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Some applications of the product of submodules in multiplication modules

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Abstract

Let R be a commutative ring with identity. Let N and K be two submodules of a multiplication R-module M. Then N=IM and K=JM for some ideals I and J of R. The product of N and K denoted by NK is defined by NK=IJM. In this paper we characterize some particular cases of multiplication modules by using the product of submodules.

Keywords: Multiplication ideal; multiplication module; prime submodule; principal ideal multiplication module; product of submodules; Quasi cyclic submodule

1. Introduction

Throughout this paper R denotes a commutative ring with identity and M denotes a unitary R-module. Also, L(R) (resp. L(M)) denotes the lattice of all ideals of R (resp. submodules of M).

For any two submodules N and K of M, the ideal {a∈R|aK⊆N} will be denoted by (N:K). Thus (0:M) is the annihilator of M. A module M is said to be faithful if (0:M) is the zero ideal of R. We say that a module M is a multiplication module [1] if every submodule of M is of the form IM, for some ideal I of R. A submodule N of M is said to be a multiple of M [2] if N=rM for some r∈R. If every submodule of M is a multiple of M, then M is said to be a principal ideal multiplication module or PI-multiplication module, for abbreviations (see [2]).

A proper submodule N of M is a prime submodule, if for any r∈R and m∈M, rm∈N implies either m∈N or r∈(N:M).

It is well-known that maximal submodules and prime submodules exist in multiplication modules (for details, see [1]). It is also well-known that if M is a multiplication R-module and P is a prime ideal of R containing (0:M) such that M≠PM, then PM is a prime submodule of M and every prime submodule of M is of the form PM for some prime ideal P of R containing (0:M) (see [1, Corollary 2.11]).

Also, if M is a finitely generated multiplication R-module and P is a prime ideal of R containing (0:M), then PM is a proper prime submodule of M (see Lemma 1.1(i), in the following).

Further PM is minimal over a submodule N of M if and only if P is minimal over the ideal (N:M) of R.

Let M be a multiplication R-module. Then for each submodule N of M, N=IM for some ideal I of R. According to [3], I is said to be a presentation ideal of N. Note that M is a multiplication R-module if and only if every submodule of M has a presentation ideal. Let N and K be submodules of a multiplication module M. Suppose N=IM and K=JM for some ideals I and J of R. The product of N and K denoted by NK is defined by NK=IJM. Observe that by [3, Theorem 3.4], the product of N and K is independent of the presentations of N and K. It should be mentioned that by [3, Proposition 3.5], the product is commutative and distributive with respect to the sum on L(M).

We will use the product of submodules in multiplication modules to find the connections between some particular types of multiplication modules, which will be introduced in the next sections.

For the convenience of the reader, some results from our references, which are used frequently in this paper, have been gathered in the following lemma.

Lemma 1.1. Let M be a non-zero R-module. Then

(i) [1, Theorem 3.1] Let M be a multiplication module. Then M is finitely generated, if and only if M≠PM, for each maximal ideal P of R containing (0:M), if and only if for any ideals A,B of R containing (0:M), the inclusion AM⊆BM implies that A⊆B.

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(ii) [1, Theorem 2.8] and [4, Proposition 4] If M is multiplication and M (resp. R) has only finitely many maximal submodules (resp. ideals), then M is cyclic**.**
(iii) [4, Proposition 5] Let M be a finitely generated module. Then M is multiplication if and only if it is locally cyclic**.**
(iv) [5, Lemma 6] If M is cyclic, then a submodule N of M is cyclic if and only if N is a multiple of M**.**
For general background and terminology, the reader is referred to [6] and [7].

2. The product of submodules in multiplication modules

According to [8], a submodule N of M is called quasi-cyclic if (B∩(K:N))N=BN∩K and (K+BN:N)=(K:N)+B for all ideals B of R and for all submodules K of M**.**

Note that N is quasi-cyclic if and only if N is finitely generated and locally cyclic if and only if N is a finitely generated multiplication submodule (by [8, Theorem 6] and Lemma 1.1(iii)).

An ideal I of R is called a quasi-principal ideal [6, Exercise 10, Page 147] (or a principal element of L(R) [9]) if I satisfies the identities (i) (A∩(B:I))I=AI∩B and (ii) (A+BI:I)=(A:I)+B, for all A, B∈L(R).

