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On the global asymptotic stability for a rational recursive sequence

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Abstract

The main objective of this paper is to study the boundedness character, the periodicity character, the convergence and the global stability of the positive solutions of the nonlinear rational difference equation

$$x_{n+1}=\left(\sum_{i=0}^{k}\alpha_i x_{n-i}\right)\Big/\left(B+\sum_{i=0}^{k}\beta_i x_{n-i}\right), \quad n=0,1,2,....$$

where the coefficients B,α_i,β_i together with the initial conditions $x_{-k},....,x_{-1},x_0$ are arbitrary positive real numbers, while k is a positive integer number.

Keywords*:* Difference equations; boundedness character; prime period two solution; global stability; convergence.

1. Introduction

The qualitative study of difference equations is a fertile research area and increasingly attracts many mathematicians. This topic draws its importance from the fact that many real life phenomena are modeled using difference equations. Examples from economy, biology, etc. can be found in [1-4]. It is known that nonlinear difference equations are capable of producing a complicated behavior regardless its order. This can be easily seen from the family $x_{n+1}=g_\mu(x_n)$, $0<\mu$, $\mu\geq 0$. This behavior ranges according to the value of μ from the existence of a bounded number of periodic solutions to chaos. There has been great interest in studying the global attractivity, boundedness character and the periodicity nature of nonlinear difference equations. For example, in the articles [5-10], closely related global convergence results were obtained which can be applied to nonlinear difference equations in proving that every solution of these equations converges to a period two solution. For other closely related results (see [11-28]), and the references cited therein. The study of these difference equations is challenging yet rewarding, and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right. Furthermore, the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations. The main objective of the present paper is to study a more general nonlinear rational difference equation which has not been discussed elsewhere. More precisely, in this paper the boundedness character, periodicity character, the convergence and the global stability of the positive solutions of the nonlinear rational difference equation

$$x_{n+1}=\left(\sum_{i=0}^{k}\alpha_i x_{n-i}\right)\Big/\left(B+\sum_{i=0}^{k}\beta_i x_{n-i}\right), \quad n=0,1,2,.... \qquad (1)$$

is investigated, where the coefficients B,α_i,β_i together with the initial conditions $x_{-k},....,x_{-1},x_0$ are arbitrary positive real numbers, while k is a positive integer number.

Definition 1. A difference equation of order $(k+1)$ is of the form

$$x_{n+1}=F(x_n,x_{n-1},....,x_{n-k}), \qquad n=0,1,2,.... \qquad (2)$$

where F is a continuous function which maps some set J^{k+1} into J and J is a set of real numbers. An equilibrium point $\tilde{x}$ of this equation

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is a point that satisfies the condition $\tilde{x} = F(\tilde{x}, \tilde{x}, ..., \tilde{x})$, that is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$ with $x_n = \tilde{x}$ for all $n \geq -k$ is a solution of that equation.

Definition 2. Let $\tilde{x} \in (0, \infty)$ be an equilibrium point of the difference equation (2). Then,

(i) An equilibrium point $\tilde{x}$ of the difference equation (2) is called locally stable if for every $0 < \varepsilon$ there exists $0 < \delta$ such that, if $x_{-k}, ..., x_{-1}, x_0 \in (0, \infty)$ with $|x_{-k} - \tilde{x}| + ... + |x_{-1} - \tilde{x}| + |x_0 - \tilde{x}| < \delta$ then $|x_n - \tilde{x}| < \varepsilon$ for all $n \geq -k$.

(ii) An equilibrium point $\tilde{x}$ of the difference equation (2) is called locally asymptotically stable if it is locally stable and there exists $0 < \gamma$ such that, if $x_{-k}, ..., x_{-1}, x_0 \in (0, \infty)$ with $|x_{-k} - \tilde{x}| + ... + |x_{-1} - \tilde{x}| + |x_0 - \tilde{x}| < \gamma$, then $\lim_{n \to \infty} x_n = \tilde{x}$.

(iii) An equilibrium point $\tilde{x}$ of the difference equation (2) is called a global attractor if for every $x_{-k}, ..., x_{-1}, x_0 \in (0, \infty)$ we have $\lim_{n \to \infty} x_n = \tilde{x}$.

(iv) An equilibrium point $\tilde{x}$ of the equation (2) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\tilde{x}$ of the difference equation (2) is called unstable if it is not locally stable.

