real Banach space C , $\{x_n\}$, S and T be as in Lemma 5. If $F \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of S and T if and only if $\liminf_{n \to \infty} d(x_n, F) = 0$, where

$$\mathrm{d(x, F) = \inf\{\|x - p\| : p \in F\}}.$$

Proof: Necessity is obvious.
Conversely, suppose that $\liminf_{n \to \infty} d(x_n, F) = 0$. As proved in Lemma 5, we have $\|x_{n+1} - p\| \le \|x_n - p\|$, which gives

$$d(x_{n+1}, F) \le d(x_n, F).$$

This implies that $\lim_{n \to \infty} d(x_n, F)$ exists and so by the hypothesis,

$$\lim_{n \to \infty} d(x_n, F) = 0 .$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in C . Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n \to \infty} d(x_n, F) = 0,$ there exists a positive integer n_0 such that $d(x_n, F) < \frac{\varepsilon}{4}, \quad \forall n \ge n_0$.
In particular, $\inf\{\left\| x_{n_0} - p \right\| : p \in F\} < \frac{\varepsilon}{2}.$ Thus there must exist $p^* \in F$ such that $\left\| x_{n_0} - p^* \right\| < \frac{\varepsilon}{2}$.
Now, for all $m, n \ge n_0$, we have

$$\left\| x_{n+m} - x_n \right\| \le \left\| x_{n+m} - p^* \right\| + \left\| x_n - p^* \right\|$$
$$\le 2\left\| x_{n_0} - p^* \right\|$$
$$\lhd 2(\frac{\varepsilon}{2}) = \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset C of a Banach space E and so it must converge to a point q in C. Now, $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(q, F) = 0$. Since F is closed, we have $q \in F$.

Khan and Fukhar-ud-din [16] introduced the so-called Condition (A') and gave a slightly improved version of it in [17] as follows:

Two mappings $S, T : C \to C$ are said to satisfy the Condition (A') if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that either $\|x - Tx\| \geq f(d(x, F))$ Or $\|x - Sx\| \geq f(d(x, F))$ for all $x \in C$.

We use the Condition (A') to study the strong convergence of $\{x_n\}$ defined in (1.3). It is worth noting that in the context of nonexpansive mappings $S, T : C \to C$, the Condition (A') is weaker than the compactness of C.

Theorem 4. Let E be a uniformly convex Banach space, C and $\{x_n\}$ be as in Lemma 6. Let $S, T : C \to C$ be two nonexpansive mappings satisfying the Condition (A'). If $F \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.

Proof: By Lemma 5, $\lim_{n\to\infty}\|x_n - x^*\|$ exists for all $x^* \in F$. Let this limit be c where $c \geq 0$. If $c = 0$, there is nothing to prove.
Suppose that $c > 0$. Now, $\|x_{n+1} - x^*\| \leq \|x_n - x^*\|$ gives

$$\inf_{x^*\in F}\|x_{n+1} - x^*\| \leq \inf_{x^*\in F}\|x_n - x^*\|,$$

which means that $d(x_{n+1}, F) \leq d(x_n, F)$ and so lim $\lim_{n\to\infty} d(x_n, F)$ exists.
By using the Condition (A'), either

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty}\|x_n - Tx_n\| = 0$$

or
$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty}\|x_n - Sx_n\| = 0$
In both cases, we have

$$\lim_{n\to\infty} f(d(x_n, F)) = 0$$

Since f is a nondecreasing function and $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. The rest of the proof follows the pattern of the above theorem and is therefore omitted.

Remark 1. All the above theorems can be proved for the iteration process (1.3) with error terms in the sense of [18] (also see [17]):

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = a_n Sx_n + b_n Tx_n + c_n u_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a_n + b_n + c_n = 1, \ \sum_{n=1}^{\infty} c_n < \infty$ and $\{u_n\}$ is a bounded sequence in C.

Remark 2. All the results obtained so far can be extended to the case of nonself-nonexpansive mappings (cf. [19]) for the scheme (1.3), with suitable changes.

Remark 3. In the light of the above remarks, several Mann-type (and Ishikawa-type) convergence results are included in the analysis of our results (see for example, [7] and [8]).

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