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A class of fourth order differential operators with transmission conditions

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Abstract

We investigate a class of fourth-order differential operators with eigenparameter dependent boundary conditions and transmission conditions. A self-adjoint linear operator A is defined in a suitable Hilbert space H such that the eigenvalues of such a problem coincide with those of A. We discuss asymptotic behavior of its eigenvalues and completeness of its eigenfunctions. Finally, we obtain the representation of its Green function.

Keywords: Differential operator; eigenvalues; eigenfunctions; Green function; completeness

1. Introduction

In recent years, more and more researchers have become interested in the discontinuous differential operator problem for its application in physics ([1, 2]). People have paid close attention to Sturm-Liouville problem of the boundary condition depending on eigenvalue parameter and studied its inverse problem, asymptotic of eigenvalues and eigenfunctions, shaking theory and so on. The various physics applications of this kind of problem are found in many literature, for example Binding ([3-5]), Hinton ([6-8]), Fulton ([9, 10]) etc., including some boundary value problems with transmission conditions that arise in the theory of heat and mass transfer (see [11-13]).

Here we consider a class of fourth-order differential operators with eigenparameter dependent boundary conditions and transmission conditions. By using the techniques of [11, 13] and some new approaches, a new linear operator A associated with the problem in an appropriate Hilbert space H is defined such that the eigenvalues of the problem coincide with those of A. Its eigenvalues and eigenfunctions are discussed, its asymptotic approximation formulas are obtained for eigenvalues, proving that the eigenfunctions of A are complete in H and its Green function is constructed, and the previous conclusions are promoted and deepened.

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In this study, we shall investigate a discontinuous eigenvalue problem which consists of differential equation

$$lu := (a(x)u''(x))'' + q(x)u(x) = \lambda u(x),\ x \in J, \quad (1)$$

where $J = [-1,0) \cup (0,1]$, $a(x) = a_1^{\,2}$ for $x \in [-1,0)$, $a(x) = a_2^{\,2}$, for $x \in (0,1], a_1 > 0$ and $a_2 > 0$ are given real numbers; the real value function $q(x) \in L^1[J, \mathrm{R}], \lambda \in \mathrm{C}$ is a complex eigenparameter; with the boundary conditions

$$l_1 u := \alpha_1 u(-1) + \alpha_2 u'''(-1) = 0, \quad (2)$$

$$l_2 u := \beta_1 u'(-1) + \beta_2 u''(-1) = 0, \quad (3)$$

the eigenparameter-dependent boundary conditions

$$l_3 u := \lambda(\gamma_1' u(1) - \gamma_2' u'''(1)) - (\gamma_1 u(1) - \gamma_2 u'''(1)) = 0, \quad (4)$$

$$l_4 u := \lambda(\tau_1' u'(1) - \tau_2' u''(1)) + (\tau_1 u'(1) - \tau_2 u''(1)) = 0, \quad (5)$$

and transmission conditions at the point of discontinuity

$$l_5 u := u(0+) - \alpha_3 u(0-) - \alpha_4 u'''(0-) = 0, \quad (6)$$

$$l_6 u := u'(0+) - \beta_3 u'(0-) - \beta_4 u''(0-) = 0, \quad (7)$$

$l_7 u := u''(0+) - \gamma_3 u'(0-) - \gamma_4 u''(0-) = 0,$ (8)

$l_8 u := u'''(0+) - \tau_3 u(0-) - \tau_4 u'''(0-) = 0,$ (9)

where $\alpha_i, \beta_i, \gamma_i, \tau_i\ \gamma'_j, \tau'_j\ (i = \overline{1,4},\ j = 1,2)$ are real numbers. We assume that

$$\theta = \begin{vmatrix} \alpha_3 & \alpha_4 \\ \tau_3 & \tau_4 \end{vmatrix} = \begin{vmatrix} \beta_3 & \beta_4 \\ \gamma_3 & \gamma_4 \end{vmatrix} > 0,$$

$$\rho_1 = \begin{vmatrix} \gamma'_1 & \gamma_1 \\ \gamma'_2 & \gamma_2 \end{vmatrix} > 0,\ \rho_2 = \begin{vmatrix} \tau'_1 & \tau_1 \\ \tau'_2 & \tau_2 \end{vmatrix} > 0,$$

and $\alpha_1^2 + \alpha_2^2 \neq 0$, $\beta_1^2 + \beta_2^2 \neq 0$.

In order to consider problem (1)-(9), we define the inner product in $L^2(J)$ as

$$\langle f, g\rangle_1 = \frac{\theta}{a_1^2}\int_{-1}^{0} f_1 \bar{g}_1 dx + \frac{1}{a_2^2}\int_0^1 f_2 \bar{g}_2 dx, \forall f, g \in L^2(J),$$

where $f_1(x) = f(x)\big|_{[-1,0)}$, $f_2(x) = f(x)\big|_{(0,1]}$. It is easy to verify that $H_1 = L^2(J, \langle \bullet, \bullet\rangle_1)$ is a Hilbert space.

2. Operator formulation

In this section, we introduce the special inner product in the Hilbert space $H := H_1 \oplus \mathbb{C} \oplus \mathbb{C}$ where $H_1 = L^2(J)$, $\mathbb{C}$ denotes the Hilbert space of complex numbers and a symmetric linear operator A defined in this Hilbert space such that (1)-(9) can be considered as the eigenvalue problem of this operator. Namely, we define an inner product in H by

$$\langle F, G\rangle = \langle f, g\rangle_1 + \frac{1}{\rho_1}\langle h, k\rangle + \frac{1}{\rho_2}\langle r, s\rangle,$$
$$f, g \in H_1,\quad h, k, r, s \in \mathbb{C},$$

for
$F := (f, h, r)$, $G := (g, k, s) \in H$.

In the Hilbert space H consider the operator A which is defined by

$$D(A) = \{(f(x), h, r) \in H \big| f_1, f'_1, f''_1, f'''_1 \in AC_{loc}((-1,0)),$$
$$f_2, f'_2, f''_2, f'''_2 \in AC_{loc}((0,1)), lf \in H_1,$$

$$l_i f = 0, i = 1,2,\overline{5,8}, h = \gamma'_1 f(1) - \gamma'_2 f'''(1), r = \tau'_1 f'(1) - \tau'_2 f''(1)\},$$
$$AF = (lf, \gamma_1 f(1) - \gamma_2 f'''(1), -(\tau_1 f'(1) - \tau_2 f''(1)),$$
$$F = (f, \gamma'_1 f(1) - \gamma'_2 f'''(1), \tau'_1 f'(1) - \tau'_2 f''(1)) \in D(A).$$

For convenience, $\forall (f, h, r) \in D(A)$, let

$$N(f) = \gamma_1 f(1) - \gamma_2 f'''(1),$$
$$N'(f) = \tau_1 f'(1) - \tau_2 f''(1),$$
$$N'''(f) = \gamma'_1 f(1) - \gamma'_2 f'''(1),$$
$$N''(f) = \tau'_1 f'(1) - \tau'_2 f''(1).$$

Now we can rewrite the considered problem (1)-(9) in the operator form $AF = \lambda F$.

Obviously, the eigenvalues and eigenfunctions of the problem (1)-(9) are defined as the eigenvalues and the first components of the corresponding eigenelements of the operator A respectively.

Lemma 2.1. The domain $D(A)$ is dense in H.