Definition 3. We say that a sequence $\{x_n\}_{n=-k}^{\infty}$ is bounded and persists if there exist positive constants m and M such that $m \leq x_n \leq M$ for all $n \geq -k$.

Definition 4. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Assume that $\tilde{a} = \sum_{i=0}^{k} \alpha_i$, $\bar{a} = \sum_{i=0}^{k} (-1)^i \alpha_i$, $\tilde{b} = \sum_{i=0}^{k} \beta_i$ and $\bar{b} = \sum_{i=0}^{k} (-1)^i \beta_i$. Then the equilibrium point $\tilde{x}$ of the difference equation (1) is the solution of the equation

$$\tilde{x} = \frac{\tilde{a}\tilde{x}}{B + \tilde{b}\tilde{x}}. \quad (3)$$

Consequently, we deduce that the equilibrium point $\tilde{x}$ of the difference equation (1) is $\tilde{x} = 0$ or $\tilde{x} = \frac{\tilde{a} - B}{\tilde{b}}$, where $B \neq \tilde{a}$.

Let $F : (0, \infty)^{k+1} \to (0, \infty)$ be a continuous function defined by

$$F(u_0, u_1, ..., u_k) = \left(\sum_{i=0}^{k} \alpha_i u_i \right) / \left(B + \sum_{i=0}^{k} \beta_i u_i \right). \quad (4)$$

Now, the linearized equation is

$$y_{n+1} = \sum_{j=0}^{k} \frac{\partial F(\tilde{x}, \tilde{x}, ..., \tilde{x})}{\partial u_j} y_{n-j},$$

which can be written in the form:

$$y_{n+1} = \sum_{j=0}^{k} b_j y_{n-j}, \quad (5)$$

where

$$b_j = (\alpha_j - \beta_j \tilde{x}) / (B + \tilde{b}\tilde{x}). \quad (6)$$

The characteristic equation of the linearized equation (5) is given by

$$\lambda^{n+1} = \sum_{j=0}^{k} b_j \lambda^{n-j}, \quad (7)$$

2. Main results

In this section we establish some results which show that the equilibrium point $\tilde{x}$ of the difference equation (1) is globally asymptotically stable and every positive solution of the difference equation (1) is bounded and convergent to the equilibrium $\tilde{x}$ and has prime period two.

Theorem 1. ([3, 28] The linearized stability theorem).

Suppose F is a continuously differentiable function defned on an open neighbourhood of the equilibrium $\tilde{x}$ Then the following statements are true.

(i) If all roots of the characteristic equation (7) of the linearized equation (5) have absolute value less

than one, then the equilibrium point $\tilde{x}$ is locally asymptotically stable.
(ii) If at least one root of Eq.(7) has absolute value greater than one, then the equilibrium point $\tilde{x}$ is unstable.
(iii) If all roots of Eq.(7) have absolute value greater than one, then the equilibrium point $\tilde{x}$ is a source.

Theorem 2. (see [29,30]). Assume that $a, b \in R$ and $k \in \{0,1,2,.....\}$. Then

$$|a| + |b| < 1, \tag{8}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + ax_n + bx_{n-k} = 0, \quad n = 0,1,2,.... \tag{9}$$

Remark 1. (see [29]). Theorem 3 can be easily extended to a general linear difference equation of the form

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0,1,2,.... \tag{10}$$

where $p_1, p_2, ..., p_k \in R$ and $k \in \{1,2,.....\}$. Then the equation (10) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1. \tag{11}$$

Theorem 3. Let $\{x_n\}_{n=-k}^{\infty}$ be a positive solution of Eq.(1). Then it is bounded and persists.
Proof. From the difference equation (1) we deduce that

$$x_{n+1} = \frac{\sum_{i=0}^{k} \alpha_i x_{n-i}}{B + \sum_{i=0}^{k} \beta_i x_{n-i}} < \frac{\sum_{i=0}^{k} \alpha_i x_{n-i}}{\sum_{i=0}^{k} \beta_i x_{n-i}} < \left(\max_{i=0,.....,k} \alpha_i \right) \left(\min_{i=0,.....,k} \beta_i \right)^{-1}. \tag{12}$$

