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ALMOST CONVERGENCE THROUGH THE GENERALIZED DE LA VALLÉE-POUSSIN MEAN*

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Abstract – Lorentz characterized the almost convergence through the concept of uniform convergence of de la Vallée-Poussin mean. In this paper, we generalize the notion of almost convergence by using the concept of invariant mean and the generalized de la Vallée-Poussin mean. We determine the bounded linear operators for the generalized σ-conservative, σ-regular and σ-coercive matrices.

Keywords – Sequence spaces; invariant mean; matrix transformation; bounded linear operators

1. INTRODUCTION AND PRELIMINARIES

We shall write w for the set of all complex sequences $x=(x_k)^{\infty}{}_{k=0}$. Let $\emptyset$, l_∞, c and c_0 denote the sets of all finite, bounded, convergent and null sequences respectively; and cs be the set of all convergent series. We write lp:= $\{x\in w : \sum_{k=0}^{\infty}|x_k|^p<\infty\}$ for $1\leq$ p $<\infty$. By e and $e^{(n)}(n\in N)$, we denote the sequences such that e_k =1 for $k = 0,1,\ldots$, and $e_n{}^{(n)}$=1 and $e_k{}^{(n)}$=0 $(k\neq n)$. For any sequence $x=(x_k)^{\infty}{}_{k=0}$, let $x^{[n]} = \sum_{k=0}^{n} x_k\, e^{(k)}$ be its n-section.

Note that l_∞, c and c_0 are Banach spaces with the sup-norm $\parallel x\parallel_\infty = \sup_k |x_k|$, and lp $(1\leq$ p $<\infty)$ are Banach spaces with the norm $\parallel x\parallel_p = (\sum_{k=0}^{\infty}|x_k|^p)^{1/\text{p}}$; while $\emptyset$ is not a Banach space with respect to any norm.

A sequence $(b^{(n)})^{\infty}{}_{n=0}$ in a linear metric space X is called $Schauder\ basis$ if for every $x\in X$, there is a unique sequence $(\beta^{(n)})^{\infty}{}_{n=0}$ of scalars such that $x=\sum_{k=0}^{\infty}\beta_k\, b^{(n)}$.

A sequence space X with a linear topology is called a K-$space$ if each of the maps $p_i : X\to C$ defined by $p_i\,(x) = x_i$ is continuous for all $i\in$ N. A K-space is called an $FK-space$ if X is complete linear metric space; a $BK-space$ is a normed FK-space. An FK-space $X\supset\emptyset$ is said to have AK if every sequence $x=(x_k)^{\infty}{}_{k=0}\in X$ has a unique representation $x=\sum_{k=0}^{\infty}x_k\, e^{(k)}$, that is, $x=\lim_{n\to\infty}x^{[n]}$.

Let X and Y be two sequence spaces and $A=(a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix of real or complex numbers. We write $=\big(A_n(x)\big)$, $A_n(x)=\sum_k a_{nk}\,x_k$, provided that the series on the right converges for each n. If $x=(x_k)\in X$ implies that $Ax\in Y$, then we say that A defines a matrix transformation from X into Y, and by (X,Y) we denote the class of such matrices [1].

The following is a very important result:

Lemma 1.1. ([2], Theorem 1.23)**.** Let X and Y be BK spaces and $B(X,Y)$ denote the set of all bounded linear operators from X into Y.

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(a) Then $(X, Y) \subset B(X, Y)$, that is every $A \in (X, Y)$ defines an operator $L_A \in B(X, Y)$ by $L_A\ (x) = Ax$ for all $x \in X$.
(b) If X has AK, then $B(X, Y) \subset (X, Y)$.
(c) We have $A \in (X, l_\infty)$ if and only if
(1.1) $\| A \|_{(X,l_\infty)} = \sup_n \| A_n \|_X^* < \infty$;
if $A \in (X, l_\infty)$ then
(1.2) $\| L_A \| = \| A \|_{(X,l_\infty)}$.

