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Wavelet solutions of the second Painleve equation

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Abstract

Dynamically adaptive numerical methods have been developed to find solutions for differential equations. The subject of wavelet has attracted the interest of many researchers, especially, in finding efficient solutions for differential equations. Wavelets have the ability to show functions at different levels of resolution. In this paper, a numerical method is proposed for solving the second Painleve equation based on the Legendre wavelet. The solutions of this method are compared with the analytic continuation and Adomian Decomposition methods and the ability of the Legendre wavelet method is demonstrated.

Keywords: Multiresolution analysis; Wavelet; Painleve equations; Legendre wavelet; Adomian Decomposition Method

1. Introduction

Recently, considerable attention has been given to using wavelets and corresponding scaling functions for solving differential equations, integral equations and integro-differential equations [1]. Differential equations arise in many countless mathematical applications. Ideally, these can be solved exactly using analytic methods: however, in many cases (especially in complicated applications), exact solutions can be difficult or impossible to obtain. Many numerical methods for solving these equations are applied, such as, finite difference methods, series expansion methods and other numerical techniques. But, wavelets have been proven to be an efficient tool in developing numerical methods.

Wavelet functions (and scaling functions) can also be used as the basic functions to approximate a given function. Another advantage of using wavelet series expansion is the construction of wavelets which can be useful for ordinary differential equations. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [2]. On the other hand, a disadvantage of wavelets is their action as differential and integral operators on the basis of which functions can be difficult to determine, depending on the choice of wavelets.

The Haar wavelet is by far the simplest wavelet, so it is natural to attempt to utilize it for solving differential equations. These are not differentiable due to the discontinuous jump in the function values. Thus, in order to avoid this difficulty, Chen and Hesiao [3] introduced a procedure to expand the derivative of the unknown function instead of the function itself on the wavelet basis. This method has been extended to nonlinear differential equations. In particular, Beylkin constructed differentiation matrices for the Daubechies wavelets. Jameson outlined this method [4] and constructed wavelet functions defined at boundaries, so the method can be used to solve differential equations with nonperiodic boundary conditions [5]. Their work utilized scaling functions as the expansion terms. Then, Coerdecker and Ivanov introduced a family of wavelets called "interpolation wavelets", which were relatively easy in integrating and producing the matrices with a convenient structure for solving matrix equations [6].

In recent years the study of Painleve equations has become more common [7]. Many situations in science and engineering reduce to Painleve equations. Its applications include plasma physics, waves, optics and mechanics. The main goal of this research, here, is to use wavelets for solving the second Painleve equation.

The Painleve equations, first considered by Painleve, are defined as follows:

$$u''(x)=f(x,u,u'), \tag{1}$$

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where f(x,u,u') is a rational function in u and u', while f is analytic in x. In this paper, the second Painleve equation will be considered in the following form,

$$u''=2u^3+xu+\alpha \quad x\in[0,1], \quad \alpha>0 \tag{2}$$

$$u(0)=1,\ u'(0)=0.$$

where u=u(x).

Rational solutions of the second Painleve equation were investigated by Murata [8] and by Yablonski [9]. Several numerical methods for approximating the solutions of the second Painleve equation have been implemented, most notable of which are; the α-method [10], Adomian Decomposition Method (ADM) [7, 11], Homotopy Perturbation Method (HPM) [7, 12] and the Legendre tau technique [7, 13].

In this work, a numerical method for solving the second Painleve equation based on the Legendre wavelet approximations is described. Working with Legendre wavelets has been chosen because these are continuous and orthonormal functions with compact support. Furthermore, Legendre wavelets have a polynomial basis, which makes them a convenient choice to work with.

The approach consists of three main stages. Firstly, the second Painleve equation will be transformed into a nonlinear integral equation. Secondly, Legendre wavelets will be applied and a system of linear equations will be obtained. Finally, the coefficients of the solution to the Painleve equation will be computed by solving the resulting linear system of equations.

This paper is organized as follows. The Legendre wavelets are described in Section 2. In Section 3, the Legendre wavelet method is developed for solving the Painleve equations. Finally, numerical results are presented in Section 4.

2. The legendre wavelets

In this section, an overview of wavelets is expressed, and a brief introduction to wavelets, the Legendre wavelets and their properties is presented.