From (12) it follows that the sequence $\{x_n\}$ is bounded from above by a positive constant M say. That is,

$$x_{n+1} < M, \quad n = 0,1,2,.... \tag{13}$$

On the other hand, we deduce from Eq. (1) and the inequality (13) that

$$\frac{\sum_{i=0}^{k} \alpha_i x_{n-i}}{B + \tilde{b} M} < \frac{\sum_{i=0}^{k} \alpha_i x_{n-i}}{B + \sum_{i=0}^{k} \beta_i x_{n-i}} = x_{n+1}$$

$$x_{n+1} \geq \frac{\tilde{a}}{B + \tilde{b} M} \min \{x_n, ..., x_{n-1}, x_{n-k}\}. \tag{14}$$

From (14) it follows that the sequence $\{x_n\}$ is bounded from below by a positive constant say, m. That is,

$$m < x_n, \quad n = 0,1,2,.... \tag{15}$$

From (13) and (15) we have

$$m < x_n < M, \quad n = 0,1,2,....$$

This shows that the sequence $\{x_n\}$ is bounded. Thus, the proof of Theorem 3 is completed.

Theorem 4. Let $\tilde{a} < B$, then all the positive solutions of Eq.(1) converges to zero.
Proof. If $\tilde{a} < B$, then from Eq.(1) we have

$$x_{n+1} < \frac{\sum_{i=0}^{k} \alpha_i x_{n-i}}{B} \leq \frac{\tilde{a}}{B} \max \{x_n, ..., x_{n-1}, x_{n-k}\}.$$

From which it follows that all positive solutions converge to zero. Thus, the proof of Theorem 4 is completed.

Theorem 5. Assume that $\tilde{a} < B$ holds. Then the equilibrium point $\tilde{x}$ of the difference equation (1) is globally asymptotically stable.

Proof: The linearized equation (5) with the equation (6) can be written in the form

$$y_{n+1} + \sum_{j=0}^{k} \left(\frac{\beta_j \tilde{x} - \alpha_j}{B + \tilde{b}\tilde{x}} \right) y_{n-j} = 0,$$

and its characteristic equation is

$$\lambda^{n+1} + \sum_{j=0}^{k} \left(\frac{\beta_j \tilde{x} - \alpha_j}{B + \tilde{b}\tilde{x}} \right) \lambda^{n-j} = 0.$$

As $\tilde{a} < B$, we get

$$\sum_{j=0}^{k}\left|\frac{\beta_j\tilde{x}-\alpha_j}{B+\tilde{b}\tilde{x}}\right| \le \frac{\left(\tilde{a}+\tilde{b}\tilde{x}\right)}{\left(B+\tilde{b}\tilde{x}\right)}<1.$$

Thus by Theorems 1,2 and Remark 1, we deduce that the equilibrium point $\tilde{x}$ of the difference equation (1) is locally asymptotically stable. It remains to be proven whether the equilibrium point $\tilde{x}$ is a global attractor. To this end, set $I=\lim_{n\to\infty}\inf x_n$ and $S=\lim_{n\to\infty}\sup x_n$, which by Theorem 3 exist and are positive numbers. Then, from the difference equation (1) we see that

$$S \le \frac{\tilde{a}S}{B+\tilde{b}I} \text{ and } I \ge \frac{\tilde{a}I}{B+\tilde{b}S}.$$

Hence

$$\left(\tilde{a}-B\right)I \le \tilde{b}IS \le \left(\tilde{a}-B\right)S.$$

From which it follows that $I \ge S$. Thus, we have $I=S$. Thus, the proof of Theorem 5 is completed.

Theorem 6. A necessary and sufficient condition for the difference equation (1) to have a positive prime period two solution is that the inequality

$$\left(\bar{a}+\tilde{a}\right)\left(\bar{b}-\tilde{b}\right)<\bar{b}\left(B+\bar{a}\right), \tag{16}$$

is valid, provided $\left(B+\bar{a}\right)<0$ and $0<\bar{b}$.

