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Common fixed points of jointly asymptotically nonexpansive mappings

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Abstract

A definition of two jointly asymptotically nonexpansive mappings S and T on uniformly convex Banach space E is studied to approximate common fixed points of two such mappings through weak and strong convergence of an Ishikawa type iteration scheme generated by S and T on a bounded closed and convex subset of E. As a consequence of the notion of two jointly asymptotically nonexpansive maps, we can relax the very commonly used strong condition "$F(S)$ and $F(T)$ has a nonempty intersection" with the weaker assumption "either $F(S)$ is nonempty or $F(T)$ is nonempty". Our convergence results are refinements and generalizations of several recent results from the current literature.

***Keywords**:* Jointly asymptotically nonexpansive mapping; Common fixed point; weak and strong convergence; iteration scheme; Condition (A^*)

1. Introduction

Throughout this paper, N will denote the set of all positive integers. Let E be a real Banach space and C a nonempty subset of E. A mappings S of C into itself is said to be asymptotically nonexpansive if for a sequence $k_n \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, $\|S^n x - S^n y\| \le k_n \|x - y\|$ holds for all $x, y \in C$ and for all $n \in N$. S is called uniformly k-Lipschitzian if for some $k > 0$, $\|S^n x - S^n y\| \le k_n \|x - y\|$ for all $x, y \in C$ and $n \in N$.

We define a pair of mappings S and T of C into itself as jointly asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$,

$$\|S^n x - T^n y\| \le k_n \|x - y\| \tag{1}$$

holds for all $x, y \in C$ and for all $n \in N$. As a special case, when $x = y$, we get $S = T$ and so we get the results for usual asymptotically nonexpansive mappings. However, for $x \neq y$, (1) remains of independent interest.

The class of asymptotically nonexpansive mappings, which is a natural generalization of the important class of nonexpansive mappings, was introduced by Goebel and Kirk [1], where it was shown that, if C is a nonempty bounded closed convex subset of a uniformly convex Banach space and $T : X \to X$ is asymptotically nonexpansive, then T has a fixed point. Moreover, the set $F(T)$ is closed and convex.

A survey of the literature regarding approximation of common fixed points of two asymptotically nonexpansive mappings S and T shows that most of the results deal with the strong and weak convergence of different iterative processes to a point in F under the assumption that $F := F(T) \cap F(S) \neq \phi$. However, for the class of mappings defined in (1), we note that $F \neq \phi$ if either $F(T)$ or $F(S)$ is nonempty. Thus our results improve several comparable results.

S and T are called jointly uniformly k-Lipschitzian if for some $k > 0$,

$$\|S^n x - T^n y\| \le k \|x - y\| \tag{2}$$

for all $x, y \in C$ and for all $n \in N$.

To approximate the common fixed points of two mappings, the following Ishikawa type two-steps iterative process is widely used (see, for example, [2-5] and references cited therein):

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$$x_1 \in C,$$
$$x_{n+1} = (1 - a_n)x_n + a_n S[(1 - b_n)x_n + b_n T x_n],$$

for all $n \in N$, where $\{a_n\}$ and $\{b_n\}$ are in [0, 1] satisfying certain conditions. Khan and Takahashi [2] considered it for two asymptotically nonexpansive mappings. Takahashi and Tamura [4] studied the above scheme for two nonexpansive mappings. Das and Debata [6] studied the above scheme for two quasi-nonexpansive mappings. Recall that *S* is asymptotically quasi-nonexpansive if *F*(*S*), the set of fixed points of *S*, is nonempty and $\|S^n x - T^n y\| \le k\|x - y\|$ or all $x, y \in C$ and $y \in F(S)$.

In this paper, we take up the problem of approximation of common fixed points of jointly asymptotically nonexpansive mappings *S* and *T* through weak and strong convergence of the sequence defined by:

$$x_1 \in C,$$
$$x_{n+1} = (1 - a_n)x_n + a_n S^n[(1 - b_n)x_n + b_n T^n x_n], \quad (3)$$

for all $n \in N$, where $\{a_n\}$ and $\{b_n\}$ are in $(0,1)$ satisfying certain conditions.

Note that when *S* = *T*, (3) reduces to modified Ishikawa iteration scheme:

$$x_1 \in C,$$
$$x_{n+1} = (1 - a_n)x_n + a_n T^n[(1 - b_n)x_n + b_n T^n x_n],$$
(4)

for all $n \in N$, where $\{a_n\}$ and $\{b_n\}$ are in $(0,1)$ satisfying certain conditions.

