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SIMULTANEOUS CONTROL OF THE SOURCE TERMS IN A VIBRATIONAL STRING PROBLEM*

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Abstract – In this paper, simultaneous control of source terms is considered in a vibrational string problem. In the considered problem, the terms to be controlled are the force and the initial velocity functions. We state the generalized (weak) solution about the considered problem. The existence and uniqueness of the solution for optimal control problem is investigated. The Frechet derivative of the functional and the Lipschitz continuity of the gradient are investigated. Minimizing sequence is obtained by the method of the projection of the gradient.

Keywords – Optimal control problem, frechet derivative, projection of the gradient

1. INTRODUCTION

Simultaneous determination of the functions in inverse hyperbolic problems has been investigated by some researchers. Bamberger, Chavent and Lailly investigated the parameter functions $\left(\rho(z),\mu(z)\right)$ in

$$\rho(z)\frac{\partial^2 y}{\partial t^2}-\frac{\partial}{\partial z}\left(\mu(z)\frac{\partial y}{\partial z}\right)=0,\ z>0,\ t>0$$

from the only boundary measurement of $y(z,t)/z=0$ in [1]. V. Isakov dealt with the problem of determining the other two coefficients functions in [2]. Gugat, Leugering and Sklyar investigated the problem of determination of $f_1(t),\ f_2(t)$ from the condition $y(x,T)=y_0(x),\ y_t(x,T)=y_1(x)$ in the problem

$$\begin{aligned}&y_{tt}(x,t)=c^2y_{xx}(x,t),\ (x,t)\in[0,L]\times[0,T]\\&y(x,0)=0,\ y_t(x,0)=0,\ x\in[0,L]\\&y(0,t)=f_1(t),\ y(L,t)=f_2(t),\ t\in[0,T]\end{aligned}$$

in [3]. A. Hasanov examined the determination of the pair $w:=\{F(x,t),\ f(t)\}$ from the final state $u(x,T)=\mu(x)$ in the problem;

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$$u_{tt} = \left(k\left(x\right)u_x\right)_x + F\left(x,t\right),\ \left(x,t\right)\in\left(0,L\right)\times\left(0,T\right)$$
$$u\left(x,0\right)=u_0\left(x\right),\ u_t\left(x,0\right)=u_1\left(x\right),\ x\in\left(0,L\right)$$
$$-k\left(0\right)u_x\left(0,t\right)=f\left(t\right),\ k\left(l\right)u_x\left(l,t\right)=0,\ t\in\left(0,T\right)$$

in [4].

Considering the inverse source problem by a control method, the reconstruction formula and regularization methods have been obtained in [5] and [6]. In [5], Mordukhovich and Raymond considered the problem of minimizing the integral functional

$$J\left(y,u\right)=\int_\Omega \varphi\left(x,y\left(T\right)\right)dx+\int_Q g\left(x,t,y\right)dxdt+\int_\Sigma h\left(s,t,u\right)dsdt$$

for the problem

$$\begin{aligned} &y_{tt}-\Delta y = f && \text{in}\quad Q:=\Omega\times\left]0,T\right[\\ &y=u && \text{on}\quad \Sigma:=\Gamma\times\left]0,T\right[\\ &y\left(0\right)=y_0,\ y_t\left(0\right)=y_1 && \text{in}\ \Omega \end{aligned}$$

M. Yamamoto in [6] considered the determination of $f\left(x\right)$ from $\left[\partial u\left(f\right)/\partial n\right]\left(x,t\right)$ in the problem

$$u''\left(x,t\right)=\Delta u\left(x,t\right)+\sigma\left(t\right)f\left(x\right),\quad \left(x\in\Omega,0<t<T\right)$$

$$u\left(x,0\right)=u'\left(x,0\right)=0,\quad \left(x\in\Omega\right)$$

$$u\left(x,t\right)=0,\quad \left(x\in\partial\Omega,0<t<T\right).$$

The stress in this paper is the force and initial velocity functions in the vibrational string problem can be controlled simultaneously from the final state of the string. The paper is organized as follows: In section 2, we give the statement of the problem with its physical motivation and state the definition of the generalized (weak) solution. In section 3, we prove the existence and uniqueness of the solution for optimal control problem by Weierstrass theorem and strict convexity of the functional. In section 4, we get the Frechet differentiability of the functional and Lipschitz constant of the gradient. In the last section, we constitute a minimizing sequence using the Lipschitz constant by the method of the projection of the gradient and prove its convergence to the optimal solution.