Proof: First, suppose that there exists positive prime period two solution ...., P,Q,P,Q,......... of the difference equation (1). We shall prove that the condition (16) holds. It follows from the difference equation (1) that if k is even, then $x_n = x_{n-k}$, and we have

$$P=\frac{\alpha_0Q+\alpha_1P+\alpha_2Q+\alpha_3P+\ldots+\alpha_kQ}{B+\beta_0Q+\beta_1P+\beta_2Q+\beta_3P+\ldots+\beta_kQ},$$

and

$$Q=\frac{\alpha_0P+\alpha_1Q+\alpha_2P+\alpha_3Q+\ldots+\alpha_kP}{B+\beta_0P+\beta_1Q+\beta_2P+\beta_3Q+\ldots+\beta_kP}.$$

While if k is odd, then $x_{n+1}=x_{n-k}$, and we have

$$P=\frac{\alpha_0Q+\alpha_1P+\alpha_2Q+\ldots+\alpha_kP}{B+\beta_0Q+\beta_1P+\beta_2Q+\beta_3P+\ldots+\beta_kP},$$

and

$$Q=\frac{\alpha_0P+\alpha_1Q+\alpha_2P+\ldots+\alpha_kQ}{B+\beta_0P+\beta_1Q+\beta_2P+\ldots+\beta_kQ}.$$

Now, we discuss the case when k is even (and in a similar way we can discuss the case when k is odd, which is omitted here). Consequently, we obtain

$$\begin{aligned}&\alpha_0Q+\alpha_1P+\alpha_2Q+\alpha_3P+\ldots+\alpha_kQ\\&=BP+\beta_0PQ+\beta_1P^2+\beta_2PQ+\beta_3P^2+\ldots+\beta_kPQ,\end{aligned} \tag{17}$$

and

$$\begin{aligned}&\alpha_0P+\alpha_1Q+\alpha_2P+\alpha_3Q+\ldots+\alpha_kP\\&=BQ+\beta_0PQ+\beta_1Q^2+\beta_2PQ+\beta_3Q^2+\ldots+\beta_kPQ.\end{aligned} \tag{18}$$

By subtracting, we deduce after some reduction that

$$P+Q=\frac{-\left(B+\bar{a}\right)}{\beta_1+\beta_3+\ldots+\beta_{k-1}}, \tag{19}$$

while, by adding we obtain

$$PQ=\frac{-\left(\alpha_0+\alpha_2+\ldots+\alpha_k\right)\left(B+\bar{a}\right)}{\bar{b}\left(\beta_1+\beta_3+\ldots+\beta_{k-1}\right)}, \tag{20}$$

where $\left(B+\bar{a}\right)<0$ and $0<\bar{b}$. Assume that P and Q are two positive distinct real roots of the quadratic equation

$$t^2-\left(P+Q\right)t+PQ=0. \tag{21}$$

Thus, we deduce that

$$\begin{aligned}&4\left(\frac{-\left(\alpha_0+\alpha_2+\ldots+\alpha_k\right)\left(B+\bar{a}\right)}{\bar{b}\left(\beta_1+\beta_3+\ldots+\beta_{k-1}\right)}\right)\\&<\left(\frac{-\left(B+\bar{a}\right)}{\beta_1+\beta_3+\ldots+\beta_{k-1}}\right)^2.\end{aligned} \tag{22}$$

From (22), we obtain $\left(\bar{a}+\tilde{a}\right)\left(\bar{b}-\tilde{b}\right)<\bar{b}\left(B+\bar{a}\right)$, and hence the condition (16) is valid. Conversely, suppose that the condition (16) is valid. Then, we deduce immediately from (16) that the inequality (22) holds. Consequently, there exists two positive distinct real numbers P and Q such that

$$P=\frac{-\left(B+\bar{a}\right)}{2\left(\beta_1+\beta_3+\ldots+\beta_{k-1}\right)}-\frac{1}{2}\sqrt{T_1}, \tag{23}$$

and

$$P=\frac{-(B+\bar{a})}{2(\beta_1+\beta_3+...+\beta_{k-1})}+\frac{1}{2}\sqrt{T_1}, \qquad (24)$$

where $0<T_1$ which is given by the formula

$$T_1=\left(\frac{-(B+\bar{a})}{\beta_1+\beta_3+...+\beta_{k-1}}\right)^2+\frac{4(\alpha_0+\alpha_2+...+\alpha_k)(B+\bar{a})}{\bar{b}(\beta_1+\beta_3+...+\beta_{k-1})}. \qquad (25)$$