2. Preliminaries

Let *E* be Banach space and let *C* be a nonempty bounded convex subset of *E*. We need the following lemma which can be found in [7].

Lemma 1. Suppose that *E* is a uniformly convex Banach space and $0 < p \le t_n \le q < 1$ for all $n \in N$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of *E* such that $\lim_{n\to\infty}\|x_n\| \le r$, $\lim_{n\to\infty}\|y_n\| \le r$ and $\lim_{n\to\infty}\|t_n x_n - (1 - t_n)y_n\| = r$ hold for some $r \ge 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

We recall that a Banach space *E* is said to satisfy Opial's condition [8] if for any sequence $\{x_n\}$ in *E*, $x_n \to x$ implies that

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_n \|y_n - x\|,$$

for all $y \in E$ with $y \ne x$. Examples of Banach spaces satisfying this condition are Hilbert spaces and all spaces $l_p (1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \ne 2$ fail to satisfy Opial's condition.

A mapping $T : C \to E$ is called demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in *C* and each $x \in E$, $x_n \to x$ and $T x_n \to y$ imply that $x \in C$ and $Tx = y$.

A Banach space *E* is said to satisfy the Kadec-Klee property if for every sequence $\{x_n\}$ in *E* converging weakly to *x* together with $\|x_n\|$ converging strongly to $\|x\|$ implies $\|x_n\|$ converges strongly to *x*. Uniformly convex Banach spaces, Banach spaces of finite dimension and reflexive locally uniform convex Banach spaces are some examples of reflexive Banach spaces which satisfy the Kadec-Klee property.

We shall use the following in our weak convergence theorem.

Lemma 2. [9] Let *C* be a nonempty bounded closed convex subset of a uniformly convex Banach space. Then there is a strictly increasing and continuous convex function $g : [0, \infty) \to [0, \infty)$ with g(0) = 0 such that, for *L*- Lipschitzian map $T : C \to C$ and for all $x, y \in C$ and $t \in [0,1]$, the following inequality holds:

$$\|T(tx + (1-t)y) - (tTx + (1-t)Ty\|$$
$$\le L g^{-1}(\|x - y\| - L^{-1}\|Tx - Ty\|).$$

Lemma 3. Let *E* be a uniformly convex Banach space such that its dual E^* satisfies the Kadec-Klee property. Assume that $\{x_n\}$ is a bounded sequence such that $\lim_{n\to\infty}\|tx_n + (q - t)p_1 - p_2\|$ exists for all $t \in [0,1]$ and for all $p_1, p_2 \in \omega_w(\{x_n\})$. Then $\omega_w(\{x_n\})$ is singleton.

Lemma 4. [10] Let *E* be a uniformly convex Banach space and let *C* be a nonempty closed convex subset of *E*. Let *T* be an asymptotically nonexpansive mapping of *C* into itself. Then 1 – *T* is demiclosed with respect to zero.

We now prove a lemma for jointly uniformly k-Lipschitzian mappings.

Lemma 5. Let E be a normed space and let C be a nonempty bounded, closed and convex subset of E. Let, for $k > 0$, S and T be jointly uniformly k-Lipschitzian mappings of C into itself. The $\lim_{n\to\infty}\|x_n - S^n x_n\| = 0$ implies $\lim_{n\to\infty}\|x_n - T^n x_n\| = 0$ and conversely.

Proof: Since $\|T^n x - S^n y\| \le \|x - y\|$ holds for all $x, y \in C$ and for all $n \in N$, therefore $\|x_n - T^n x_n\| \le \|x_n - S^n x_n\| + \|S^n x_n - T^n x_n\| \le \|x_n - S^n x_n\| + k\,\|x_n - x_n\|$.

Hence $\lim_{n\to\infty}\|x_n - S^n x_n\| = 0$ implies that $\lim_{n\to\infty}\|x_n - T^n x_n\| = 0$ and conversely.

The above lemma enables us to take only one of the limits $\lim_{n\to\infty}\|x_n - S^n x_n\|$ or $\lim_{n\to\infty}\|x_n - T^n x_n\|$ to be equal to zero to prove the following important lemma. However, this is not the case for two individually asymptotically nonexpansive mappings where we essentially need both of these limits to be equal to zero.