Proof: Let $F = (f, h, r) \in H$, $F \perp D(A)$ and $\widetilde{C}_0^\infty$ be a functional set such that

$$\phi(x) = \begin{cases} \varphi_1(x), x \in [-1,0), \\ \varphi_2(x), x \in (0,1], \end{cases}$$

where $\varphi_1(x) \in C_0^\infty[-1,0)$ and $\varphi_2(x) \in C_0^\infty(0,1]$. Since $\widetilde{C}_0^\infty \oplus 0 \oplus 0 \subset D(A))$ $(0 \in \mathbb{C})$, and $U = (u(x), 0, 0) \in \widetilde{C}_0^\infty \oplus 0 \oplus 0$ is orthogonal to F, namely

$$\langle F, U\rangle = \frac{\theta}{a_1^2}\int_{-1}^{0} f(x)\bar{u}(x)dx + \frac{1}{a_2^2}\int_0^1 f(x)\bar{u}(x)dx = \langle f, u\rangle_1.$$

We can learn $f(x)$ is orthogonal to $\widetilde{C}_0^\infty$ in H_1, this implies $f(x) = 0$. So for all $G_1 = (g(x), k, 0) \in D(A)$, $\langle F, G_1\rangle = \frac{1}{\rho_1}\langle h, k\rangle = \frac{1}{\rho_1} h\bar{k} = 0$. Thus $h = 0$ since k can be chosen arbitrarily. Similarly $r = 0$. So $F = (0,0,0)$, which proves the assertion.

Theorem 2.2. The operator A is self-adjoint in H.

Proof: Let $F, G \in D(A)$. By two partial integrations we obtain

$$\langle AF, G\rangle = \langle F, AG\rangle + \theta[f\bar{g}]_{-1}^{0-} + [f\bar{g}]_{0+}^{1} - \frac{1}{\rho_1}(N(f)\overline{N'''(g)}$$
$$- N'''(f)\overline{N(g)} - \frac{1}{\rho_2}(N''(f)\overline{N'(g)} - N'(f)\overline{N''(g)},$$

where

$$[f\bar{g}](x) = f'''(x)\overline{g(x)} - f(x)\overline{g'''(x)} + f'(x)\overline{g''(x)} - f''(x)\overline{g'(x)} \cdot$$

Since f and g satisfy the boundary conditions (2) and (3), it follows that $[f\bar{g}](-1)=0$. From the transmission conditions (6)-(9), we get

$$[f\bar{g}](0+) = \theta[f\bar{g}](0-).$$

Further, it is easy to verify that

$$[f\bar{g}](1) = \frac{1}{\rho_1}(N(f)\overline{N'''(g)} - N'''(f)\overline{N(g)} + \frac{1}{\rho_2}(N''(f)\overline{N'(g)} - N'(f)\overline{N''(g)}.$$

Then we have $\langle AF, G\rangle = \langle F, AG\rangle$, so A is symmetric.

It remains to show that if $\langle AF, W\rangle = \langle F, U\rangle$ for all $F = (f, N'''(f), N''(f)) \in D(A)$, then $W \in D(A)$ and $AW = U$, where $W = (w(x), h, r)$, $U = (u(x), k, s)$, i.e., (i) w_1, w_1', w_1'', $w_1''' \in AC_{loc}((-1,0))$, $w_2, w_2', w_2'', w_2''' \in AC_{loc}((0,1))$, $lw \in H_1$; (ii) $h = \gamma_1' w(1) - \gamma_2' w'''(1)$, $r = \tau_1' w'(1) - \tau_2' w''(1)$; (iii) $l_i w = 0, i = 1,2,\overline{5,8}$; (iv) $u(x) = lw$; (v) $k = \gamma_1 w(1) - \gamma_2 w'''(1)$, $s = -(\tau_1 w'(1) - \tau_2 w''(1))$.

For an arbitrary point $F \in \widetilde{C}_0^\infty \oplus 0 \oplus 0 \subset D(A)$ such that

$$\frac{\theta}{a_1^2}\int_{-1}^0 (lf)\bar{w}dx + \frac{1}{a_2^2}\int_0^1 (lf)\bar{w}dx = \frac{\theta}{a_1^2}\int_{-1}^0 f\bar{u}dx + \frac{1}{a_2^2}\int_0^1 f\bar{u}dx,$$

that is $\langle lf, w\rangle_1 = \langle f, u\rangle_1$. According to normal Sturm-Liouville theory, (i) and (iv) hold. By (iv), the equation $\langle AF, W\rangle = \langle F, U\rangle, \forall F \in D(A)$, becomes

$$\langle lf, w\rangle_1 = \langle f, lw\rangle_1 - \frac{N(f)\bar{h} - N'''(f)\bar{k}}{\rho_1} + \frac{N'(f)\bar{r} - N''(f)\bar{s}}{\rho_2}.$$

However,

$$\langle lf, w\rangle_1 = \langle f, lw\rangle_1 + \theta[f\bar{w}]_{-1}^{0-} + [f\bar{w}]_{0+}^1.$$

So

$$-\frac{N(f)\bar{h}}{\rho_1} + \frac{N'''(f)\bar{k}}{\rho_1} + \frac{N'(f)\bar{r}}{\rho_2} + \frac{N''(f)\bar{s}}{\rho_2} = \theta[f\bar{w}](0-) - \theta[f\bar{w}](-1) + [f\bar{w}](1) - [f\bar{w}](0+). \quad (10)$$

By Naimark's Patching Lemma [14], there is an $F \in D(A)$ such that

$f^{(i)}(-1) = f^{(i)}(0-) = f^{(i)}(0+) = 0, (i = 0,1,2,3)$,
$f(1) = \gamma_2', f'(1) = f''(1) = 0, f'''(1) = \gamma_1'$.

Thus $N'(f) = N''(f) = N'''(f) = 0$. Then from (10) the equality $h = \gamma_1' w(1) - \gamma_2' w'''(1)$. Further, $\exists F \in D(A)$, such that

$f^{(i)}(-1) = f^{(i)}(0-) = f^{(i)}(0+) = 0, (i = 0,1,2,3)$,
$f'(1) = \tau_2', f''(1) = \tau_1', f(1) = f'''(1) = 0$.

Thus $N(f) = N'(f) = N'''(f) = 0$. Then from (10) the quality $r = \tau_1' w'(1) - \tau_2' w''(1)$. So (ii) holds.Similarly one proves (v). Next, choose $F \in D(A)$ so that

$$f^{(i)}(1) = f^{(i)}(0-) = f^{(i)}(0+) = 0, (i = 0,1,2,3),$$
$$f(-1) = \alpha_2, f'(-1) = f''(-1) = 0, f'''(-1) = -\alpha_1.$$

Then
$N(f) = N'(f) = N''(f) = N'''(f) = 0$. So from (10), we get $\alpha_1 w(-1) + \alpha_2 w'''(-1) = 0$. Similarly $\beta_1 w'(-1) + \beta_2 w''(-1) = 0$. Let $F \in D(A)$ satisfy

$f^{(i)}(-1) = f^{(i)}(1) = f'(0-) = f''(0-) = f(0+) = f'(0+) = f''(0+) = 0$,
$(i = 0,1,2,3)$, $f(0-) = -\alpha_3$, $f'''(0-) = \alpha_4$, $f'''(0+) = \theta$,

then $N(f) = N'(f) = N''(f) = N'''(f) = 0$. From (10), we have $w(0+) = \alpha_3 w(0-) + \alpha_4 w'''(0-) = 0$. Using the same method one proves $l_6 w = l_7 w = l_8 w = 0$.

Corollary 2.3. All eigenvalues of the problem (1)-(9) are real, and if λ_1, λ_2 be two different eigenvalues, then the corresponding eigenfunctions $f(x)$ and $g(x)$ of this problem are orthogonal in the sense of

$$\frac{\theta}{a_1^2}\int_{-1}^0 f\bar{g} + \frac{1}{a_2^2}\int_0^1 f\bar{g} + \frac{1}{\rho_1}(\gamma_1' f(1) - \gamma_2' f'''(1))(\gamma_1'\bar{g}(1) - \gamma_2'\bar{g}'''(1))$$
$$+ \frac{1}{\rho_2}(\tau_1' f'(1) - \tau_2' f''(1))(\tau_1'\bar{g}'(1) - \tau_2''\bar{g}''(1)) = 0 \cdot$$

3. Asymptotic formula for eigenvalues

Lemma 3.1. Let the real valued function $q(x) \in C[-1,1]$ and $f_i(\lambda)$ ($i = \overline{1,4}$) be given entire functions. Then for $\forall \lambda \in \mathrm{C}$, the equation

(1) has a unique solution $u(x,\lambda)$, satisfying the initial conditions
$u(-1)= f_1(\lambda)$, $u'(-1)= f_2(\lambda)$, $u''(-1)= f_3(\lambda)$, $u'''(-1)= f_4(\lambda)$,
(or $u(1)= f_1(\lambda)$, $u'(1)= f_2(\lambda)$, $u''(1)= f_3(\lambda)$, $u'''(-1)= f_4(\lambda)$).