2. STATEMENT OF THE PROBLEM

In this paper, we consider the problem of simultaneous control of the functions $w:=\left\{f(x,t),h\left(x\right)\right\}$ in the problem

$$\rho(x)u_{tt}=(k(x)u_x)_x+f(x,t),\ (x,t)\in\Omega_T=(0,l)\times\left(0,T\right]\subset\mathbb{R}^2 \tag{1}$$

$$u(x,0)=g(x),\ u_t(x,0)=h(x),\ x\in\ (0,l) \tag{2}$$

$$u\left(0,t\right)=0,\ u\left(l,t\right)=\ 0,\ \ t\in\left(0,T\right]. \tag{3}$$

Using the functional

$$J_\alpha(w) = \int_0^l \left[u(x,T;w) - y(x)\right]^2 dx + \alpha \|w\|_W^2 . \tag{4}$$

For $\alpha > 0.$ we will deal with the problem

$$J_\alpha(w_*) = \inf_{w \in W} J_\alpha(w). \tag{5}$$

The problem (1)-(3) models the vibration of a flexible string made of nonhomogeneous material. Here the functions $\rho(x) > 0$ and $k(x) > 0$ represent the density and flexural rigidity of the string. $f(x,t)$ is the applied force function to the string. According to the Hooke's law, the function $T(x,t) := k(x)u_x(x,t)$ defines the tension of the flexible string. $g(x)$ and $h(x)$ state the initial position and the initial velocity, respectively. The $x = 0$ and $x = l$ ends of the string are fixed.
Here, the functions satisfy the conditions;

$$f(x,t) \in L_2(\Omega_T),\ g(x) \in W_2^1[0,l],\ h(x) \in L_2[0,l],\ y(x) \in L_2(0,l) \tag{6}$$

and

$$0 < \rho_0 \le \rho(x) \le \rho_1,\ \ 0 < k_0 \le k(x) \le k_1,\ \ \overset{0}{\forall} x \in [0,l]. \tag{7}$$

The space W_2^1 is a Sobolev space whose functions with their first order generalized derivatives belong to L_2 space.
The admissible control set is defined as follows:

$$W := \left\{ w = \{f,h\} : \|w\|_W \le b \right\} \subset L_2(\Omega_T) \times L_2[0,l]. \tag{8}$$

The inner product in the space W is defined as

$$(w_1, w_2)_W := \int_0^T \int_0^l f_1(x,t)\, f_2(x,t)\, dxdt + \int_0^l h_1(x) h_2(x)\, dx,\ \forall\ w_1, w_2 \in W , \tag{9}$$

and the norm of an element is

$$\|w\|_W = \left(\int_0^T \int_0^l \left[f(x,t) \right]^2 dxdt + \int_0^l \left[h(x) \right]^2 dx \right)^{1/2} . \tag{10}$$

Definition 2.1. The function $u \in \overset{0}{W}{}_2^1$ with $u(x,0) = g(x)$, which satisfies the following equation for all $\eta \in \hat{\overset{0}{W}}{}_2^1$, is called the generalized solution of the problem (1)-(3),

$$\iint_{\Omega_T} \left(-\rho(x) u_t \eta_t + k(x) u_x \eta_x \right) dxdt = \int_0^l \rho(x) h(x) \eta(x,0)\, dx + \iint_{\Omega_T} f(x,t) \eta\, dxdt . \tag{11}$$

Here, the space $\overset{0}{W}{}_2^1$ consists of those elements of W_2^1 which vanish for $t = 0$ and $\hat{\overset{0}{W}}{}_2^1$ consists of those elements of $\overset{0}{W}{}_2^1$ which vanish for $t = T$.
Now, for $w \in W$ the following theorem is valid;

Theorem 2.2. Let the conditions (6), (7) hold. Then for each $w \in W$, there is a generalized solution of the problem (1)-(3), and for this solution the estimate

$$\|u\|^2_{H^1(\Omega_T)} \le c_5\left(\|g\|^2_{H^1[0,l]} + \|w\|^2_W\right) \tag{12}$$

is valid.