Let σ be a one-to-one mapping from the set N of natural numbers into itself. A continuous linear functional φ on l_∞ is said to be an *invariant mean* or a σ- *mean* if and only if (i) $\varphi(x) \geq 0$ *if* $x \geq 0$ $(i.e. x_k \geq 0$ for all $k)$, (ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \cdots)$, (iii) $\varphi(x) = \varphi((x_k))$ for all $x \in l_\infty$.

Throughout this paper we consider the mapping σ which has no finite orbits, that is, $\sigma^p(k) \neq k$ for all integer $k \geq 0$ and $p \geq 1$, where $\sigma^p(k)$ denotes the pth iterate of σ at k. Note that, a σ -mean extends the limit functional on the space c in the sense that $\varphi(x) = \lim x$ for all $x \in c$, (cf [3]). Consequently, $c \subset V_\sigma$, the set of bounded sequences all of whose σ -means are equal. We say that a sequence $x = (x_k)$ is σ – *convergent* if and only if $x \in V_\sigma$. Using this concept, Schaefer [4] defined and characterized σ -*conservative*, σ -*regular* and σ-*coercive matrices*. If σ is translation then V_σ is reduced to the set f of almost convergent sequences [5]. The idea of σ-convergence for double sequences was introduced in [6] and further studied recently in [7]. In [8-12] we study various classes of four dimensional matrices, e.g. σ -regular, σ -conservative, regularly σ -conservative, boundedly σ -conservative and σ -coercive matrices.

In this paper, we define (σ, λ)-*convergence*, i.e. the σ -convergence through the concept of uniform convergence of the generalized de la Vallée-Poussin means. We also generalize the above matrices by using the concept of (σ, λ)-convergence and determine the associated bounded linear operators for these matrix classes.

2. (σ, λ)-CONVERGENCE

Actually Lorentz [5] characterized the almost convergence (and hence the σ-convergence) through the concept of uniform convergence of de la Vallée -Poussin means. In this paper, we define the σ-convergence through the concept of uniform convergence of the generalized de la Vallée -Poussin means and we call it the (σ, λ)-convergence, which is more general than almost convergence and σ-convergence both.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{m+1} \leq \lambda_m$, $\lambda_1 = 0$, $\rho_m(x) = \frac{1}{\lambda_m} \sum_{j \in I_m} x_j$ is called the *generalized de la Vallée – Poussin mean*, where $Im = [m - \lambda_m + 1, m]$.

A sequence $x = (x_k)$ of real numbers is said to be (σ, λ)-*convergent* to a number L if and only if $x \in V_\sigma^\lambda$, where

$$V_\sigma^\lambda = \{x \in l_\infty : \lim_{m\to\infty} t_{mn}(x) = L, \text{uniformly in } n; L = (\sigma, \lambda) - \lim x\},$$

$t_{mn}(x) = \frac{1}{\lambda_m} \sum_{j \in I_m} x_{\sigma^j(n)}$.
We denote by V_σ^λ the space of all (σ, λ)-convergent sequences. Note that
(i) if $\sigma(n) = n + 1$, then V_σ^λ is reduced to the space $f\_$ (cf [13]),
(ii) and if $\lambda_m = m$, then V_σ^λ is reduced to the space V_σ,
(iii) if $\sigma(n) = n + 1$ and $\lambda_m = m$, then V_σ^λ is reduced to the space f,
(iv) $c \subset V_\sigma^\lambda \subset l_\infty$.

Remark 2.1. It is easy to see that V_σ^λ is a BK space with $\| x \| = \sup m, n\ |tmn(x)|$.

Remark 2.2. Note that a convergent sequence is (σ, λ)-convergent but converse need not hold, e.g. let $\lambda_m = m$, $\sigma(n) = n+1$ and the sequence $x = (x_k)$ be defined by

$$x_k = \begin{cases} 1; & \text{if k is odd,} \\ -1; & \text{if k is even,} \end{cases}$$

then $x = (x_k)$ is (σ, λ)-convergent to 0 but not convergent.