Proof: In terms of existence and uniqueness in ordinary differential equation theory, we can conclude this conclusion.

Let $\varphi_{11}(x,\lambda)$ be the solution of equation (1) on the interval $[-1,0)$, satisfying the initial conditions

$$\varphi_{11}(-1)=\alpha_2,\ \varphi'_{11}(-1)=\varphi''_{11}(-1)=0,$$
$$\varphi'''_{11}(-1)=-\alpha_1.$$

By virtue of Lemma 3.1, after defining this solution we can define the solution $\varphi_{12}(x,\lambda)$ of equation (1) on the interval $(0,1]$ by the initial conditions

$$\varphi_{12}(0)=\alpha_3\varphi_{11}(0)+\alpha_4\varphi'''_{11}(0),$$
$$\varphi'_{12}(0)=\beta_3\varphi'_{11}(0)+\beta_4\varphi''_{11}(0),$$
$$\varphi''_{12}(0)=\gamma_3\varphi'_{11}(0)+\gamma_4\varphi''_{11}(0),$$
$$\varphi'''_{12}(0)=\tau_3\varphi_{11}(0)+\tau_4\varphi'''_{11}(0).$$

Again let $\varphi_{21}(x,\lambda)$ still be the solution of equation (1) on the interval $[-1,0)$, satisfying the initial conditions

$$\varphi_{21}(-1)=0,\ \varphi'_{21}(-1)=\beta_2,\ \varphi''_{21}(-1)=-\beta_1,$$
$$\varphi'''_{21}(-1)=0.$$

After defining this solution, we can also define the solution $\varphi_{22}(x,\lambda)$ of equation (1) on the interval $(0,1]$ by the initial conditions

$$\varphi_{22}(0)=\alpha_3\varphi_{21}(0)+\alpha_4\varphi'''_{21}(0),$$
$$\varphi'_{22}(0)=\beta_3\varphi'_{21}(0)+\beta_4\varphi''_{21}(0),$$
$$\varphi''_{22}(0)=\gamma_3\varphi'_{21}(0)+\gamma_4\varphi''_{21}(0),$$
$$\varphi'''_{22}(0)=\tau_3\varphi_{21}(0)+\tau_4\varphi'''_{21}(0).$$

Analogously we shall define the solutions $\chi_{12}(x,\lambda)$, $\chi_{11}(x,\lambda)$ by initial conditions

$$\chi_{12}(1)=\lambda\gamma'_2-\gamma_2,\qquad \chi'_{12}(1)=\chi''_{12}(1)=0,$$
$$\chi'''_{12}(1)=\lambda\gamma'_1-\gamma_1,$$
$$\chi_{11}(0)=\frac{\tau_4\chi_{12}(0)-\alpha_4\chi'''_{12}(0)}{\theta},$$
$$\chi'_{11}(0)=\frac{\gamma_4\chi'_{12}(0)-\beta_4\chi''_{12}(0)}{\theta},$$
$$\chi''_{11}(0)=\frac{\beta_3\chi''_{12}(0)-\gamma_3\chi'_{12}(0)}{\theta},$$
$$\chi'''_{11}(0)=\frac{\alpha_3\chi'''_{12}(0)-\tau_3\chi_{12}(0)}{\theta}.$$

In addition, we shall define the solutions $\chi_{22}(x,\lambda)$, $\chi_{21}(x,\lambda)$, satisfying the initial conditions

$$\chi_{22}(1)=0,\qquad \chi'_{22}(1)=\lambda\tau'_2+\tau_2,$$
$$\chi''_{22}(1)=\lambda\tau'_1+\tau_1,\ \chi'''_{22}(1)=0,$$
$$\chi_{21}(0)=\frac{\tau_4\chi_{22}(0)-\alpha_4\chi'''_{22}(0)}{\theta},$$
$$\chi'_{21}(0)=\frac{\gamma_4\chi'_{22}(0)-\beta_4\chi''_{22}(0)}{\theta},$$
$$\chi''_{21}(0)=\frac{\beta_3\chi''_{22}(0)-\gamma_3\chi'_{22}(0)}{\theta},$$
$$\chi'''_{21}(0)=\frac{\alpha_3\chi'''_{22}(0)-\tau_3\chi_{22}(0)}{\theta}.$$

Let us consider the Wronskians

$$W_1(\lambda):=\begin{vmatrix}\varphi_{11}(x,\lambda) & \varphi_{21}(x,\lambda) & \chi_{11}(x,\lambda) & \chi_{21}(x,\lambda)\\ \varphi'_{11}(x,\lambda) & \varphi'_{21}(x,\lambda) & \chi'_{11}(x,\lambda) & \chi'_{21}(x,\lambda)\\ \varphi''_{11}(x,\lambda) & \varphi''_{21}(x,\lambda) & \chi''_{11}(x,\lambda) & \chi''_{21}(x,\lambda)\\ \varphi'''_{11}(x,\lambda) & \varphi'''_{21}(x,\lambda) & \chi'''_{11}(x,\lambda) & \chi'''_{21}(x,\lambda)\end{vmatrix}$$

and

$$W_2(\lambda):=\begin{vmatrix}\phi_{12}(x,\lambda) & \phi_{22}(x,\lambda) & \chi_{12}(x,\lambda) & \chi_{22}(x,\lambda)\\ \phi'_{12}(x,\lambda) & \phi'_{22}(x,\lambda) & \chi'_{12}(x,\lambda) & \chi'_{22}(x,\lambda)\\ \phi''_{12}(x,\lambda) & \phi''_{22}(x,\lambda) & \chi''_{12}(x,\lambda) & \chi''_{22}(x,\lambda)\\ \phi'''_{12}(x,\lambda) & \phi'''_{22}(x,\lambda) & \chi'''_{12}(x,\lambda) & \chi'''_{22}(x,\lambda)\end{vmatrix},$$

which are independent of x and are entire functions. This sort of calculation gives $W_2(\lambda)=\theta^2 W_1(\lambda)$. Now we may introduce, in consideration, the characteristic function as $W(\lambda)=W_1(\lambda)$.

Theorem 3.2. The eigenvalues of the problem (1)-(9) consist of the zeros of the function $W(\lambda)$.

Proof: Let $u(x)$ be any eigenfunction corresponding to eigenvalue λ_0. Then the function $u(x)$ may be represented in the form

$$u(x)=\begin{cases} c_1\phi_{11}(x,\lambda_0)+c_2\phi_{21}(x,\lambda_0) \\ +c_3\chi_{11}(x,\lambda_0)+c_4\chi_{21}(x,\lambda_0), & x\in[-1,0); \\ c_5\phi_{12}(x,\lambda_0)+c_6\phi_{22}(x,\lambda_0) \\ +c_7\chi_{12}(x,\lambda_0)+c_8\chi_{22}(x,\lambda_0), & x\in(0,1], \end{cases} \quad (11)$$

where at least one of the constants $c_i\,(i=\overline{1,8})$ is not zero.