Lemma 6. Let E be a normed space and let C be a nonempty bounded, closed and convex subset of E. Let, for $k > 0$, S and T be jointly uniformly k-Lipschitzian mappings of C into itself. Define a sequence $\{x_n\}$ as in (3) where $\{a_n\}$ and $\{b_n\}$ are in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. If either $\lim_{n\to\infty}\|x_n - S^n x_n\| = 0$ or $\lim_{n\to\infty}\|x_n - T^n x_n\| = 0$, then $\lim_{n\to\infty}\|x_n - Sx_n\| = 0 = \lim_{n\to\infty}\|x_n - Tx_n\|$.

Proof: Set

$$c_n = \|x_n - T^n x_n\|$$

and

$$d_n = \|x_n - S^n x_n\|$$

for all $n \in N$. Also put, for simplicity, $y_n = (1 - b_n)x_n + b_n T^n x_n,\ n \in N,$ so that (3) becomes

$$x_{n+1} = (1 - a_n)x_n + a_n S^n y_n$$

and

$$\begin{aligned}
\|x_{n+1} - x_n\| &= a_n\|x_n - S^n y_n\| \\
&\le \|x_n - S^n x_n\| \\
&\le \|x_n - T^n x_n\| + \|T^n x_n - S^n y_n\| \\
&\le c_n + k\|x_n - y_n\| \\
&\le c_n + k b_n\|x_n - T^n x_n\| \\
&\le (1+k)\,c_n.
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+1} - Sx_{n+1}\| &= \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - Sx_{n+1}\| \\
&\le c_{n+1} + k\|x_{n+1} - T^n x_n\| \\
&\le c_{n+1} + k\,(\|x_{n+1} - x_n\| + \|x_n - S^n x_n\| \\
&\quad + \|S^n x_n - T^n x_{n+1}\|) \\
&\le c_{n+1} + k(\|x_{n+1} - x_n\| + \|x_n - S^n x_n\| \\
&\quad + k\|x_n - x_{n+1}\|) \\
&= c_{n+1} + k\,[(k+1)\,\|x_{n+1} - x_n\| + d_n] \\
&\le c_{n+1} + k\,[(k+1)^2 c_n + d_n].
\end{aligned}$$

By Lemma 5, $\lim_{n\to\infty} d_n = 0$ implies $\lim_{n\to\infty} c_n = 0$, therefore

$$\limsup_{n\to\infty}\|x_{n+1} - Sx_{n+1}\| \le 0.$$

Hence

$$\lim_{n\to\infty}\|x_n - Sx_n\| \le 0.$$

Similarly it can be shown that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$

With the help of the above lemmas, we also prove the following lemma needed in the sequel.

Lemma 7. Let E be a normed space and let C be a nonempty bounded, closed and convex subset of E. Let, for $k > 0$, S and T be jointly uniformly k-Lipschitzian mappings of C into itself. Define a sequence $\{x_n\}$ as in (3) where $\{a_n\}$ and $\{b_n\}$ are in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. If either $\lim_{n\to\infty}\|x_n - S^n x_n\| = 0$ or $\lim_{n\to\infty}\|x_n - T^n x_n\| = 0$, then $\lim_{n\to\infty}\|Tx_n - Sx_n\| = 0.$

Proof: If either $\lim_{n\to\infty}\|x_n - S^n x_n\| = 0$ or $\lim_{n\to\infty}\|x_n - T^n x_n\| = 0$, then by Lemma 6,

$\lim_{n\to\infty}\|x_n - Sx_n\| = 0 = \lim_{n\to\infty}\|x_n - Tx_n\| = 0.$

Therefore,

$\limsup_{n\to\infty}\|Sx_n - Tx_n\| \le \lim_{n\to\infty}\|Sx_n - x_n\| + \lim_{n\to\infty}\|Tx_n - x_n\|$ $=0$

and so $\lim_{n\to\infty}\|Sx_n - Tx_n\| = 0.$

3. Weak and Strong Convergence Theorems

We first prove a lemma which, in fact, constitutes a considerably large part of the proofs of both weak and strong convergence theorems. It is noted that if p is a fixed point of one of the mappings S and T satisfying (1), then it becomes the common fixed point of these mappings. Thus in the sequel we will replace the usual assumption $F(S) \cap F(T) \neq \emptyset$ by a weaker assumption of either $F(S) \neq \emptyset$ or $F(T) \neq \emptyset$.

Lemma 8. Let E be a uniformly convex Banach space and let C be its bounded, closed and convex subset. Let S and T be two mappings from C into itself satisfying $\|S^n x - S^n y\| \le k_n \|x - y\|$ for all $n \in N$, where $k_n \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$.