3. (σ, λ)-CONSERVATIVE MATRICES

Definition 3.1. An infinite matrix $A = (a_{nk})$ is said to be (σ, λ)-*conservative* if and only if $Ax \in V_\sigma^\lambda$ for all $x = (x_k) \in c$. We denote this by $A \in (c, V_\sigma^\lambda)$.

Definition 3.2. We say that infinite matrix $A = (a_{nk})$ is said to be (σ, λ)-*regular* if and only if $A = (a_{nk})$ is (σ, λ)-conservative and (σ, λ)- $\lim Ax = \lim x$ for all $x = (x_k) \in c$. We denote this by $A \in (c, V_\sigma^\lambda)_{reg}$.

Remark 3.1. If we take $\lambda_n = n$, then V_σ^λ is reduced to the space and (σ, λ)-conservative and (σ, λ)-regular matrices are respectively reduced to σ-conservative and σ-regular matrices (cf [4]); and in addition, if $\sigma(n) = n+1$ then the space V_σ^λ is reduced to the space f of almost convergent sequences (cf [5]) and these matrices are reduced to the almost conservative and almost regular matrices (cf [14]) respectively.

In the following theorem we characterize (σ, λ)-conservative and (σ, λ)-regular matrices and find the associated bounded linear operators.

Theorem 3.1. (a) A matrix $A = (a_{nk})$ is (σ, λ)-conservative, i.e. $A \in (c, V_\sigma^\lambda)$ if and only if it satisfies the following conditions

(i) $\|A\|_{(l_\infty, l_\infty)} = \sup_n \sum_k |a_{nk}| < \infty$;

(ii) $a_{(k)} = (a_{nk})_{n=1}^\infty \in V_\sigma^\lambda$, for each k;

(iii) $a = \left(\sum_k a_{nk}\right)_{n=1}^\infty \in V_\sigma^\lambda$.

In this case, the (σ, λ)-limit of Ax is

$$\lim x \left[u - \sum_k u_k\right] + \sum_k x_k u_k,$$

Where $u = (\sigma, \lambda)\text{-}\lim\ a$ and $u_k = (\sigma, \lambda)\text{-}\lim\ a_k$, $k = 1,2, \dots$.

(b) $A \in (c, V_\sigma^\lambda)$ defines an operator $L_A \in \mathcal{B}(c, V_\sigma^\lambda)$ by $L_A(x) = Ax$ for all $x \in c$, and $\|L_A\| = \|A\|_{(l_\infty, l_\infty)}$.

Proof: (a) It is quite similar to that of Theorem 1 of Schaefer [4] once we take

$$t_{mn}(x) = \frac{1}{\lambda_m} \sum_{k=1}^\infty \sum_{j \in I_m} a_{\sigma^j(n),k} x_k.$$

(b) It follows directly from Lemma 1.1(a). Since V_σ^λ is a BK space and $(c, V_\sigma^\lambda) \subset (c, l_\infty) \subset (l_\infty, l_\infty)$, we get by Lemma 1.1(c) $\|L_A\| = \|A\|_{(l_\infty, l_\infty)}$.

This completes the proof of the theorem.

Now, we deduce the following.

Corollary 3.2. $A = (a_{nk})$ is (σ, λ)-regular if and only if the conditions (i), (ii) with(σ, λ)-limit zero for each k, and (iii) with (σ, λ)-limit 1 of Theorem 3.1 hold. In this case, the (σ, λ)-limit of Ax is $\sum_k x_k u_k$.

4. (σ, λ)-COERCIVE MATRICES

Definition 4.1. A matrix $A = (a_{nk})$ is said to be (σ, λ)-*coercive* if and only if $Ax \in V_\sigma^\lambda$ for all $x = (x_k) \in l_\infty$, and this is denoted by $A \in (l_\infty, V_\sigma^\lambda)$.

Remark 4.1. If we then take $\lambda_n = n$, (σ, λ)-coercive matrices are reduced to σ-coercive matrices (cf [4]); and in addition, if $\sigma(n) = n + 1$ then these matrices are reduced to the almost coercive matrices (cf [15]).