Consider the true function

$$l_v(u(x))=0, v=\overline{1,8} \quad (12)$$

as the homogenous system of linear equations in the variables $c_i\,(i=\overline{1,8})$.

$$l_1 u = c_3(\alpha_1\chi_{11}(-1)+\alpha_2\chi_{11}'''(-1))+c_4(\alpha_1\chi_{21}(-1)+\alpha_2\chi_{21}'''(-1))=0,$$

$$l_2 u = c_3(\beta_1\chi_{11}'(-1)+\beta_2\chi_{11}''(-1))+c_4(\beta_1\chi_{21}'(-1))+\beta_2\chi_{21}''(-1))=0,$$

$$l_3 u = c_5((\lambda\gamma_1'+\gamma_1)\phi_{12}(1)-(\lambda\gamma_2'+\gamma_2)\phi_{12}'''(1))$$
$$+c_6((\lambda\gamma_1'+\gamma_1)\phi_{22}(1)-(\lambda\gamma_2'+\gamma_2)\phi_{22}'''(1))=0,$$

$$l_4 u = c_5((\lambda\tau_1'+\tau_1)\phi_{12}'(1)-(\lambda\tau_2'+\tau_2)\phi_{12}''(1))$$
$$+c_6((\lambda\tau_1'+\tau_1)\phi_{22}'(1)-(\lambda\tau_2'+\tau_2)\phi_{22}''(1))=0,$$

$$l_5 u = -c_1\phi_{12}(0)-c_2\phi_{22}(0)-c_3\chi_{12}(0)-c_4\chi_{22}(0)$$
$$+c_5\phi_{12}(0)+c_6\phi_{22}(0)+c_7\chi_{12}(0)+c_8\chi_{22}(0)=0,$$

$$l_6 u = -c_1\phi_{12}'(0)-c_2\phi_{22}'(0)-c_3\chi_{12}'(0)-c_4\chi_{22}'(0)$$
$$+c_5\phi_{12}'(0)+c_6\phi_{22}'(0)+c_7\chi_{12}'(0)+c_8\chi_{22}'(0)=0,$$

$$l_7 u = -c_1\phi_{12}''(0)-c_2\phi_{22}''(0)-c_3\chi_{12}''(0)-c_4\chi_{22}''(0)$$
$$+c_5\phi_{12}''(0)+c_6\phi_{22}''(0)+c_7\chi_{12}''(0)+c_8\chi_{22}''(0)=0,$$

$$l_8 u = -c_1\phi_{12}'''(0)-c_2\phi_{22}'''(0)-c_3\chi_{12}'''(0)-c_4\chi_{22}'''(0)$$
$$+c_5\phi_{12}'''(0)+c_6\phi_{22}'''(0)+c_7\chi_{12}'''(0)+c_8\chi_{22}'''(0)=0.$$

So it follows that the determinant of the system (12) is

$$\begin{vmatrix} 0 & 0 & l_1\chi_{11} & l_1\chi_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & l_2\chi_{11} & l_2\chi_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_3\phi_{12} & l_3\phi_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & l_4\phi_{12} & l_4\phi_{22} & 0 & 0 \\ -\phi_{12}(0) & -\phi_{22}(0) & -\chi_{12}(0) & -\chi_{22}(0) & \phi_{12}(0) & \phi_{22}(0) & \chi_{12}(0) & \chi_{22}(0) \\ -\phi_{12}'(0) & -\phi_{22}'(0) & -\chi_{12}'(0) & -\chi_{22}'(0) & \phi_{12}'(0) & \phi_{22}'(0) & \chi_{12}'(0) & \chi_{22}'(0) \\ -\phi_{12}''(0) & -\phi_{22}''(0) & -\chi_{12}''(0) & -\chi_{22}''(0) & \phi_{12}''(0) & \phi_{22}''(0) & \chi_{12}''(0) & \chi_{22}''(0) \\ -\phi_{12}'''(0) & -\phi_{22}'''(0) & -\chi_{12}'''(0) & -\phi_{22}'''(0) & \phi_{12}'''(0) & \phi_{22}'''(0) & \chi_{12}'''(0) & \chi_{22}'''(0) \end{vmatrix}.$$

In the following, we will get the asymptotic approximation formula of its eigenvalues. Here for simplicity, we let $a_1=a_2=1$.

Lemma 3.4. Let $\lambda=s^4$, $s=\sigma+it$. Then

$$\frac{d^k}{dx^k}\phi_{11}(x,\lambda)=\frac{\alpha_2}{2}\frac{d^k}{dx^k}\cos s(x+1)$$
$$+\frac{\alpha_2}{4}\frac{d^k}{dx}(e^{s(x+1)}+e^{-s(x+1)})+O(|s|^{k-1}e^{|s|(x+1)}), \quad (17)$$

$$\frac{d^k}{dx^k}\phi_{12}(x,\lambda)=\frac{\alpha_4\phi_{11}'''(0)}{2}\frac{d^k}{dx^k}\cos sx+(\frac{\alpha_4\phi_{11}'''(0)}{4}$$
$$+\frac{\beta_4\phi_{11}''(0)}{4s})\frac{d^k}{dx}(e^{sx}+e^{-sx})+O(|s|^{k+2}e^{|s|(x+1)}), \quad (18)$$

$$\frac{d^k}{dx^k}\phi_{21}(x,\lambda)=\frac{\beta_2}{2s}\frac{d^k}{dx^k}\sin s(x+1)$$
$$+\frac{\beta_2}{4s}\frac{d^k}{dx}(e^{s(x+1)}-e^{-s(x+1)})+O(|s|^{k-2}e^{|s|(x+1)}), \quad (19)$$

$$\frac{d^k}{dx^k}\phi_{22}(x,\lambda)=\frac{\alpha_4\phi_{21}'''(0)}{2}\frac{d^k}{dx^k}\cos sx$$
$$+(\frac{\alpha_4\phi_{21}'''(0)}{4}+\frac{\beta_4\phi_{21}''(0)}{4s})\frac{d^k}{dx}(e^{sx}+e^{-sx})+O(|s|^{k+1}e^{|s|(x+1)}), \quad (20)$$

$k=\overline{0,3}$. Each of these asymptotic equalities hold uniformly for x as $|\lambda|\to\infty$.

Proof: Let $\varphi_{11}(x,\lambda)=e^{|s|(x+1)}F(x,\lambda)$. We can easily get that $F(x,\lambda)$ is bounded. So $\varphi_{11}(x,\lambda)=O(e^{|s|(x+1)})$. Substituting it into (13) and differentiating it with respect to x for $k=\overline{0,3}$, we obtain (17). Next according to transmission conditions (6)- (9),

$$\varphi_{12}(0)\approx\alpha_4\varphi_{11}'''(0),\varphi_{12}'(0)\approx\beta_4\varphi_{11}''(0),$$
$$\varphi_{12}''(0)\approx\gamma_4\varphi_{11}'(0),\varphi_{12}'''(0)\approx\tau_4\varphi_{11}'''(0),$$
$$(|\lambda|\to\infty).$$

Substituting these asymptotic expressions into (14) for $k=0$, we get

$$\varphi_{12}(x,\lambda)=\frac{\alpha_4\varphi_{11}'''(0)}{2}\cos sx+(\frac{\alpha_4\varphi_{11}'''(0)}{4}+\frac{\beta_4\varphi_{11}''(0)}{4s})(e^{sx}+e^{-sx})$$
$$+\frac{1}{2s^3}\int_0^x\frac{d^k}{dx^k}(\sin s(x-y)-e^{s(x-y)}+e^{-s(x-y)})q(y)\phi_{12}(y,\lambda)dy+O(|s|^2e^{|s|(x+1)}). \quad (21)$$

Multiplying through by $|s|^{-3}e^{|s|(x+1)})$, and denoting

$$F_{12}(x,\lambda):=O(|s|^{-3}e^{-|s|(x+1)})\varphi_{12}(x,\lambda).$$

Denoting $M(\lambda):=\max_{x\in[0,1)} F_{12}(x,\lambda)$ from the last formula, it follows that

$$M(\lambda)\le\frac{3|\alpha_2\alpha_4|}{4}+\frac{3|\alpha_2\beta_4|}{8}+\frac{M(\lambda)}{2}\int_0^x q(y)dy+M_0$$

for some $M_0>0$. From this, it follows that $M(\lambda)=O(1)$ as $|\lambda|\to\infty$, so $\varphi_{12}(x,\lambda)=O(|s|^3 e^{|s|(x+1)})$. Substituting this back into the integral on (21) yields (18) for $k=0$. The other assertions can be proved similarly.