Proof: The proof follows from the Galerkin method [7].
Since the problem is ill-posed for $\alpha = 0$, we use the parameter $\alpha > 0$ as the regularization parameter, which is the strong convexity constant, which guarantees the uniqueness and stability of the regularized solution. This parameter can be found for numerical investigations by regularization method such as Tikhonov regularization ([8]-[9]).

3. THE EXISTENCE AND UNIQUENESS OF THE OPTIMAL SOLUTION

In this section, we prove that the problem (5) has a solution. Here, we show that the problem satisfies the conditions of Weierstrass theorem. The space $L_2(\Omega_T)\times L_2[0,l]$ is a Hilbert space and the set W is a weakly compact set in this space. If we can show that the functional $J_\alpha(w)$ is weakly semicontinuous from below on the set W, then according to the generalized Weierstrass theorem[10], we will prove the existence of the minimum. In addition, if the functional $J_\alpha(w)$ is strictly convex then the minimum will be unique (Theorem 38.C. in [10]).
Let's consider the functional

$$J_0(w) = \int_0^l \left[u(x,T;w) - y(x)\right]^2 dx$$

and

$$J_\alpha(w) = J_0(w) + \alpha\|w\|^2_W .$$

If the functional $J_\alpha(w)$ is convex and continuous on W then it is weakly semicontinuous from below. Now, let's prove the continuity of $J_\alpha(w)$ on W. To do this it is enough to show the continuity of $J_0(w)$.

Theorem 3.1. Let $w \in W$ be an arbitrary element and Δw be an increment for w such that $w + \Delta w \in W$. So, the following inequality is valid;

$$\left|\Delta J_0(w)\right| = \left|J_0(w+\Delta w) - J_0(w)\right| \le s_0\left(\|\Delta w\|_w + \|\Delta w\|^2_w\right). \tag{13}$$

Here the number s_0 is independent of Δw.

Proof: For $w+\Delta w = \{f+\Delta f, h+\Delta h\}$, we show by $u_\Delta(x,t;w) = u(x,t;w+\Delta w)$ and the increment function $\Delta u(x,t;w) = u_\Delta(x,t;w) - u(x,t;w)$ satisfies the problem;

$$\rho\Delta u_{tt} = (k\Delta u_x)_x + \Delta f \tag{14}$$

$$\Delta u(x,0) = 0 \quad \Delta u_t(x,0) = \Delta h \tag{15}$$

$$\Delta u(0,t) = \Delta u(l,t) = 0 . \tag{16}$$

The variation of $J_0(w)$ is

$$\Delta J_0(w)=J_0(w+\Delta w)-J_0(w)=\int_0^l\left\{\left[u_\Delta(x,T;w)-y\right]^2-\left[u(x,T;w)-y\right]^2\right\}dx$$
$$=\int_0^l 2\left[u(x,T;w)-y\right]\Delta u(x,T)\,dx+\int_0^l\left[\Delta u(x,T)\right]^2dx \tag{17}$$

With Cauchy-Bunyakovsky inequality, we can write that

$$\left|\Delta J_0(w)\right|\le 2\left\|u(x,T;w)-y\right\|_{L_2[0,l]}\left\|\Delta u(x,T)\right\|_{L_2[0,l]}+\left\|\Delta u(x,T)\right\|^2_{L_2[0,l]} \tag{18}$$

Lemma 3.2. For the function $\Delta u(x,T)$, the following estimate is valid;

$$\left\|\Delta u(x,T)\right\|^2_{L_2[0,l]}\le s_2\left(\left\|\Delta h\right\|^2_{L_2[0,l]}+\left\|\Delta f\right\|^2_{L_2(\Omega_T)}\right)=s_2\left\|\Delta w\right\|^2_W,\quad s_2=3s_1^2 \tag{19}$$

Here, $s_1=\max\left\{T\sqrt{\frac{r_0}{\rho_0}},\frac{T\sqrt{T}}{\rho_0\sqrt{3}}\right\}$ and $r_0=\max\left\{\rho_1,k_1\right\}$.