An ideal I of R is quasi-principal if and only if it is finitely generated and locally principal (see [9, Theorem 2] or [10, Theorem 4])).

Lemma 2.1. Suppose M is a faithful quasi-cyclic R-module. Let N_1 and N_2 be quasi-cyclic submodules of M with $(0:(N_1:M))=0$. If $(N_1+N_2)(N_1\cap N_2)=N_1N_2$, then (N_1+N_2) is quasi-cyclic.

Proof: Assume that $(N_1+N_2)(N_1\cap N_2)=N_1N_2$. Since M is a finitely generated faithful multiplication R-module, it follows that $(N_1:M)$ and $(N_2:M)$ are the presentation ideals of N_1 and N_2. Now, $N_1N_2=N_1(N_1\cap N_2)+N_2(N_1\cap N_2)$. As $(N_1\cap N_2)\subseteq N_1$ and N_1 is quasi-cyclic, we have $(N_1\cap N_2)=IN_1$ for some $I\in L(R)$. So $N_1N_2=N_1(IN_1)+N_2(IN_1)$.
Then $N_1N_2=(N_1:M)(N_2:M)M=$
$(N_1:M)I(N_1:M)M+I(N_2:M)(N_1:M)M$,
thus by Lemma 1.1(iii), $(N_1:M)(N_2:M)=(N_1:M)(I(N_1:M)+I(N_2:M))$. By [11, Lemma 1.4], $(N_1:M)$ is quasi-principal and $(N_1:M)$ has zero annihilator. Therefore $(N_1:M)$ is a cancellation ideal, so $(N_2:M)=I(N_1:M)+I(N_2:M)$ and hence $N_2=IN_1+IN_2$. As N_2 is quasi-cyclic, we have $R=((IN_1+IN_2):N_2)=I+(IN_1:N_2)$. Let P be a maximal ideal of R. As R_P is local, it follows that $R_P=I_P$ or $R_P=(IN_1:N_2)_P$. If $R_P=I_P$, then $(N_1)_P\subseteq(N_2)_P$ since $IN_1\subseteq N_2$. If $R_P=(IN_1:N_2)_P$, then $(N_2)_P\subseteq(N_1)_P$. In any case, $(N_1+N_2)_P$ is cyclic in M_P. Therefore N_1+N_2 is locally cyclic and hence N_1+N_2 is quasi-cyclic. This completes the proof of the lemma.

The following lemma studies the behavior of the product of submodules under the localization.

Lemma 2.2. Let M be a finitely generated multiplication R-module and let P be a maximal ideal of R. Then
(i) For every $N,K\in L(M)$, $N_PK_P=(NK)_P$.
(ii) For every $N\in L(M)$ and any positive integer m, $(N_P)^m=(N^m)_P$.

Proof: (i) Since the R_P-module M_P is a multiplication module, by definition, $(N_P:M_P)$ and $(K_P:M_P)$ are presentation ideals of N_P and K_P in M_P, respectively. So $N_PK_P=(N_P:M_P)(K_P:M_P)M_P$. As M is finitely generated, $(N_P:M_P)=(N:M)_P$, and $(K_P:M_P)=(K:M)_P$. Hence $N_PK_P=(N:M)_P(K:M)_PM_P=((N:M)(K:M)M)_P=(NK)_P$, since $(N:M)$ and $(K:M)$ are presentation ideals of N and K respectively.
(ii) The assertion follows from (i).

Recall that a module M is said to be distributive if the lattice $L(M)$ is a distributive lattice. Also, M is said to be a valuation module if any two submodules of M are comparable.

Lemma 2.3. [12, Theorem 2.16] Suppose M is an R-module. Then the following statements are equivalent.
(i) M is a distributive module.
(ii) M is a locally valuation module.

Theorem 2.4. Suppose M is a multiplication R-module such that $(0:M)$ is a prime ideal. Then the following statements are equivalent.

(i) M is a distributive module.
(ii) $R/(0:M)$ is a prüfer domain.
(iii) For every $N,K,L\in L(M)$, $N(K\cap L)=NK\cap NL$.
(iv) For every $N,K,L\in L(M)$, $(N+K)(N\cap K)=NK$.

Proof: According to [1, Proposition 3.4], if $PM\neq M$, for any minimal prime ideal P over $(0:M)$, then M is finitely generated. Hence if we put $R'=R/(0:M)$, then M is a finitely generated faithful multiplication R'-module.