Theorem 3.5. Let $\lambda=s^4$, $s=\sigma+it$. Then the characteristic function $W(\lambda)$ has the following asymptotic representations:

Case 1 $\gamma_2'\neq 0$, $\tau_2'\neq 0$,

$$W(\lambda)=\frac{\alpha_2\beta_2\alpha_4\beta_4\gamma_2'\tau_2's^{16}}{32\theta^2}(1-e^{-s}\sin s-e^{s}\cos s)[(e^{s}-e^{-s})\cos s - (e^{s}+e^{-s})\sin s]+O\left(|s|^{15}e^{2|s|}\right),$$

Case 2 $\gamma_2'\neq 0$, $\tau_2'=0$,

$$W(\lambda)=\frac{\alpha_2\beta_2\alpha_4\beta_4\gamma_2'\tau_1's^{15}}{16\theta^2}(1-e^{-s}\sin s-e^{s}\cos s)(e^{s}-e^{-s})\sin s+O\left(|s|^{14}e^{2|s|}\right),$$

Case 3 $\gamma_2'=0$, $\tau_2'\neq 0$,

$$W(\lambda)=-\frac{\alpha_2\beta_2\alpha_4\beta_4\gamma_1'\tau_2's^{13}}{16\theta^2}(1-e^{-s}\sin s-e^{s}\cos s)(e^{s}+e^{-s})\cos s+O\left(|s|^{12}e^{2|s|}\right),$$

Case 4 $\gamma_2'=0$, $\tau_2'=0$,

$$W(\lambda)=-\frac{\alpha_2\beta_2\alpha_4\beta_4\gamma_1'\tau_1's^{12}}{32\theta^2}(1-e^{-s}\sin s-e^{s}\cos s)[(e^{s}-e^{-s})\cos s + (e^{s}+e^{-s})\sin s+O\left(|s|^{11}e^{2|s|}\right).$$

Proof: The proof is obtained by substituting the asymptotic equalities $\frac{d^k}{dx^k}\varphi_{12}(1,\lambda)$ and $\frac{d^k}{dx^k}\varphi_{22}(1,\lambda)$ into the representation

$$\theta^2 W(\lambda):=\begin{vmatrix}\varphi_{12}(1,\lambda) & \varphi_{22}(1,\lambda) & \lambda\gamma_2'-\gamma_2 & 0\\ \varphi_{12}'(1,\lambda) & \varphi_{22}'(1,\lambda) & 0 & \lambda\tau_2'+\tau_2\\ \varphi_{12}''(1,\lambda) & \varphi_{22}''(1,\lambda) & 0 & \lambda\tau_1'+\tau_1\\ \varphi_{12}'''(1,\lambda) & \varphi_{22}'''(1,\lambda) & \lambda\gamma_1'-\gamma_1 & 0\end{vmatrix}=$$

$$\left(\frac{\alpha_2\beta_2\alpha_4 s^3}{64}[\tau_4(e^{2s}+2+e^{-2s})+\beta_4(e^{2s}-e^{-2s})]+O(|s|^2e^{2|s|})\right)\begin{vmatrix}\cos s & e^{s}+e^{-s} & s^4\gamma_2'-\gamma_2 & 0\\ -s\sin s & s(e^{s}-e^{-s}) & 0 & s^4\tau_2'+\tau_2\\ -s^2\cos s & s^2(e^{s}+e^{-s}) & 0 & s^4\tau_1'+\tau_1\\ s^3\sin s & s^3(e^{s}-e^{-s}) & s^4\gamma_1'-\gamma_1 & 0\end{vmatrix}.$$

Corollary 3.6. The eigenvalues of the boundary value problem (1)-(9) are bounded below.

Proof: By putting $s^2=it$ $(t>0)$ in the above formulas it follows that $W(-t^2)\to\infty$ as $t\to\infty$. Consequently, $W(\lambda)\neq 0$ for λ negative and sufficiently large in modulus.

Since the eigenvalues are coincident with the zeros of the entire function $W(\lambda)$, it follows that they have no finite limits. Moreover, all eigenvalues are real and bounded below by Corollary 2.3 and 3.6. Therefore, we may renumber them as $\lambda_1\le\lambda_2\le\lambda_3\le\ldots$ by counting their multiplicities. Below we shall denote $\lambda_n=s_n^4$ for sufficiently large n.

Theorem 3.7. Let $a_1=a_2=1$, the following asymptotic formulas hold for the eigenvalues of the boundary value transmission problem (1)-(9):

Case 1 $\gamma_2'\neq 0$, $\tau_2'\neq 0$,

$$\sqrt[4]{\lambda_n}=(n-\frac{3}{4})\pi+O\left(\frac{1}{n}\right),$$

Case 2 $\gamma_2'\neq 0$, $\tau_2'=0$,

$$\sqrt[4]{\lambda_n}=n\pi+O\left(\frac{1}{n}\right),$$

Case 3 $\gamma_2'=0$, $\tau_2'\neq 0$,

$$\sqrt[4]{\lambda_n}=(n+\frac{1}{2})\pi+O\left(\frac{1}{n}\right),$$

Case 4 $\gamma_2'=0$, $\tau_2'=0$,

$$\sqrt[4]{\lambda_n}=(n+\frac{3}{4})\pi+O\left(\frac{1}{n}\right).$$

All these asymptotic formulas hold uniformly for x.

Proof: If $s > 0$, by applying the known Rouche theorem, we can obtain these conclusions (cf. [15, Theorem 2.3]). The proofs of other cases for $s < 0$ and $\mathrm{Im}s \neq 0$ are similar.

4. Completeness of eigenfunction

Theorem 4.1. The operator A has only point spectrum, i.e., $\sigma(A) = \sigma_p(A)$.

Proof: It suffices to prove that if γ is not an eigenvalue of A, then $\gamma \in \rho(A)$. Hence we investigate the equation $(A-\gamma)Y = F \in H$, where $\gamma \in \mathrm{R}, F = (f, h, r)$.

Consider the initial-value problem

$$\begin{cases} ly - \gamma y = f, x \in J, \\ \alpha_1 y(-1) + \alpha_2 y'''(-1) = 0, \\ \beta_1 y'(-1) + \beta_2 y''(-1) = 0, \\ y(0+) = \alpha_3 y(0-) + \alpha_4 y'''(0-), \\ y'(0+) = \beta_3 y'(0-) + \beta_4 y''(0-), \\ y''(0+) = \gamma_3 y'(0-) + \gamma_4 y''(0-), \\ y'''(0+) = \tau_3 y(0-) + \tau_4 y'''(0-). \end{cases}$$

Let $u(x)$ be the solution of the equation $lu - \gamma u = 0$ satisfying

$$u(-1) = \alpha_2, u'(-1) = \beta_2, u''(-1) = -\beta_1,$$
$$u'''(-1) = -\alpha_1,$$
$$u(0+) = \alpha_3 u(0-) + \alpha_4 u'''(0-),$$
$$u'(0+) = \beta_3 u'(0-) + \beta_4 u''(0-),$$
$$u''(0+) = \gamma_3 u'(0-) + \gamma_4 u''(0-),$$
$$u'''(0+) = \tau_3 u(0-) + \tau_4 u'''(0-).$$

In fact,

$$u(x) = \begin{cases} u_1(x), & x \in [-1,0); \\ u_2(x), & x \in (0,1], \end{cases}$$

where $u_1(x)$ is the unique solution of the initial-value problem

$$\begin{cases} a_1{}^2 u^{(4)} + q(x)u = \gamma u, x \in [-1,0), \\ u_1(-1) = \alpha_2, u_1'(-1) = \beta_2, \\ u_1''(-1) = -\beta_1, u_1'''(-1) = -\alpha_1, \end{cases}$$

and $u_2(x)$ is the unique solution of the problem

$$\begin{cases} a_2{}^2 u^{(4)} + q(x)u = \gamma u, x \in (0,1], \\ u_2(0) = \alpha_3 u_1(0) + \alpha_4 u_1''', u_2'(0) = \beta_3 u_1'(0) + \beta_4 u_1''(0), \\ u_2''(0) = \gamma_3 u_1'(0) + \gamma_4 u_1''(0), u_2'''(0) = \tau_3 u_1(0) + \tau_4 u_1'''(0). \end{cases}$$