2.1. Wavelets

Wavelets are a family of functions which are derived from the family of scaling functions $\{\phi_{j,k} : k\in Z\}$ where

$$\phi(x)=\sum_k a_k\phi(2x-k). \tag{3}$$

For the continuous wavelet, the following form can be represented:

$$\psi_{a,b}(x)=|a|^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right) \quad a,b\in R,\ a\neq 0, \tag{4}$$

where a and b are dilation and translation parameters respectively, such that $\psi(x)$ is a single wavelet function.

The discrete values are put for a and b in the initial form of the continuous wavelets, i.e.:

$$\begin{aligned} &a=a_0^{-j},\ a_0>1,\ b_0>0,\\ &b=nb_0a_0^{-j},\ j,k\in Z. \end{aligned} \tag{5}$$

Then, a family of discrete wavelets can be constructed as follows:

$$\psi_{j,k}(x)=|a_0|^{\frac{j}{2}}\psi(a_0^jx-nb_0), \tag{6}$$

where, $\{\psi_{j,k}\}_{j\in Z}$ is the wavelet basis for $L^2(R)$. Therefore, a wavelet basis will be constructed in the following stage for $a_0=2$ and $b_0=1$. Hence, the family of functions $\{\psi_{j,k}\}_{j\in Z}$ is in the following form:

$$\psi_{j,k}(x)=2^{\frac{j}{2}}\psi(2^jx-k). \tag{7}$$

So, $\{\psi_{j,k}(x)\}_{j\in Z}$ constitutes an orthonormal basis in $L^2(R)$, where $\psi(x)$ is a single function.

2.2. Legendre Wavelets and their Properties

The Legendre wavelets are in the following form:

$$\psi_{k,m}(x)=\begin{cases}\sqrt{m+\frac{1}{2}}\,2^{\frac{j}{2}}p_m(2^jx-k) & \frac{k-1}{2^j}\le x<\frac{k}{2^j},\\ 0 & \text{otherwise},\end{cases}$$

where $m=0,1,2,...,M-1$ and $k=1,2,...,2^{j-1}$. The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, then by (7) the wavelets $\{\psi_{k,m}\}$ form an orthonormal basis for $L^2[0,1]$.

In the above formulation of Legendre wavelets, the Legendre polynomials are in the following form:

$$p_0=1,$$
$$p_1=x,$$
$$p_{m+1}(x)=\frac{(2m+1)}{(m+1)}xp_m(x)-\frac{m}{m+1}p_{m-1}(x),$$

and $\{p_{m+1}(x)\}$ are the orthogonal functions of order $\mathbf{m}$, which is named the well-known shifted Legendre polynomials on the interval [0, 1]. Note that, in the general form of Legendre wavelets, the dilation parameter is $a=2^{-j}$ and the translation parameter is $b=n2^{j}$.

2.3. Function Approximation

A given function f(x) with the domain [0, 1] can be approximated by:

$$f(x)=\sum_{k=1}^{\infty}\sum_{m=0}^{\infty} c_{k,m}\psi_{k,m}(x). \tag{8}$$

If the infinite series in equation (8) is truncated, then this equation can be written as:

$$f(x)\simeq\sum_{k=1}^{2^{j-1}}\sum_{m=0}^{M-1}c_{k,m}\psi_{k,m}(x)=C^T\cdot\Psi(x), \tag{9}$$

where C and $\Psi(x)$ are matrices of size $(2^{j-1}M\times 1)$ as follows**:**

$$C=[c_{1,0},c_{1,1},\ldots,c_{1,M-1},c_{2,0},c_{2,1},\ldots,$$
$$c_{2,M-1},\ldots,c_{2^{k-1},0},c_{2^{k-1},1},\ldots,c_{2^{k-1},M-1}]^T,$$
$$\Psi(x)=[\psi_{1,0},\psi_{1,1},\ldots,\psi_{1,M-1},\psi_{2,0},\psi_{2,1},\ldots,$$
$$\psi_{2,M-1},\ldots,\psi_{2^{k-1},0},\psi_{2^{k-1},1},\ldots,\psi_{2^{k-1},M-1}]^T.$$

3. Legendre wavelet method to solve the painleve equations

Consider the general form of Painleve equation given in (1). Let L_λ be the following integral operator,

$$L_\lambda(f)=\int_0^x x^{-\lambda}\int_0^x t^\lambda\ f(t)\ dt\ dx. \tag{10}$$

If $f(x,u,u')=g(x,u)+h(x)$, then:

$$L_\lambda(f)=L_\lambda(g)+L_\lambda(h), \tag{11}$$

and

$$L_\lambda(u''(x))=L_\lambda(f(x,u,u')),$$

where $x\in[0,1]$.