Thus, P and Q represent two positive distinct real roots of the quadratic equation (21). Now, we are going to prove that P and Q are positive prime period two solutions of the difference equation (1). To this end, we assume that

$$x_{-k}=P, \qquad x_{-k+1}=Q,......., \qquad x_{-1}=Q, \quad \text{and} \quad x_0=P.$$

We wish to show that

$$x_1=Q, \quad \text{and} \quad x_2=P.$$

To this end, we deduce from the difference equation (1) that

$$x_1=\frac{\alpha_0x_0+\alpha_1x_{-1}+...+\alpha_kx_{-k}}{B+\beta_0x_0+\beta_1x_{-1}+...+\beta_kx_{-k}}=\frac{P(\alpha_0+\alpha_2+...+\alpha_k)+Q(\alpha_1+\alpha_3+...+\alpha_{k-1})}{B+P(\beta_0+\beta_2+...+\beta_k)+Q(\beta_1+\beta_3+...+\beta_{k-1})}. \qquad (26)$$

Dividing the denominator and numerator of (26) by $\frac{-(B+\bar{a})}{(\beta_1+\beta_3+...+\beta_{k-1})}$ and using (23)-(25) we obtain

$$x_{n+1}=\frac{\left[1+\sqrt{K_1}\right](\alpha_0+\alpha_2+...+\alpha_k)}{K_2+\left[1+\sqrt{K_1}\right](\beta_0+\beta_2+...+\beta_k)+\left[1-\sqrt{K_1}\right](\beta_1+\beta_3+...+\beta_{k-1})}+\frac{\left[1-\sqrt{K_1}\right](\alpha_1+\alpha_3+...+\alpha_{k-1})}{K_2+\left[1+\sqrt{K_1}\right](\beta_0+\beta_2+...+\beta_k)+\left[1-\sqrt{K_1}\right](\beta_1+\beta_3+...+\beta_{k-1})}$$

$$x_{n+1}=\frac{\left[(\alpha_0+\alpha_2+...+\alpha_k)+(\alpha_1+\alpha_3+...+\alpha_{k-1})\right]}{K_2+\left[1+\sqrt{K_1}\right](\beta_0+\beta_2+...+\beta_k)+\left[1-\sqrt{K_1}\right](\beta_1+\beta_3+...+\beta_{k-1})}+\frac{\left[(\alpha_0+\alpha_2+...+\alpha_k)-(\alpha_1+\alpha_3+...+\alpha_{k-1})\right]\sqrt{K_1}}{K_2+\left[1+\sqrt{K_1}\right](\beta_0+\beta_2+...+\beta_k)+\left[1-\sqrt{K_1}\right](\beta_1+\beta_3+...+\beta_{k-1})}$$

$$x_{n+1}=\frac{(\tilde{a}+\bar{a})\sqrt{K_1}}{\left[\tilde{b}-\frac{2B(\beta_1+\beta_3+...+\beta_{k-1})}{(B+\bar{a})}\right]+\bar{b}\sqrt{K_1}}, \qquad (27)$$

where

$$K_1=1-\left[\frac{(\bar{a}+\tilde{a})(\bar{b}-\tilde{b})}{\bar{b}(B+\bar{a})}\right],$$

$$K_2=\frac{-2B(\beta_1+\beta_3+...+\beta_{k-1})}{(B+\bar{a})}, \qquad (28)$$

and from the condition (16) we deduce that $0<K_1$. Multiplying the denominator and numerator of (27) by

$$\left(\tilde{b}-\frac{2B(\beta_1+\beta_3+...+\beta_{k-1})}{(B+\bar{a})}\right)-\bar{b}\sqrt{K_1},$$

then, we have

$$x_1=\frac{\tilde{a}\left[\tilde{b}-\frac{2B(\beta_1+\beta_3+...+\beta_{k-1})}{(B+\bar{a})}\right]-\bar{a}\bar{b}K_1}{\left[\tilde{b}-\frac{2B(\beta_1+\beta_3+...+\beta_{k-1})}{(B+\bar{a})}\right]^2-\bar{b}^2K_1}+\frac{\left[\tilde{b}\bar{a}-\tilde{a}\bar{b}-\bar{a}\frac{2B(\beta_1+\beta_3+...+\beta_{k-1})}{(B+\bar{a})}\right]\sqrt{K_1}}{\left[\tilde{b}-\frac{2B(\beta_1+\beta_3+...+\beta_{k-1})}{(B+\bar{a})}\right]^2-\bar{b}^2K_1}.$$