Let

$$w(x) = \begin{cases} w_1(x), & x \in [-1,0); \\ w_2(x), & x \in (0,1] \end{cases}$$

be a solution of $lw - \gamma w = f$ satisfying

$$\alpha_1 w(-1) + \alpha_2 w'''(-1) = 0,$$
$$\beta_1 w'(-1) + \beta_2 w''(-1) = 0,$$
$$w(0+) = \alpha_3 w(0-) + \alpha_4 w'''(0-),$$
$$w'(0+) = \beta_3 w'(0-) + \beta_4 w''(0-) \quad (22)$$
$$w''(0+) = \gamma_3 w'(0-) + \gamma_4 w''(0-),$$
$$w'''(0+) = \tau_3 w(0-) + \tau_4 w'''(0-).$$

Then (22) has the general solution

$$y(x) = \begin{cases} du_1 + w_1, & x \in [-1,0); \\ du_2 + w_2, & x \in (0,1], \end{cases} \quad (23)$$

where $d \in \mathrm{C}$.

Since γ is not an eigenvalue of (1)-(9), we have

$$\gamma(\gamma_1' u_2(1) - \gamma_2' u_2'''(1)) - (\gamma_1 u_2(1) - \gamma_2 u_2'''(1)) \neq 0, \quad (24)$$

or

$$\gamma(\tau_1' u_2'(1) - \tau_2' u_2''(1)) + (\tau_1 u_2'(1) - \tau_2 u_2''(1)) \neq 0. \quad (25)$$

The second component of $(A-\gamma)Y = F$ involves the equation

$$\gamma_2 y'''(1) - \gamma_1 y(1) + \gamma(\gamma_1' y(1) - \gamma_2' y'''(1)) = h. \quad (26)$$

Substituting (23) into (26), we get

$$(\gamma_2 u_2'''(1) - \gamma_1 u_2(1) + \gamma(\gamma_1' u_2(1) - \gamma_2' u_2'''(1)))d$$
$$= h + \gamma_1 w_2(1) - \gamma_2 w_2'''(1) + \gamma(\gamma_2' w_2'''(1) - \gamma_1' w_2(1)).$$

In view of (24), we know that d is a unique solution.

The third component of $(A-\gamma)Y=F$ involves the equation

$$\tau_2 y''(1)-\tau_1 y'(1)-\gamma(\tau_1' y'(1)-\tau_2' y''(1))=k\,. \quad (27)$$

Substituting (23) into (27), we get

$$\begin{aligned}&(\tau_2 u_2''(1)-\tau_1 u_2'(1)+\gamma(\tau_2' u_2''(1)-\tau_1' u_2'(1)))d\\ &=k+\tau_1 w_2'(1)-\tau_2 w_2''(1)+\gamma(\tau_1' w_2'(1)-\tau_2' w_2''(1)).\end{aligned}$$

In view of (25), we know that d is a unique solution.

Thus if γ is not an eigenvalue of (1)-(9), d is uniquely solvable. Hence y is uniquely determined.

The above arguments show that $(A-\gamma I)^{-1}$ is defined on all of H. We get that $(A-\gamma I)^{-1}$ is bounded by Theorem 2.3 and the Closed Graph Theorem. Thus $\gamma\in\rho(A)$. Hence, $\sigma(A)=\sigma_p(A)$.

Lemma 4.2. The operator A has compact resolvent, i.e., for each $\delta\in \mathbb{R}/\sigma_p(A)$, $(A-\delta I)^{-1}$ is compact on H (cf. [12], Theorem 3.2).

By the above lemmas and the spectral theorem for compact operator, we obtain the following theorem:

Theorem 4.3. The eigenfunctions of the problem (1)-(9), augmented to become eigenfunctions of A, are complete in H, i.e., if we let $\{\Phi_n=(\varphi_n(x),N'''(\varphi_n),N''(\varphi_n);n\in \mathbb{N}\}$ be a maximum set of orthonormal eigenfunctions of A, where $\{(\varphi_n(x);n\in\mathbb{N}\}$ are eigenfunctions of the problem (1)-(9), then for all $F\in H$, $F=\sum_{n=1}^{\infty}\langle F,\Phi_n\rangle\Phi_n$.

5. Green function

Let us consider the following differential equation

$$\begin{aligned}&(a(x)u''(x))''+q(x)u(x)\\ &-\lambda u(x)=-f(x), x\in J,\end{aligned} \quad (28)$$

where $J=[-1,0)\cup(0,1]$, $a(x)=a_1^{\;2}$ for $x\in[-1,0)$ and $a(x)=a_2^{\;2}$, for $x\in(0,1]$, $a_1>0$ and $a_2>0$ are given real numbers; together with the eigenparameter-dependent boundary conditions and transmission conditions (1)-(9).

We can represent the general solution (11) of homogeneous differential equation (1), appropriate to equation (28). By applying the standard method of variation of constants, we shall search the general solution of the non-homogeneous differential equation (28) in the form

$$U(x)=\begin{cases}C_1(x,\lambda)\phi_{11}(x,\lambda)+C_2(x,\lambda)\phi_{21}(x,\lambda)\\ +C_3(x,\lambda)\chi_{11}(x,\lambda)+C_4(x,\lambda)\chi_{21}(x,\lambda), & x\in[-1,0);\\ C_5(x,\lambda)\phi_{12}(x,\lambda)+C_6(x,\lambda)\phi_{22}(x,\lambda)\\ +C_7(x,\lambda)\chi_{12}(x,\lambda)+C_8(x,\lambda)\chi_{22}(x,\lambda), & x\in(0,1],\end{cases} \quad (29)$$

where the functions $C_i(x,\lambda)(i=\overline{1,8})$ satisfy the linear system of equation

$$\begin{cases}C_1'(x,\lambda)\phi_{11}(x,\lambda)+C_2'(x,\lambda)\phi_{21}(x,\lambda)\\ +C_3'(x,\lambda)\chi_{11}(x,\lambda)+C_4'(x,\lambda)\chi_{21}(x,\lambda)=0,\\ C_1'(x,\lambda)\phi_{11}'(x,\lambda)+C_2'(x,\lambda)\phi_{21}'(x,\lambda)\\ +C_3'(x,\lambda)\chi_{11}'(x,\lambda)+C_4'(x,\lambda)\chi_{21}'(x,\lambda)=0,\\ C_1'(x,\lambda)\phi_{11}''(x,\lambda)+C_2'(x,\lambda)\phi_{21}''(x,\lambda)\\ +C_3'(x,\lambda)\chi_{11}''(x,\lambda)+C_4'(x,\lambda)\chi_{21}''(x,\lambda)=0,\\ C_1'(x,\lambda)\phi_{11}'''(x,\lambda)+C_2'(x,\lambda)\phi_{21}'''(x,\lambda)\\ +C_3'(x,\lambda)\chi_{11}'''(x,\lambda)+C_4'(x,\lambda)\chi_{21}'''(x,\lambda)=f(x),\end{cases}$$

for $x\in[-1,0)$, and

$$\begin{cases}C_5'(x,\lambda)\phi_{12}(x,\lambda)+C_6'(x,\lambda)\phi_{22}(x,\lambda)\\ +C_7'(x,\lambda)\chi_{12}(x,\lambda)+C_8'(x,\lambda)\chi_{22}(x,\lambda)=0,\\ C_5'(x,\lambda)\phi_{12}'(x,\lambda)+C_6'(x,\lambda)\phi_{22}'(x,\lambda)\\ +C_7'(x,\lambda)\chi_{12}'(x,\lambda)+C_8'(x,\lambda)\chi_{22}'(x,\lambda)=0,\\ C_5'(x,\lambda)\phi_{12}''(x,\lambda)+C_6'(x,\lambda)\phi_{22}''(x,\lambda)\\ +C_7'(x,\lambda)\chi_{12}''(x,\lambda)+C_8'(x,\lambda)\chi_{22}''(x,\lambda)=0,\\ C_5'(x,\lambda)\phi_{12}'''(x,\lambda)+C_6'(x,\lambda)\phi_{22}'''(x,\lambda)\\ +C_7'(x,\lambda)\chi_{12}'''(x,\lambda)+C_8'(x,\lambda)\chi_{22}'''(x,\lambda)=f(x),\end{cases}$$