Since f is analytic in x and f(x,u,u') is a rational function in u and u', then $u(x)\in\ \oplus W_j=L^2([0,1])$ where $L^2[0,1]$ is a Hilbert space and $\{\psi_{k,m}\}_{k\in Z}$ is an orthonormal basis in $L^2([0,1])$. So,

$$u(x)=\sum_{k=1}^{2^{j-1}}\sum_{m=0}^{M-1}c_{k,m}\psi_{k,m}(x) \tag{12}$$

where clearly $\{\psi_{k,m}\}$ are the Legendre wavelets. In other words, this method represents the solution u(x) as an expansion of Legendre wavelets at the j-level of resolution.

Inserting the expansion of u(x) in (11). One obtains a system of linear equations which will be solved to compute the coefficients $c_{k,m}$ by using the given initial conditions. Then, the solutions of the second Painleve equation will be obtained. u(x) is approximated as:

$$u(x)=C^T\cdot\Psi(x), \tag{13}$$

Then one has:

$$L_\lambda(\frac{d^2}{dx^2}[C^T\cdot\Psi(x)])$$
$$=L_\lambda(f(x,[C^T\cdot\Psi(x)],\frac{d}{dx}[C^T\cdot\Psi(x)])). \tag{14}$$

Setting $\lambda=0$ and

$$L_0(f)=\int_0^x\int_0^x f(t)\ dt\ dx,$$

$$C^T\cdot\Psi(x)=1+\int_0^x\int_0^x f(x,[C^T\cdot\Psi(t)],\frac{d}{dt}[C^T\cdot\Psi(t)])\ dt\ dx. \tag{15}$$

3.1. Solutions for the Second Painleve Equation

In this section, the wavelet method is applied to solve the second Painleve equation. Let L_λ be in the following form:

$$L_\lambda(\cdot)=\int_0^x x^{-\lambda}\int_0^x t^\lambda(\cdot)\ dt\ dx. \tag{16}$$

By applying L_λ to the second Painleve equation, one can get:

$$L_\lambda(u'')=L_\lambda(2u^3+xu+\alpha).$$

Then for $\lambda=0$ and $\alpha=1$

$$u=1+\frac{x^2}{2}+\int_0^x\int_0^x(2(u(t))^3+tu(t))\,dt\,dx. \quad (17)$$

Let

$$x_i = x_0 + ih \quad i=1,2,... \qquad x_0 = 0 \quad (18)$$

By collocating the above equation at point x_i, one gets

$$C^T\cdot\Psi(x_i)=1+\frac{x_i}{2}+\int_0^{x_i}\int_0^{x_i}(2[C^T\cdot\Psi(t)]^3+t[C^T\cdot\Psi(t)])\,dt\,dx. \quad (19)$$

where Ψ is a matrix of Legendre wavelets. Consider the initial conditions:

$$\begin{cases}\sum_{k=1}^{2^{j-1}}\sum_{m=0}^{M-1} c_{k,m}\psi_{k,m}(0)-1=0,\\ \sum_{k=1}^{2^{j-1}}\sum_{m=0}^{M-1} c_{k,m}[\frac{d}{dx}\psi_{k,m}(x)](0)=0.\end{cases}$$

The solutions u(x) of this equation will be obtained by computing the coefficients $\{c_{k,m}\}$.

The Legendre wavelet method, with j=1, M=3 and h=0.05 are applied to the second Painleve equation. The numerical solutions of this equation are computed and presented in Table 1. Similarly, for $\alpha=2$ one obtains

$$L_\lambda(u'')=L_\lambda(2u^3+xu+2),$$

where for $\lambda=0$, the following form can be represented:

$$u=1+2\frac{x^2}{2}+\int_0^x\int_0^x(2(u(t))^3+tu(t))\,dt\,dx. \quad (20)$$

By collocating the above equation at points x_i and inserting u(x) in (20), the numerical values of u(x) will be obtained.These numerical results are presented in Table 2 for $\alpha=2$.

4. Numerical results

In order to compare the numerical results obtained by the Legendre wavelet method, the ADM is also implemeted for the second Painleve equation. The numerical results are presented in Tables 1-2.