(i) $\Leftrightarrow$ (ii) Note that $M_P\cong R'_P$, for each prime ideal P of R'**.** Now the proof follows from Lemma 2.3((i) $\Leftrightarrow$ (ii)).

(i) $\Rightarrow$ (iii) By Lemma 2.3, any two submodules are locally comparable. Therefore $N_P(K_P\cap L_P)=$

$N_P K_P \cap N_P L_P$ for every maximal ideal P of R'. So by Lemma 2.2, $(N(K \cap L))_P = (NK \cap NL)_P$ for any maximal ideal P of R'. Therefore (iii) holds.

(iii) $\Rightarrow$ (iv) We have $NK \supseteq (N+K)(N \cap K) = ((N+K)N) \cap ((N+K)K) \supseteq NK$. Therefore (iv) holds.

(iv) $\Rightarrow$ (i) By Lemma 2.1, every finitely generated submodule is quasi-cyclic and hence M is a distributive module.

The proof of the following lemma is easy and it left to the reader.

Lemma 2.5. Let M be an R-module and $N = IM \neq M$, where $I \in L(R)$. Then

(i) If M is a multiplication module and I is a maximal ideal of R, then N is a maximal submodule of M.

(ii) If N is a maximal submodule of M, then $(N: M)$ is a maximal ideal of R. The converse is correct if M is a multiplication module.

The following result will give us a condition under which a maximal submodule of a finitely generated valuation module is cyclic.

Proposition 2.6. Let M be a non-zero finitely generated valuation R-module, where R is a local ring. Suppose N is a maximal submodule such that $N \neq N^2$. Then N is cyclic.

Proof: By hypothesis, M is cyclic, and by Lemma 2.5(ii), $(N: M) = P$ is the maximal ideal of R. Note that $N^2 = P^2M \neq N = PM$, thus $P^2 + (0: M) \subset P$. Choose $a \in P \backslash P^2 + (0: M)$. Then $aM \subseteq N$ and $aM \nsubseteq N^2$. As M is cyclic and aM is a multiple of M, by Lemma 1.1(iv) aM is cyclic and so $aM = Rx$ for some $x \in M$. We show that $N = Rx$. Clearly, $Rx = aM \subseteq N$. If $Rx \neq N$, then choose an element $y \in N \backslash Rx$. Thus $Rx \subset Ry$, and so $Rx = I(Ry)$ for some proper ideal $I \in L(R)$. Hence $aM = Rx = I(Ry) \subseteq IN = IPM \subseteq P^2M = N^2$, which is a contradiction. Therefore, $N = Rx$ and hence N is cyclic.

The following lemma is a key result for proving the main theorem of this paper (Theorem 2.10).

Lemma 2.7. Let M be a non-zero Noetherian cyclic R-module, where R is a local ring with maximal ideal P. Suppose N is a maximal submodule of M. If N is cyclic, then every non-zero submodule of M is a power of N.

Proof: Note that $N = PM$. So by [13, Proposition 4.6, page 390], $\bigcap_{n=1}^{\infty} N^n = \bigcap_{n=1}^{\infty} (P^nM) = 0$. Let $0 \neq K \in L(M)$. Then there exists a positive integer m such that $K \subseteq N^m$ and $K \nsubseteq N^{m+1}$. As M and N are cyclic, Lemma 1.1(iv) implies that N is a multiple of M, and so N^m is a multiple of M and consequently again by Lemma 1.1(iv), N^m is cyclic. Now as $K \subseteq N^m$ and N^m is multiplication (cyclic), it follows that $K = IN^m$ for some ideal I of R. If $I \subseteq P$, then $K = IN^m = IP^mM \subseteq P^{m+1}M = N^{m+1}$, a contradiction. Therefore $K = N^m$, and so, K is cyclic.

Let $N, K \in L(M)$. We denote $[N, K] = \{L \in L(M) | N \subseteq L \subseteq K\}$. Also, it is defined as $rad\ N = \cap \{K \in Spec\ M | N \subseteq K\}$. If no prime submodule of M contains N, then it is defined as $rad\ N = M$.

If $N \in L(M)$ is primary and $rad\ N = L$ is a prime submodule, then we say that N is an L-primary submodule of M (see [14]).