for $x\in(0,1]$. Because the characteristic function $W(\lambda)\neq 0$, $C_i'(x,\lambda)(i=\overline{1,8})$ can be solved. After the appropriate calculations, we can obtain the following relation:

$$U(x)=\begin{cases}\int_{-1}^{0}k_1(x,\xi,\lambda)f(\xi)d\xi+C_1\phi_{11}(x,\lambda)+C_2\phi_{21}(x,\lambda)\\ +C_3\chi_{11}(x,\lambda)+C_4\chi_{21}(x,\lambda), & x\in[-1,0);\\ \int_{0}^{1}k_2(x,\xi,\lambda)f(\xi)d\xi+C_5\phi_{12}(x,\lambda)+C_6\phi_{22}(x,\lambda)\\ +C_7\chi_{12}(x,\lambda)+C_8\chi_{22}(x,\lambda), & x\in(0,1].\end{cases} \quad (30)$$

Here, $C_i\,(i=\overline{1,8})$ are arbitrary constants and

$$k_1(x,\xi,\lambda)=\begin{cases}\dfrac{Z_1(x,\xi,\lambda)}{W_1(\lambda)}, & -1\le\xi\le x\le 0,\\ 0, & -1\le x\le\xi\le 0,\end{cases},$$

$$k_2(x,\xi,\lambda)=\begin{cases}\dfrac{Z_2(x,\xi,\lambda)}{W_2(\lambda)}, & 0\le x\le\varsigma\le 1,\\ 0, & 0\le\xi\le x\le 1,\end{cases}$$

With

$$Z_1(x,\xi,\lambda):=\begin{vmatrix}\phi_{11}(\xi,\lambda) & \phi_{21}(\xi,\lambda) & \chi_{11}(\xi,\lambda) & \chi_{21}(\xi,\lambda)\\ \phi'_{11}(\xi,\lambda) & \phi'_{21}(\xi,\lambda) & \chi'_{11}(\xi,\lambda) & \chi'_{21}(\xi,\lambda)\\ \phi''_{11}(\xi,\lambda) & \phi''_{21}(\xi,\lambda) & \chi''_{11}(\xi,\lambda) & \chi''_{21}(\xi,\lambda)\\ \phi_{11}(x,\lambda) & \phi_{21}(x,\lambda) & \chi_{11}(x,\lambda) & \chi_{21}(x,\lambda)\end{vmatrix},$$

$$Z_2(x,\xi,\lambda):=\begin{vmatrix}\phi_{12}(\xi,\lambda) & \phi_{22}(\xi,\lambda) & \chi_{12}(\xi,\lambda) & \chi_{22}(\xi,\lambda)\\ \phi'_{12}(\xi,\lambda) & \phi'_{22}(\xi,\lambda) & \chi'_{12}(\xi,\lambda) & \chi'_{22}(\xi,\lambda)\\ \phi''_{12}(\xi,\lambda) & \phi''_{22}(\xi,\lambda) & \chi''_{12}(\xi,\lambda) & \chi''_{22}(\xi,\lambda)\\ \phi_{12}(x,\lambda) & \phi_{22}(x,\lambda) & \chi_{12}(x,\lambda) & \chi_{22}(x,\lambda)\end{vmatrix}.$$

In the following, for convenience, set $k_i(0)=k_i(0,\xi,\lambda)$, $K_i(0)=K_i(0,\xi,\lambda)$ $(i=1,2)$, etc.. Substituting (30) for (2)-(9), we can have

$$C_1=\frac{1}{W_2(\lambda)}\begin{vmatrix}\int_{-1}^{0}(\alpha_3k_1(0)+\alpha_4k_1'''(0))f(\xi)d\xi-\int_0^1k_2(0)f(\xi)d\xi & -\phi_{22}(0) & \chi_{12}(0) & \chi_{22}(0)\\ \int_{-1}^{0}(\beta_3k_1'(0)+\beta_4k_1''(0))f(\xi)d\xi-\int_0^1k_2'(0)f(\xi)d\xi & -\phi'_{22}(0) & \chi'_{12}(0) & \chi'_{22}(0)\\ \int_{-1}^{0}(\tau_3k_1(0)+\tau_4k_1'''(0))f(\xi)d\xi-\int_0^1k_2''(0)f(\xi)d\xi & -\phi''_{22}(0) & \chi''_{12}(0) & \chi''_{22}(0)\\ \int_{-1}^{0}(\gamma_3k_1'(0)+\gamma_4k_1''(0))f(\xi)d\xi-\int_0^1k_2'''(0)f(\xi)d\xi & -\phi'''_{22}(0) & \chi'''_{12}(0) & \chi'''_{22}(0)\end{vmatrix},$$

$$C_2=\frac{1}{W_2(\lambda)}\begin{vmatrix}-\phi_{12}(0) & \int_{-1}^{0}(\alpha_3k_1(0)+\alpha_4k_1'''(0))f(\xi)d\xi-\int_0^1k_2(0)f(\xi)d\xi & \chi_{12}(0) & \chi_{22}(0)\\ -\phi'_{12}(0) & \int_{-1}^{0}(\beta_3k_1'(0)+\beta_4k_1''(0)f(\xi)d\xi-\int_0^1k_2'(0)f(\xi)d\xi & \chi'_{12}(0) & \chi'_{22}(0)\\ -\phi''_{12}(0) & \int_{-1}^{0}(\tau_3k_1(0)+\tau_4k_1'''(0))f(\xi)d\xi-\int_0^1k_2''(0)f(\xi)d\xi & \chi''_{12}(0) & \chi''_{22}(0)\\ -\phi'''_{12}(0) & \int_{-1}^{0}(\gamma_3k_1'(0)+\gamma_4k_1''(0))f(\xi)d\xi-\int_0^1k_2'''(0)f(\xi)d\xi & \chi'''_{12}(0) & \chi'''_{22}(0)\end{vmatrix},$$

$$C_7=\frac{1}{W_2(\lambda)}\begin{vmatrix}\phi_{12}(0) & \phi_{22}(0) & \int_{-1}^{0}(\alpha_3k_1(0)+\alpha_4k_1'''(0))f(\xi)d\xi-\int_0^1k_2(0)f(\xi)d\xi & \chi_{22}(0)\\ \phi'_{12}(0) & \phi'_{22}(0) & \int_{-1}^{0}(\beta_3k_1'(0)+\beta_4k_1''(0))f(\xi)d\xi-\int_0^1k_2'(0)f(\xi)d\xi & \chi'_{22}(0)\\ \phi''_{12}(0) & \phi''_{22}(0) & \int_{-1}^{0}(\tau_3k_1(0)+\tau_4k_1'''(0))f(\xi)d\xi-\int_0^1k_2''(0)f(\xi)d\xi & \chi''_{22}(0)\\ \phi'''_{12}(0) & \phi'''_{22}(0) & \int_{-1}^{0}(\gamma_3k_1'(0)+\gamma_4k_1''(0))f(\xi)d\xi-\int_0^1k_2'''(0)f(\xi)d\xi & \chi'''_{22}(0)\end{vmatrix},$$