Table 1. The comparison of the values obtained for u(x) by the analytic continuation [7], ADM and the Legendre wavelet method with $\alpha=1$.

x	The values of u(x)obtained by the method of analytic continuation [7]	The values of u(x) obtained by ADM	the values of u(x) obtained by the Legendre wavelet method
0	1	1	1
0.05	1.0038	1.00377556945843	1.0037755831
0.10	1.0152	1.01524353738588	1.0152441160
0.15	1.0347	1.03470887678813	1.0347143511
0.20	1.0626	1.06261465111813	1.0626429943
0.25	1.0996	1.09956760325147	1.0996732195
0.30	1.1464	1.14637603460243	1.1466969087
0.35	1.2041	1.20410448048055	1.2049555680
0.40	1.2742	1.27415228539083	1.2762089939
0.45	1.3584	1.35836736627797	1.3600321862
0.50	1.4592	1.45921344816914	1.4603783632

Table 2. The comparison of the values obtained for u(x) by analytic continuation [7], ADM and the Legendre wavelet method with $\alpha=2$.

x	The values of u(x)obtained by the analytic continuation method [7]	The values of u(x) obtained by ADM	the values of u(x) obtained by the Legendre wavelet method
0	1	1	1
0.05	1.0050	1.00502714606259	1.0050271609
0.10	1.0203	1.02026919477424	1.0202698545
0.15	1.0461	1.04609205682859	1.0460985293
0.20	1.0830	1.08304976093849	1.0830842957
0.25	1.1319	1.13192491551257	1.1320572554
0.30	1.1938	1.19379024536885	1.1940037587
0.35	1.2701	1.27009977531541	1.2702297562
0.40	1.3628	1.36282365175205	1.3656460735
0.45	1.4746	1.47464972088494	1.4713002282
0.50	1.6093	1.60929103954460	1.6044667906

As was observed, the numerical results show that the wavelet solutions are good approximations for the exact solutions of the second Painleve equation (Figs. 1 & 2).

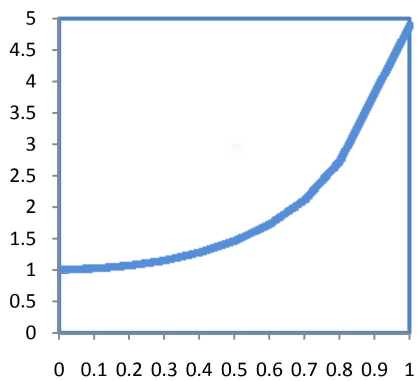

Fig. 1. Plot of the 20-th approximation of u(x) for $\alpha=1$

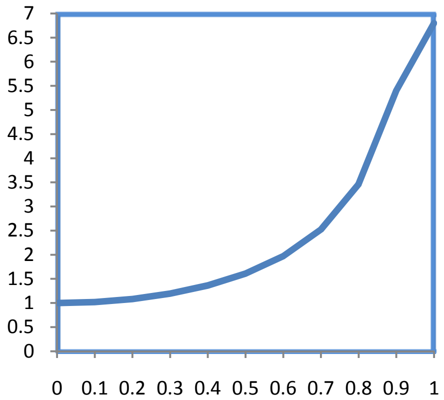

Fig. 2. Plot of the 20-th approximation of u(x) for α=2

5. Conclusions

Several numerical methods are implemented for solving Painleve equations, and the wavelet techniques have been described as a numerical tool for the fast and accurate solution of differential equations [14, 15].

The aim of the present work was to apply the Legendre wavelet method for solving the second Painleve equation. Numerical results proved the ability of the proposed technique in comparison with other methods.

The Adomian Decomposition Procedure was based on the search for a solution in the form of a series with the computed components [11]. Furthermore, the Legendre wavelets had the capability to approximate the solutions in different levels of resolution. Then, by changing the dilation and translation parameters, a good approximation with a few terms of basis functions was obtained. Also, the amount of calculations in the Legendre wavelet method was lower than the ADM.

Note that in the Legendre wavelet method, the values of integrals were evaluated very exactly because, the basis of Legendre wavelets were polynomials of different orders. Thus, by using this method, the problem was transformed to a nonlinear integral equation and good approximations were obtained for the solutions.

In future works, the aims will be to develop wavelet methods for solving other differential equations which are defined in multidimensions and other Painleve equations.

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