We prove the following lemma which will be used in our next theorem.

Lemma 4.1. Let $B(n) = (b_{mk}(n))$, $n = 0,1,2, ...$ be a sequence of infinite matrices such that
(i) $\|B(n)\| < H < +\infty$ for all n; and
(ii)$\lim_m b_{mk}(n) = 0$ for each k, uniformly in n.
Then

$$\lim_m \sum_k b_{mk}(n) x_k = 0 \text{ uniformly in } n \text{ for each } x \in l_\infty \tag{1}$$

If and only if

$$\lim_m \sum_k |b_{mk}(n)| = 0 \text{ uniformly in } n. \tag{2}$$

Proof: Let (2) hold and $x \in l_\infty$. Then, since

$$\left|\sum_k b_{mk}(n) x_k\right| \le \|x\|_\infty \sum_k |b_{mk}(n)|,$$

condition (1) holds clearly.

Conversely suppose that (1) holds but (2) does not hold. Let

$$\lim_m \sum_k |b_{mk}(n)| = \lambda > 0 \text{ for all } n.$$

For fixed n, let us write $b(m, k)$ in place of $b_{mk}(n)$. Let for a given $\varepsilon > 0$,

$$N(\varepsilon) = \left\{m \in \mathbb{N}: \sum_k |b(m.k)| > \lambda - \varepsilon\right\}.$$

Then by (i) and (ii) there exist increasing sequences of integers $m_r \in N(1/r)$ and k_r such that

$$\begin{cases} \sum_{k \le k_{r-1}} |b(m_r, k)| < \frac{1}{r}, \\ \sum_{k > k_r} |b(m_r, k)| < \frac{1}{r}. \end{cases} \tag{3}$$

Now define $x \in l_\infty$ such that $k_{r-1} < k < k_r$,

$$x_k = \begin{cases} 1 \ ; \text{ if } b(m_r, k) \ge 0, \\ -1; \text{ if } b(m_r, k) < 0. \end{cases}$$

Then for all $m_r \in N\left(\frac{1}{r}\right)$,

$$\begin{aligned}\sum_k b(m_r,k)x_k &= \sum_{k\le k_{r-1}} b(m_r,k)x_k + \sum_{k_{r-1}<k\le k_r} b(m_r,k)x_k + \sum_{k>k_r} b(m_r,k)x_k \\ &\ge \sum_{k_{r-1}<k\le k_r} b(m_r,k)x_k - \|x\|_\infty \sum_{k\le k_{r-1}} |b(m_r,k)| - \|x\|_\infty \sum_{k>k_r} |b(m_r,k)| \\ &\ge \sum_{k_{r-1}<k\le k_r} b(m_r,k)x_k - \frac{2}{r} \\ &= \sum_{k_{r-1}<k\le k_r} |b(m_r,k)| - \frac{2}{r} \\ &= \sum_k |b(m_r,k)| + \sum_{k\le k_{r-1}} |b(m_r,k)| + \sum_{k>k_r} |b(m_r,k)| - 2/r \\ &\ge \sum_k |b(m_r,k)| - \frac{4}{r}.\end{aligned}$$

Therefore,

$$\lim_r \sum_k b(m_r,k)x_k \ge \lim_r \sum_k |b(m_r,k)|$$

and (1) implies that

$$\lim_m \sum_k |b_{mk}(n)| = 0 \text{ uniformly in } n.$$

This completes the proof of the lemma.

Now, we characterize (σ,λ)-coercive matrices and obtain the bounded linear operator for these matrices.

Theorem 4.1. (a) A matrix $A = a_{nk}$ is (σ,λ)-coercive, i.e. $A \in (l_\infty, V_\sigma^\lambda)$ if and only if (i) and (ii) of Theorem 3.1 hold, and

$$\text{(iii)} \quad \lim_m \sum_{k=1}^{\infty} \left| \sum_{j\in I_m} a_{\sigma^j(n),k} - u_k \right| \text{ uniformly in } n.$$

In this case, the (σ,λ)-limit of Ax is

$$\sum_k x_k u_k \qquad \forall\, x \in l_\infty,$$

Where $u_k = (\sigma,\lambda)\text{-lim } a_k$.