$$C_8=\frac{1}{W_2(\lambda)}\begin{vmatrix}\phi_{12}(0) & \phi_{22}(0) & \chi_{12}(0) & \int_{-1}^{0}(\alpha_3k_1(0)+\alpha_4k_1'''(0))f(\xi)d\xi-\int_0^1k_2(0)f(\xi)d\xi\\ \phi'_{12}(0) & \phi'_{22}(0) & \chi'_{12}(0) & \int_{-1}^{0}(\beta_3k_1'(0)+\beta_4k_1''(0))f(\xi)d\xi-\int_0^1k_2'(0)f(\xi)d\xi\\ \phi''_{12}(0) & \phi''_{22}(0) & \chi''_{12}(0) & \int_{-1}^{0}(\tau_3k_1(0)+\tau_4k_1'''(0))f(\xi)d\xi-\int_0^1k_2''(0)f(\xi)d\xi\\ \phi'''_{12}(0) & \phi'''_{22}(0) & \chi'''_{12}(0) & \int_{-1}^{0}(\gamma_3k_1'(0)+\gamma_4k_1''(0))f(\xi)d\xi-\int_0^1k_2'''(0)f(\xi)d\xi\end{vmatrix},$$

$$C_3=C_4=C_5=C_6=0.$$

Finally, by substituting the coefficients $C_i\,(i=\overline{1,8})$ in (30), the following formulae is obtained for $U(x)$:

$$U(x)=\begin{cases}\int_{-1}^{0}k_1(x,\xi,\lambda)f(\xi)d\xi+C_1\phi_{11}(x,\lambda) & x\in[-1,0);\\ +C_2\phi_{21}(x,\lambda), & \\ \int_0^1k_2(x,\xi,\lambda)f(\xi)d\xi+C_7\chi_{12}(x,\lambda) & x\in(0,1].\\ +C_8\chi_{22}(x,\lambda), & \end{cases}$$

Let

$$K(x,\xi,\lambda)=\begin{cases}K_1(x,\xi,\lambda), & -1\le x<0,\\ K_2(x,\xi,\lambda), & 0<x\le 1,\end{cases},$$

where

$$K_1(x,\xi,\lambda)=\begin{cases}k_1(x,\xi,\lambda), & -1\le x<0,\\ 0, & 0<x\le 1,\end{cases},$$

$$K_2(x,\xi,\lambda)=\begin{cases}0, & -1\le x<0,\\ k_2(x,\xi,\lambda), & 0<x\le 1.\end{cases}$$

Let

$$B(x,\xi,\lambda)=\begin{cases}B_1(x,\xi,\lambda), & -1\le x<0,\\ B_2(x,\xi,\lambda), & 0<x\le 1,\end{cases}$$

where

$$B_1(x,\xi,\lambda)\equiv\begin{vmatrix}\phi_{12}(0) & \phi_{22}(0) & \chi_{12}(0) & \chi_{22}(0) & \alpha_3K_1(0)+\alpha_4K_1'''(0)-K_2(0)\\ \phi'_{12}(0) & \phi'_{22}(0) & \chi'_{12}(0) & \chi'_{22}(0) & \beta_3K_1'(0)+\beta_4K_1''(0)-K_2'(0)\\ \phi''_{12}(0) & \phi''_{22}(0) & \chi''_{12}(0) & \chi''_{22}(0) & \tau_3K_1(0)+\tau_4K_1'''(0)-K_2''(0)\\ \phi'''_{12}(0) & \phi'''_{22}(0) & \chi'''_{12}(0) & \chi'''_{22}(0) & \gamma_3K_1'(0)+\gamma_4K_1''(0)-K_2'''(0)\\ \phi_{11}(x) & \phi_{21}(x) & 0 & 0 & 0\end{vmatrix},$$

$$B_2(x,\xi,\lambda)\equiv\begin{vmatrix}\phi_{12}(0) & \phi_{22}(0) & \chi_{12}(0) & \alpha_3K_1(0)+\alpha_4K_1'''(0)-K_2(0) & \chi_{22}(0)\\ \phi'_{12}(0) & \phi'_{22}(0) & \chi'_{12}(0) & \beta_3K_1'(0)+\beta_4K_1''(0)-K_2'(0) & \chi'_{22}(0)\\ \phi''_{12}(0) & \phi''_{22}(0) & \chi''_{12}(0) & \tau_3K_1(0)+\tau_4K_1'''(0)-K_2''(0) & \chi''_{22}(0)\\ \phi'''_{12}(0) & \phi'''_{22}(0) & \chi'''_{12}(0) & \gamma_3K_1'(0)+\gamma_4K_1''(0)-K_2'''(0) & \chi'''_{22}(0)\\ 0 & 0 & \chi_{12}(x) & 0 & \chi_{22}(x)\end{vmatrix}.$$

Then

$$U(x)=\int_{-1}^{1}\left(K(x,\xi,\lambda)+\frac{1}{W_2(\lambda)}B(x,\xi,\lambda)\right)f(\xi)d\xi. \qquad (31)$$

Thus, the resolvent of the boundary-value transmission problem is obtained. We can find the Green function from the resolvent (31). Namely, denoting

$$G(x,\xi,\lambda) = K(x,\xi,\lambda) + \frac{1}{W_2(\lambda)} B(x,\xi,\lambda).$$

We can rewrite the resolvent (31) in the next form

$$U(x) = \int_{-1}^{1} G(x,\xi,\lambda) f(\xi) dy.$$

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References

[1] Demirci, M., Akdoğan, Z. & Mukhtarov, O. Sh. (2004). Asymptotic behavior of eigenvalues and eigenfuctions of one discontinuous boundary-value problem. *International Journal of Computational Cognition*, *2*(3), 101-113.
[2] Buschmann, D., Stolz, G. & Weidmann, J. (1995). One-dimensional Schrődinger operators with local point interactions. *J. Reine Angew. Math*, *467*, 169-186.
[3] Binding, P. A., Browne, P. J. & Watson, B. A. (2002). Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter, II. *Journal of Computational and Applied Mathematics*, *148*, 147-168.
[4] Binding, P. A., Browne, P. J. & Watson, B. A. (2000). Inverse spectral problems for Sturm-Liouville equations with eigenparameter dependent boundary conditions. *J. London Math. Soc*., *62*(1), 161-182.
[5] Binding, P. A., Browne, P. J. & Watson, B. A. (2001). Transformations between Sturm-Liouville problems with eigenvalue dependent and independent Boundary conditions. *Bull. London Math Soc*., *33*, 749-757.
[6] Hinton, D. B. & Shaw, J. K. (1990). Differential operators with spectral parameter incompletely in the boundary conditions. *Funkcial. Ekvac*., *33*, 363-385.
[7] Hinton, D. B. (1990). An expansion theorem for an eigenvalue problem with eigenvalue parameter in the boundary condition. Quart. *J. Math, Oxford*, *2*, 41, 189-224.
[8] Hinton, D. B. (1979). An expansion theorem for an eigenvalue problem with eigenvalue parameter in the boundary condition. *Quart J. Math, Oxford*, *30*, 33-42.
[9] Fulton, C. (1980/81). Singular eigenvalue problems with eigenvalue parameter contained in the boundary conditions. Proc. Roy. Soc. Edinburgh, *87*(1), 1-34.
[10] Fulton, C. (1977). Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions. *Proceedings of the Royal Society of Edinburgh*, *77*A, 293-308.
[11] Kadakal, M., Mukhtarov, O. Sh. & Muhtarov, F. S. (2005). Some spectral problems of Sturm-Liouville problem with transmission conditions. *IJST, Trans. A*, *49*(A2), 229-245.
[12] Aiping, W. (2006). Research on Weidmann conjecture and differential operators with transmission conditions [Ph D Thesis], Inner Mongolia University，66-74. [in Chinese]
[13] Kadakal, M. & Mukhtarov, O. Sh. (2006). Discontinuous Sturm-Liouville problems containing eigenparameter in the boundary condition. *Acta Mathematical sinic, English Series Sep*. *22*(5), 1519-1528.
[14] Naimark, M. A. (1968). *Linear Differential Operators*, *Part II*. London, Harrap.
[15] Zhijiang, C. (1986). Ordinary Differential Operator. Shanghai Press. [in Chinese].