Proof: Proof can be obtained via the processes in [4].
Since $y(x)\in L_2[0,l]$, applying $\varepsilon-\text{Cauchy}$ inequality for (18), we get

$$\left|\Delta J_0(w)\right|\le s_0\left(\left\|\Delta w\right\|_w+\left\|\Delta w\right\|_w^2\right).$$

Here the number s_0 is independent of Δw. Theorem 3.1 has been proven.
So, the functional $J_\alpha(w)$ is continuous on W, then it is weakly semicontinuous from below.

Theorem 3. 2. The functional $J_\alpha(w)$ is strongly convex on W.

Proof: We know that the set W is convex [11]. From the linearity of the boundary value problem and uniqueness of the solution, we can write that

$$u\left(x,t;\beta w_1+(1-\beta)w_2\right)=\beta u(x,t;w_1)+(1-\beta)u(x,t;w_2)$$

for each $w_1,w_2\in W$ and $\beta\in[0,1]$. Now, we will show that the following inequality is valid with the constant $\chi>0$,

$$J_\alpha\left(\beta w_1+(1-\beta)w_2\right)\le\beta J_\alpha(w_1)+(1-\beta)J_\alpha(w_2)-\chi\beta(1-\beta)\left\|w_1-w_2\right\|_W^2$$

for $\forall\ w_1,w_2\in W$ and for $\forall\ \beta\in[0,1]$.
From the convexity of the function $|u-y|^2$ to u, we have

$$J_\alpha\left(\beta w_1+(1-\beta)w_2\right)=\int_0^l\left[\psi\left(x,T;\beta w_1+(1-\beta)w_2\right)-y(x)\right]^2dx+\alpha\left\|\beta w_1+(1-\beta)w_2\right\|_W^2$$
$$\le\beta\left\{\int_0^l\left[\psi(x,T;w_1)-y(x)\right]^2dx+\alpha\left\|w_1\right\|_W^2\right\}$$
$$+(1-\beta)\left\{\int_0^l\left[\psi(x,T;w_2)-y(x)\right]^2dx+\alpha\left\|w_2\right\|_W^2\right\}$$
$$-\alpha\beta(1-\beta)\left\|w_1-w_2\right\|_W^2.$$

So, the cost functional is strongly convex with $\chi = \alpha$.

We proved that the cost functional is strongly convex (so strictly convex) for strong convexity constant $\alpha > 0$ on the weakly compact set W. So according to the generalized Weierstrass theorem [8], the optimal solution $w_* \in W$ to the problem

$$J_\alpha(w_*) = \inf_{w \in W} J_\alpha(w)$$

exists and is unique.

4. FRECHET DIFFERENTIABILITY OF THE FUNCTIONAL AND LIPSCHITZ CONTINUITY OF GRADIENT

In this section, we use the definition

$$\Delta J_\alpha(w) = J_\alpha(w + \Delta w) - J_\alpha(w) = \left(J'_\alpha(w), \Delta w\right)_w + O\left(\|\Delta w\|_w^2\right)$$

of Frechet differentiability of $J_\alpha(w)$,

Theorem 4.1. Let the conditions (6)–(7) hold. Therefore, the functional $J_\alpha(w) \in C^1(W)$ can be Frechet differentiable. Moreover, this derivation can be defined as

$$J'_\alpha(w) = \left\{\eta(x,t;w) + 2\alpha f,\ p(x)\eta(x,0;w) + 2\alpha h\right\} \tag{20}$$

via the solution of adjoint boundary value problem;

$$\rho\eta_{tt} = (k\eta_x)_x \tag{21}$$

$$\eta(x,T) = 0 \tag{22}$$

$$\rho\eta_t(x,T) = -2\left[u(x,T) - y\right] \tag{23}$$

$$k(0)\eta(0,t) = 0,\ \ k(l)\eta(l,t) = 0 \tag{24}$$

Proof: Let's consider the increment of the functional $J_\alpha(w)$;

$$\begin{aligned}\Delta J_\alpha(w) = J_\alpha(w+\Delta w) - J_\alpha(w) &= \int_0^l \left\{\left[u_\Delta(x,T;w) - y\right]^2 - \left[u(x,T;w) - y\right]^2\right\} dx \\ &+ \|w+\Delta w\|_W^2 - \|w\|_W^2 \\ &= \int_0^l 2\left[u(x,T;w) - y\right]\Delta u(x,T)\,dx + \int_0^l \left[\Delta u(x,T)\right]^2 dx \\ &+ \int_0^T\int_0^l 2\alpha f\Delta f dx dt + \int_0^l 2\alpha h\Delta h dx + \alpha\|\Delta w\|_W^2 .\end{aligned}$$

Now we must estimate the term $2\int_0^l \left[u(x,T;w) - y\right]\Delta u(x,T)\,dx$, whose relation with $\Delta J_0(w)$ can be seen in (17).