After some reduction, we deduce that

$$x_1=\frac{\frac{-(B+\bar{a})}{2(\beta_1+\beta_3+...+\beta_{k-1})}\left(1+\sqrt{K_1}\right)T_2}{T_2}=\frac{-(B+\bar{a})\left(1+\sqrt{K_1}\right)}{2(\beta_1+\beta_3+...+\beta_{k-1})}=\frac{-(B+\bar{a})}{2(\beta_1+\beta_3+...+\beta_{k-1})}+\frac{1}{2}\sqrt{T_1}=Q.$$

where

$$T_2=2(\alpha_1+...+\alpha_{k-1})(\beta_0+...+\beta_k)-2(\alpha_0+...+\alpha)(\beta_1+...+\beta_{k-1})+\frac{2B\bar{a}(\beta_1+\beta_3+...+\beta_{k-1})}{(B+\bar{a})}$$

Similarly, we can show that

$$x_2=\frac{\alpha_0x_1+\alpha_1x_0+...+\alpha_kx_{-(k-1)}}{B+\beta_0x_1+\beta_1x_0+...+\beta_kx_{-(k-1)}}=\frac{Q(\alpha_0+\alpha_2+...+\alpha_k)+P(\alpha_1+\alpha_3+...+\alpha_{k-1})}{B+Q(\beta_0+\beta_2+...+\beta_k)+P(\beta_1+\beta_3+...+\beta_{k-1})}=P.$$

By using the mathematical induction, we have $x_n=P$ and $x_{n+1}=Q$ for all $n\geq -k$.

Thus the difference equation (1) has positive prime period two solutions $\ldots\ldots\ldots, P, Q, P, Q, \ldots\ldots\ldots$ Hence the proof of Theorem 6 is completed.

3. Numerical examples of the solutions of Eq.(1)

In order to illustrate the results of the previous subsections and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to nonlinear difference equation(1).

Example 1. Fig. 1, shows that Eq.(1) has prime period two solution if

$k = 1,\ x_{-1} = 0.297,\ x_0 = 5.583,\ B = 0.3,\ \alpha_0 = 2,$

$\alpha_1 = 20,\ \beta_0 = 10,\ \beta_1 = 3.$

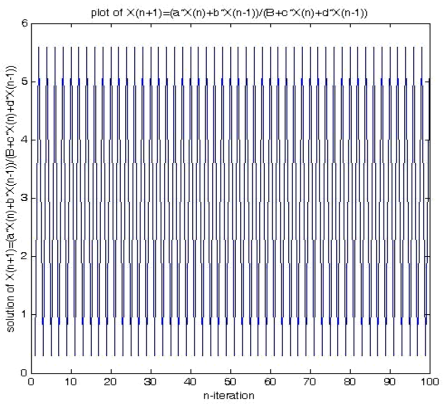

Fig. 1.

Example 2. Fig. 2, shows that the solution of Eq.(1) has global stability if

$k = 1,\ x_{-1} = 1,\ x_0 = 2,\ B = 10,\ \alpha_0 = 2,$

$\alpha_1 = 0.5,\ \beta_0 = 3,\ \beta_1 = 0.25.$

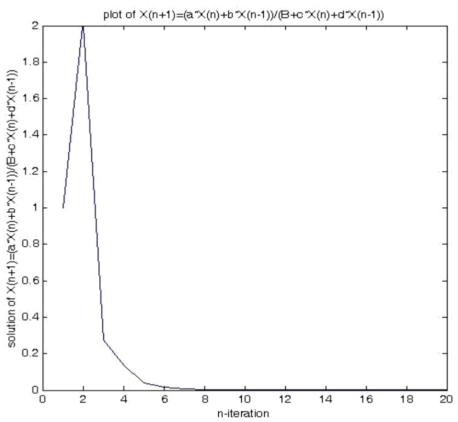

Fig. 2.

Note that example 1 verifies Theorem 6 which show that Eq.(1) has prime period two solution. But example 2 verifies Theorem 5, which shows that the solution of Eq.(1) has globally asymptotic stable.

4. Conclusion

In this paper, we have studied the boundedness character, the periodicity character, as well as the global attractivity of the positive solution of the rational recursive sequence (1) which play an important role in economy and biology.

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