(b) $A \in (l_\infty, V_\sigma^\lambda)$ defines an operator $L_A \in \mathcal{B}(l_\infty, V_\sigma^\lambda)$ by $L_A(x) = Ax$ for all $x \in l_\infty$, and $\|L_A\| = \|A\|_{(l_\infty,l_\infty)}$.

Proof: (a) *Sufficiency*. Same as in Theorem 3 of [4].

Necessity. Let A be (σ,λ)-coercive matrix. This implies that A is (σ,λ)-conservative, then we have condition (i) and (ii) from Theorem 3.1. Now we have to show that (iii) holds.

Suppose that for some n, we have

$$\limsup_n \sum_{k=1}^{\infty} \left| \sum_{j\in I_m} \left[a_{\sigma^j(n),k} - u_k\right] \right| \Big/ \lambda_m = N > 0.$$

Since $\|A\|$ is finite, N is also finite. We observe that since $\sum_{k=1}^{\infty}|u_k| < \infty$ and A is (σ,λ)-coercive, the matrix $B = (b_{nk})$, where $b_{nk} = a_{nk} - u_k$, is also a (σ,λ)-coercive matrix. By an argument similar to that of Theorem 2.1 in [15], one can find $x \in l_\infty$ for which $Bx \notin V_\sigma^\lambda$. This contradiction implies the necessity of (iii).

Now, we use Lemma 4.1 to show that this convergence is uniform in n. Let

$$h_{mk}(n) = \sum_{j\in I_m} \left[a_{\sigma^j(n),k} - u_k\right]/\lambda_m$$

and let $H(n)$ be the matrix $(h_{mk}(n))$. It is easy to see that $\|H(n)\| \le 2\|A\|_{(l_\infty,l_\infty)}$ for every n; and from condition (ii)

$$\lim_m h_{mk}(n) = 0 \text{ for each } k, \text{uniformly in } n.$$

For any $x \in l_\infty$

$$\lim_m \sum_{j\in I_m} h_{mk}(n) x_k = (\sigma,\lambda)\text{-}\lim Ax - \sum_{k=1}^{\infty} u_k x_k$$

and the limit exists uniformly in n, since $Ax \in V_\sigma^\lambda$. Moreover, this limit is zero since

$$\left|\sum_{k=1}^{\infty} h_{mk}(n)x_k\right| \le \|x\|_\infty \sum_{k=1}^{\infty} \left|\sum_{j\in I_m} \left[a_{\sigma^j(n),k} - u_k\right]\right| \Big/ \lambda_m.$$

Hence

$$\lim_m \sum_{k=1}^{\infty} |h_{mk}(n)| = 0 \text{ uniformly in } n;$$

i.e. the condition (iii) holds.

(b) Observe that $\left(l_\infty, V_\sigma^\lambda\right) \subset (l_\infty, l_\infty)$ and the proof is the same as that of Theorem 3.1 (b). This completes the proof of the theorem

5. (l_1, V_σ^λ)-MATRICES

We prove the following theorem:

Theorem 5.1. (a) We have (l_1, V_σ^λ)=$\mathcal{B}(l_1, V_\sigma^\lambda)$ and $A \in (l_1, V_\sigma^\lambda)$ if and only if

$$(i)\ \ \|A\| = \sup_{m,n,k} \left| \frac{1}{\lambda_m} \sum_{j\in I_m} a_{\sigma^j(n),k} \right| < \infty, and$$

the condition (ii) of Theorem 3.1 hold.
(b) If $A \in (l_1, V_\sigma^\lambda)$ then $\|L_A\| = \|A\|$.

Proof: Since l_1 has AK, Lemma 1.1 (b) yields the first part.