Lemma 2.8. Let M be a non-zero Noetherian cyclic R-module, where R is a local ring with maximal ideal P. Suppose N is a maximal submodule of M. Then the following statements are equivalent.
(i) M is a valuation module.
(ii) $[N^2, N]$ is totally ordered.
(iii N is cyclic.
(iv) There are no submodules strictly between N and N^2.

Proof: (i) $\Rightarrow$ (ii) The proof is obvious.

(ii) $\Rightarrow$ (iii) Let $N = \sum_{i=1}^{n} Rx_i$ for some $x_1, x_2, \cdots, x_n \in N$. Then $N = \sum_{i=1}^{n} (Rx_i + N^2)$, so by (ii), $N = Rx_i + N^2$, for some $x_i \in N$. As $N = PM$, we have $N^2 = P^2M = PN$, and thus $N = Rx_i + PN$, so by Nakayama's Lemma, $N = Rx_i$ and hence (iii) holds.

(iii) $\Rightarrow$ (iv) The assertion follows from Lemma 2.7.

(iv) $\Rightarrow$ (i) Note that $N = PM$. If $N = N^2$, then $N = PN$, so by Nakayama's Lemma, $N = 0$. Consequently, M is a valuation module. Now assume that $N \neq N^2$. Then by (iv), $N = Rx + N^2 = Rx + PN$ for some $x \in N \backslash N^2$. So by Nakayama's Lemma, $N = Rx$. Therefore N is cyclic. Now the result follows from Lemma 2.7.

A well-known result states that if I is a maximal ideal of a ring R, then for every positive integer k, each ideal of R between I and I^k is an I-primary ideal. Part (i) and (ii) of the following lemma is the module version of this result.

Lemma 2.9. Suppose M is a non-zero finitely generated multiplication R-module and N is a maximal submodule of M with $(N: M) = P$. Then
(i) N^k is N-primary for all positive integers k.
(ii) For every positive integer k and for any $L \in [N^k, N]$, L is N-primary.
(iii) If $K \in L(M)$ is N-primary, then $K = (K_P)^c$.

(iv) For any $N \in L(M)$, the interval $[N^2, N]$ in $L(M)$ is totally ordered if and only if the interval $[(N^2)_P, N_P]$ in $L(M_P)$ is totally ordered.

Proof: (i) Note that $(N^k : M)M = N^k = P^k M = (P^k + (0:M))M$, and hence by Lemma 1.1(iii), $(N^k : M) = P^k + (0:M)$. We have $\sqrt{(N^k : M)} = \sqrt{P^k + (0:M)} = P$, and by Lemma 2.5(ii), P is a maximal ideal, consequently N^k is primary. According to [15, Theorem 3], if M is a finitely generated multiplication module and $(0:M) \subseteq I \in L(R)$, then $rad(IM) = \sqrt{I}M$. Thus $rad\, N^k = rad(P^k M) = rad((P^k + (0:M))M) = \sqrt{P^k + (0:M)}M = PM = N$ and hence N^k is an N-primary submodule.

(ii) Suppose $L \in [N^k, N]$, for some positive integer k. As M is a multiplication module, we have $L = IM$ for some $I \in L(R)$. Since $(P^k + (0:M))M = P^k M = N^k \subseteq L = IM = (I + (0:M))M \subseteq N = PM$ and M is a non-zero finitely generated multiplication R-module, Lemma 1.1(iii) implies that $P^k + (0:M) \subseteq I + (0:M) \subseteq P$. Hence $\sqrt{I + (0:M)} = P$ and P is a maximal ideal of R, so $I + (0:M)$ is a P-primary ideal. Consequently $L = (I + (0:M))M$ is an N-primary submodule, similar to the proof of part (i).

(iii) Let $K \in L(M)$ be N-primary. By [15, Theorem 3], $\sqrt{(K:M)}M = rad((K:M)M) = rad\, K = N = PM$. So by Lemma 1.1(iii), $\sqrt{(K:M)} = P$ and P is a maximal ideal. Therefore K is a primary submodule with $\sqrt{(K:M)} = P$, which implies that $K = (K_P)^c$.

(iv) Suppose the interval $[N^2, N]$ in $L(M)$ is totally ordered, and consider $K, L \in [(N^2)_P, N_P]$. Then by part (iii), $K^c, L^c \in [((N^2)_P)^c, (N_P)^c] = [N^2, N]$, and hence by hypothesis, $K^c \subseteq L^c$ or $L^c \subseteq K^c$. So $K = (K^c)_P \subseteq (L^c)_P = L$ or $L = (L^c)_P \subseteq (K^c)_P = K$. Therefore, the interval $[(N^2)_P, N_P]$ in $L(M_P)$ is totally ordered.