If we multiply the adjoint boundary value problem by Δu and integrate over Ω_T, we find

$$\int_0^l\int_0^T \rho\eta_{tt}\Delta u dt dx = \int_0^l\int_0^T (k\eta_x)_x \Delta u dt dx .$$

By integration by parts, we get

$$\iint\left[\rho\Delta u_{tt}-(k\Delta u_x)_x\right]\eta dxdt+\int_0^l \rho\eta(x,0)\Delta h dx=\int_0^l 2\left[u(x,T)-y\right]\Delta u(x,T)dx. \tag{25}$$

After multiplying (14)- (16) by $\eta(x,t)$ and integrating over Ω_T, if we subtract this from (25), we have

$$\int_0^l \rho\eta(x,0)\Delta h dx+\int_0^T\int_0^l \Delta f\,\eta dxdt=2\int_0^l\left[u(x,T;w)-y\right]\Delta u(x,T)dx$$

Considering (19) here, we write

$$\Delta J_\alpha(w)=\int_0^T\int_0^l \eta\Delta f dxdt+\int_0^l \rho\eta(x,0;w)\Delta h dx+\int_0^T\int_0^l 2\alpha f\Delta f dxdt+\int_0^l 2\alpha h\Delta h dx+o\left(\|\Delta w\|_w^2\right)$$

so the Frechet derivative is

$$J'_\alpha(w)=\left\{\eta(x,t;w)+2\alpha f,\ \rho(x)\eta(x,0;w)+2\alpha h\right\}.$$

Hence, theorem 4.1 has been proven.

Theorem 4.2. Let the conditions of Theorem 4.1 hold. Then the estimate

$$\left\|J'_\alpha(w+\Delta w)-J'_\alpha(w)\right\|_w\le L\|\Delta w\|_w,\ \ \forall w,w+\Delta w\in W \tag{26}$$

is valid. Here,

$$L=\sqrt{2qs_2+8\alpha^2},\ q:=\max\left\{\frac{4l^2T}{\rho_0},\frac{4\rho_1^2l^2}{\rho_0}\right\},\ s_2=3\left(\max\left\{T\sqrt{\frac{r_0}{\rho_0}},\frac{T\sqrt{T}}{\rho_0\sqrt{3}}\right\}\right)^2 \tag{27}$$

and

$$\left\|J'_\alpha(w+\Delta w)-J'_\alpha(w)\right\|_w^2=\iint_{\Omega_T}\left[\Delta\eta(x,t;w)+2\alpha\Delta f\right]^2dxdt+\int_0^l\left[\rho(x)\Delta\eta(x,0;w)+2\alpha\Delta h\right]^2dx \tag{28}$$

Proof: We can rewrite (28) with the inequality $(a+b)^2\le 2(a^2+b^2)$ such that

$$\begin{aligned}\left\|J'_\alpha(w+\Delta w)-J'_\alpha(w)\right\|_w^2&=\iint_{\Omega_T}\left[\Delta\eta(x,t;w)+2\alpha\Delta f\right]^2dxdt+\int_0^l\left[\rho(x)\Delta\eta(x,0;w)+2\alpha\Delta h\right]^2dx\\&\le 2\iint_{\Omega_T}\left[\Delta\eta(x,t;w)\right]^2dxdt+2\iint_{\Omega_T}\left[2\alpha\Delta f\right]^2dxdt\\&+2\int_0^l\left[\rho(x)\Delta\eta(x,0;w)\right]^2dx+2\int_0^l\left[2\alpha\Delta h\right]^2dx.\end{aligned}$$