Sufficiency. Let the conditions hold. For $x = (x_k) \in l_1$, we see that

$$\lim_{m\to\infty} \frac{1}{\lambda_m} \sum_{k=1}^{\infty} \sum_{j \in I_m} a_{\sigma^j(n),k} x_k = \sum_{k=1}^{\infty} a_k x_k \text{, uniformly in } n \tag{4}$$

it also converges absolutely. Furthermore, $\frac{1}{\lambda_m} \sum_{k=1}^{\infty} \sum_{j \in I_m} a_{\sigma^j(n),k} x_k$ converges absolutely for each m, n. Given ε>0, there exists $k_o = k_o(\varepsilon)$ such that

$$\sum_{k>k_o} |x_k| < \varepsilon. \tag{5}$$

By (ii), we can find $m_o \in \mathbb{N}$ such that

$$\left|\sum_{k>k_o} \left[\frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma^j(n),k} - a_k\right] x_k\right| < \infty, \tag{6}$$

for all $m > m_o$, uniformly in n. Now

$$\left|\sum_{k=1}^{\infty} \left[\frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma^j(n),k} - a_k\right] x_k\right| \le \left|\sum_{k \le k_o} \left[\frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma^j(n),k} - a_k\right] x_k\right| + \sum_{k>k_o} \left|\frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma^j(n),k} - a_k\right| |x_k|,$$

for all $m > m_o$, uniformly in n, by (5), (6) and (i). Hence (4) holds.

Necessity. Let us define a continuous linear functional Q_{mn} on l_1 by

$$Q_{mn}(x) = \frac{1}{\lambda_m} \sum_k \sum_{j \in I_m} a_{\sigma^j(n),k}\, x_k.$$

Now

$$|Q_{mn}(x)| \le \sup_k \left|\frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma^j(n),k}\right| \|x_k\|_1$$

and hence

$$\|Q_{mn}\| \le \sup_k \left|\frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma^j(n),k}\right|. \tag{7}$$

For any fixed $k \in \mathbb{N}$, define $x = (x_i)$ by

$$x_i = \begin{cases} sgn\left(\frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma^j(n),k}\right) \text{; for } i = k \\ 0 \text{; for } i \ne k. \end{cases}$$

Then $\|x\|_1 = 1$, and

$$|Q_{mn}(x)| = \left|\frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma^j(n),k}\, x_k\right| = \left|\frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma^j(n),k}\right| \|x\|_1,$$

So that

$$\|Q_{mn}\| \ge \sup_k \left|\frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma^j(n),k}\right|. \tag{8}$$

Now, by (7) and (8)

$$\|Q_{mn}\| = \sup_k \left| \frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma^j(n),k} \right|.$$

Since $A \in (l_1, V_\sigma^\lambda)$, we have

$$\sup_{m,n} |Q_{mn}(x)| = \sup_{m,n} \left| \frac{1}{\lambda_m} \sum_k \sum_{j \in I_m} a_{\sigma^j(n),k}\, x_k \right| < \infty.$$

Therefore, by the uniform boundedness principle, we have

$$\sup_{m,n} \|Q_{mn}(x)\| = \sup_{m,n,k} \left| \frac{1}{\lambda_m} \sum_{j \in I_m} a_{\sigma^j(n),k} \right| < \infty.$$

(b) If $A \in (l_1, V_\sigma^\lambda)$ then

$$\|L_A(x)\| = \sup_{m,n} |t_{mn}(Ax)| \leq \|A\| \|x\|_1,$$

Which implies that $\|L_A(x)\| \leq \|A\|$. Also, $L_A \in \mathcal{B}(l_1, V_\sigma^\lambda)$ implies that

$$\|L_A(x)\| = \|Ax\| \leq \|L_A\| \|x\|_1,$$

and it follows from $\|e^{(k)}\|_1 = 1$ for all k that $\|A\| \leq \|L_A\|$. Hence $\|L_A\| = \|A\|$.

This completes the proof of the theorem.

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