Conversely, assume that the interval $[(N^2)_P, N_P]$ in $L(M_P)$ is totally ordered. Suppose $A, B \in [N^2, N]$. Then evidently $A_P, B_P \in [(N^2)_P, N_P]$, so either $A_P \subseteq B_P$ or $B_P \subseteq A_P$. By (ii), A and B are N-primary submodules, and so by (iii), either $A = (A_P)^c \subseteq (B_P)^c = B$ or $B = (B_P)^c \subseteq (A_P)^c = A$. Therefore the interval $[N^2, N]$ in $L(M)$ is totally ordered.

Definition 1. An R-module M is said to be a general quasi-cyclic module if every submodule of M is quasi-cyclic.

Recall that an R-module M is called a cyclic submodule module (CSM), if every submodule of M is cyclic.

Evidently every CSM is a general quasi-cyclic module. But the converse is not true since in a ring R, quasi-principal ideals need not be principal ideals.

It is well-known that R is a general ZPI-ring if and only if every ideal is quasi-principal [16, Theorem 2.2]).

General ZPI-rings are examples of general quasi-cyclics, particularly consider $R = M = Z[\sqrt{-5}]$. So M is a general quasi-cyclic R-module, but it is not a CSM, as R is not a principal ideal ring.

Theorem 2.10. Suppose M is a non-zero finitely generated multiplication R-module. Then the following statements are equivalent.

(i) M is a locally PI-multiplication module.
(ii) M is distributive and locally Noetherian module.
(iii) M is a locally CSM.
(iv) M is locally Noetherian and for every maximal submodule N of M, the interval $[N^2, N]$ is totally ordered.
(v) M is locally Noetherian and for every maximal submodule N of M, there are no submodules strictly between N^2 and N.
(vi) M is a locally general quasi-cyclic module.

Proof: (i) $\Rightarrow$ (ii) Let $N \in L(M)$. Since M is locally cyclic, by Lemma 1.1(iv), N is locally cyclic, so by [4, Proposition 6] M is distributive and a locally Noetherian module.

(ii) $\Leftrightarrow$ (iii) The assertion follows from Lemma 2.3 and [4, Proposition 6].

(iii) $\Rightarrow$ (iv) The proof follows from Lemma 2.8.

(iv) $\Rightarrow$ (v) Suppose N is a maximal submodule of M. Then $N = PM$ for some maximal ideal P of R. Suppose $N^2 \subseteq L \subseteq N$ for some $L \in L(M)$. Consider the R_P-module M_P. By Lemma 2.9, the interval $[(N^2)_P, N_P]$ in $L(M_P)$ is totally ordered. So by Lemma 2.7 and Lemma 2.8, $L_P = (N^m)_P$ for some positive integer m. Again by Lemma 2.9, L and N^m are N-primary submodules and so $L = N^m$. Consequently, $N^2 = L$ or $L = N$. Thus (v) holds.

(v) $\Rightarrow$ (i) Suppose P is a maximal ideal of R. If $PM = M$, then $M_P = 0_P$, by Nakayama's Lemma. Now assume that $PM \neq M$. Then by Lemma 2.5(i), $N = PM$ is a maximal submodule of M, so N_P is a maximal submodule of the R_P-module M_P. Suppose $(N^2)_P \subseteq L \subseteq N_P$ for some $L \in L(M)$. Then by Lemma 2.9 parts (i) and (iiii), we have $N^2 = ((N^2)_P)^c \subseteq L^c \subseteq (N_P)^c = N$, so either $N^2 = L^c$ or $L^c = N$ and hence either $(N^2)_P = (L^c)_P = L$ or $L = (L^c)_P = N_P$. Therefore there are no submodules strictly between N_P and $(N^2)_P$. Now by Lemma 2.7 and Lemma 2.8, N_P is cyclic and every non zero submodule of M_P is a power of N_P.

As M_P is cyclic, by Lemma 1.1(iv), M_P is a PI-multiplication module. Thus (i) holds.

(iii) $\Leftrightarrow$ (vi) The proof follows from [8, Theorem 5]. This completes the proof of the theorem.

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