If we use the increment $w+\Delta w$ in the solution of adjoint boundary value problem and show by $\eta_\Delta = \eta(x,t;w+\Delta w)$, we get the following problem for $\Delta\eta = \eta(x,t;w+\Delta w) - \eta(x,t;w)$;

$$\rho\Delta\eta_{tt} = (k\Delta\eta_x)_x \tag{29}$$

$$\Delta\eta(x,T) = 0 \tag{30}$$

$$\rho\Delta\eta_t(x,T) = -2\Delta u(x,T;w) \tag{31}$$

$$k(0)\Delta\eta(0,t) = 0 \;\; k(l)\Delta\eta(l,t) = 0. \tag{32}$$

If we multiply the both sides of (29) by $\Delta\eta_t$, integrate over $[0,l]$ and use boundary conditions, we have

$$\int_0^l \left[\rho\Delta\eta_{tt} - (k\Delta\eta_x)_x\right]\Delta\eta_t dx = 0$$

$$\int_0^l \left[\rho\Delta\eta_{tt}\Delta\eta_t + k\Delta\eta_x\Delta\eta_{xt}\right]dx = \int_0^l (k\Delta\eta_x\Delta\eta_t)_x\, dx$$

$$\frac{1}{2}\frac{d}{dt}\left\{\int_0^l \left[\rho(\Delta\eta_t)^2 + k(\Delta\eta_x)^2\right]dx\right\} = k(l)\Delta\eta_x(l,t)\Delta\eta_t(l,t) - k(0)\Delta\eta_x(0,t)\Delta\eta_t(0,t).$$

Integrating the last equality over $[t,T]$ we obtain,

$$\frac{1}{2}\int_0^l \left\{\rho\left[\Delta\eta_t(x,T)\right]^2 + k\left[\Delta\eta_x(x,T)\right]^2\right\}dx = \frac{1}{2}\int_0^l \left\{\rho\left[\Delta\eta_t(x,t)\right]^2 + k\left[\Delta\eta_x(x,t)\right]^2\right\}dx$$

$$\int_0^l \left(\rho\Delta\eta_t^2 + k\Delta\eta_x^2\right)dx = \int_0^l \frac{4\left[\Delta u(x,T)\right]^2}{\rho(x)}dx \tag{33}$$

$$\int_0^l \Delta\eta_x^2 dx \le \frac{4}{\rho_0}\int_0^l \left[\Delta u(x,T)\right]^2 dx.$$

For the term $\Delta\eta_x$, we have the following inequality;

$$\frac{2}{l^2}\left\|\Delta\eta(x,t)\right\|^2_{L_2[0,l]} \le \left\|\Delta\eta_x\right\|^2_{L_2[0,l]},\ \forall t\in[0,T].$$

Substituting this into (33), we write

$$\frac{2}{l^2}\int_0^l \Delta\eta^2 dx \le \frac{4}{\rho_0}\int_0^l \left[\Delta u(x,T)\right]^2 dx$$

$$\left\|\Delta\eta\right\|^2_{L_2[0,l]} \le \frac{2l^2}{\rho_0}\left\|\Delta u(x,T)\right\|^2_{L_2[0,l]},\ \forall t\in[0,T]$$

and integrating over $[0,T]$, we get

$$\int_0^T\int_0^l \left[\Delta\eta(x,t;w)\right]^2 dxdt \le \frac{2l^2T}{\rho_0}\left\|\Delta u(x,T)\right\|^2_{L_2[0,l]}.$$

Moreover, we can write that

$$\int_0^l \left[\rho(x)\Delta\eta(x,0)\right]^2 dx \le \rho_1^2 \int_0^l \left[\Delta\eta(x,0)\right]^2 dx \le \frac{2\rho_1^2 l^2}{\rho_0}\left\|\Delta u(x,T)\right\|_{L_2[0,l]}^2$$

Hence

$$2\int_0^T\int_0^l \left[\Delta\eta(x,t;w)\right]^2 dxdt + 2\int_0^l \left[\rho(x)\Delta\eta(x,0)\right]^2 dx \le \frac{4l^2T}{\rho_0}\left\|\Delta u(x,T)\right\|_{L_2[0,l]}^2 + \frac{4\rho_1^2 l^2}{\rho_0}\left\|\Delta u(x,T)\right\|_{L_2[0,l]}^2 \tag{34}$$

and taking as $q := \max\left\{\frac{4l^2T}{\rho_0}, \frac{4\rho_1^2 l^2}{\rho_0}\right\}$, we obtain

$$\left\|J'_\alpha(w+\Delta w) - J'_\alpha(w)\right\|_w^2 \le q\left\|\Delta u(x,T)\right\|_{L_2[0,l]}^2 + 8\alpha^2\left\|\Delta w\right\|_W^2$$

With (19) one can write

$$\left\|J'_\alpha(w+\Delta w) - J'_\alpha(w)\right\|_w^2 \le 2qs_2\left\|\Delta w\right\|_w^2 + 8\alpha^2\left\|\Delta w\right\|_W^2$$

and for $L = \sqrt{2qs_2 + 8\alpha^2}$, we have

$$\left\|J'_\alpha(w+\Delta w) - J'_\alpha(w)\right\|_w \le L\left\|\Delta w\right\|_w .$$

Hence, theorem 4.2 has been proven.

5. MINIMIZING SEQUENCE AND ITS CONVERGENCE

We set the minimizing sequence according to the method of projection of the gradient [10], by

$$w_{k+1} = P_W\left(w_k - \beta_k J'_\alpha(w_k)\right), \quad k = 0,1,2,... \tag{35}$$

where $P_W\left(w_k - \beta_k J'_\alpha(w_k)\right)$ is the projection of the element $w_k - \beta_k J'_\alpha(w_k)$ in the set W.

Theorem 5.1. Let $w_0 \in W$ be the initial point. The sequence $\{w_k\}$ defined by (35) converges to the unique minimum element w_* of the functional $J_\alpha(w)$ for $\beta_k = \beta \in \left(0, 4\alpha L^{-2}\right)$. Here, L is the Lipschitz constant for $J'_\alpha(w)$. Moreover, for the minimizing sequence the following inequality holds;

$$\left\|w_k - w_*\right\| \le \left\|w_0 - w_*\right\|.q^k, \; k = 0,1,...$$

where $q = \left(1 - 4\alpha\beta + \beta^2 L^2\right)^{1/2} \in (0,1)$.

Proof: We define the mapping $A: W \to W$ with

$$Aw = P_W\left(w - \beta J'_\alpha(w)\right).$$

It can easily be shown that the mapping P_W is a contraction mapping, namely;

$$\left\|P_W(u) - P_W(v)\right\| \le \left\|u - v\right\|; \; \forall u,v \in W.$$

We must show that the mapping A also holds this property, while $0 < \beta < 4\alpha L^{-2}$. We know that

$$\left\langle J'_\alpha(u) - J'_\alpha(v), u - v\right\rangle \ge 2\alpha\left\|u - v\right\|^2, \; \forall u,v \in W. \tag{36}$$

So, using (36) and Lipschitz continuity of the functional, we write the following:

$$\begin{aligned}\|Au - Av\|^2 &= \left\|P_W\left(u - \beta J'_\alpha(u)\right) - P_W\left(v - \beta J'_\alpha(v)\right)\right\|^2 \\ &\le \left\|(u-v) - \beta\left(J'_\alpha(u) - J'_\alpha(v)\right)\right\|^2 \\ &= \|u-v\|^2 - 2\beta\left\langle J'_\alpha(u) - J'_\alpha(v), u-v\right\rangle + \beta^2\left\|J'_\alpha(u) - J'_\alpha(v)\right\|^2 \\ &\le \left(1 + \beta^2 L^2 - 4\beta\alpha\right)\|u-v\|^2\end{aligned}$$

Then we have

$$\|Au - Av\| \le q\|u - v\|$$

by $q = \left(1 + \beta^2 L^2 - 4\beta\alpha\right)^{1/2}$. The condition $\beta < 4\alpha L^{-2}$ gives $q \in (0,1)$.

The statement (35) can be written as $w_{k+1} = Aw_k$. Then by the contraction mapping principle, the sequence $\{w_k\}$ converges to the fixed point $w_* = Aw_*$ of the operator A with the factor q. We know that the minimum element w_* is unique. So we get the following:

$$\begin{aligned}\|w_k - w_*\| = \|Aw_{k-1} - Aw_*\| &\le q\|w_{k-1} - w_*\| \\ &\le q^2\|w_{k-2} - w_*\| \\ &\vdots \\ &\le q^k\|w_0 - w_*\|\end{aligned}$$

Hence, the theorem 5.1. has been proven.

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