Define a sequence $\{x_n\}$ in C as:

$$x_1 \in C,$$
$$x_{n+1} = (1-a_n)\, x_n + a_n S^n\left[(1-b_n)x_n + b_n T^n x_n\right],$$

for all $n \in N$ where $\{a_n\}$ and $\{b_n\}$ are in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. If $F(T) \neq \emptyset$, then $\lim_{n\to\infty}\|x_n - Sx_n\| = 0 = \lim_{n\to\infty}\|x_n - Tx_n\|$.

Proof: Let $p \in F(T)$ and put $y_n = (1 - b_n)x_n + b_n T^n x_n$ for the sake of simplicity. A straightforward calculation gives

$$\|x_{n+1} - p\| = \|(1 - a_n)\,(x_n - p) + a_n(S^n y_n - p)\|$$
$$\le \left[(1 - a_n) + a_n k_n (1 - b_n) + a_n k_n^2 b_n\right]\|x_n - p\|.$$

Setting $v_n = (1 - a_n) + a_n k_n (1 - b_n) + a_n k_n^2 b_n$, the above expression exists for all $n \in N$. By induction, $\|x_{n+m} - p\| \le \left(\prod_{i=1}^{n+m-1} v_i\right)$ for all $n, m \in N$. Also, note that $\sum_{n=1}^{\infty} v_n < \infty$, we obtain $\lim_{n\to\infty}\prod_{i=n}^{\infty} v_i = 1$ and hence $\lim_{n\to\infty}\|x_n - p\|$ exists. Let $\lim_{n\to\infty}\|x_n - p\| = c$ where $c \ge 0$ is a real number. If $c = 0$, the result is evident. So we assume $c > 0$. Now

$$\|T^n x_n - p\| \le k_n \|x_n - p\|$$

for all $n \in N$ so

$$\limsup_{n\to\infty}\|T^n x_n - p\| \le c.$$

Also,

$$\|y_n - p\| = \|(1 - b_n)\,(x_n - p) + b_n\,(T^n x_n - p)\|$$
$$\le (1 - b_n)\|x_n - p\| + k_n b_n \|x_n - p\|$$
$$= \|x_n - p\| + (k_n - 1) b_n \|x_n - p\|$$
$$\le \|x_n - p\| + (k_n - 1)\|x_n - p\|$$

gives

$$\limsup_{n\to\infty}\|y_n - p\| \le c. \tag{5}$$

Next, $\|S^n y_n - p\| \le k_n \|y_n - p\|$ gives by virtue of (5) and $k_n \to 1$ as $n \to \infty$ that $\limsup_{n\to\infty}\|S^n y_n - p\| \le c$.

Moreover, $c = \lim_{n\to\infty}\|x_{n+1} - p\|$ means that

$$\lim_{n\to\infty}\|(1-a_n)(x_n - p) + a_n(S^n y_n - p)\| = c.$$

Applying Lemma 1,

$$\lim_{n\to\infty}\|S^n y_n - x_n\| = 0. \tag{6}$$

Now

$$\|x_n - p\| \le \|x_n - S^n y_n\| + \|S^n y_n - p\|$$
$$\le \|x_n - S^n y_n\| + k_n \|y_n - p\|$$

yields

$$c \le \liminf_{n\to\infty}\|y_n - p\|. \tag{7}$$

By (5) and (7), we obtain

$$\lim_{n\to\infty}\|y_n - p\| = c. \tag{8}$$

That is,

$$\lim_{n\to\infty}\|(1-b_n)(x_n - p) + b_n(T^n y_n - p)\| = c.$$

Again by Lemma 1, we get

$$\lim_{n\to\infty}\|T^n x_n - x_n\| = 0. \tag{9}$$

Lemma 6 combined with (6), (8) and (9) now reveals that

$$\lim_{n\to\infty}\|S x_n - x_n\| = 0 = \lim_{n\to\infty}\|T x_n - x_n\|.$$
(10)

Lemma 9. Let E be a uniformly convex Banach space and C its nonempty closed convex subset. Let $S,T : C \to C$ be jointly asymptotically nonexpansive mappings and $\{x_n\}$ as defined in (3). Then, $p_1, p_2 \in F(T)$, $\lim_{n\to\infty}\|t x_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$.

Proof: By Lemma 8, $\lim_{n\to\infty}\| x_n + p\|$ exists for all $p \in F$ and so $\{x_n\}$ is bounded. Hence we may assume that C is bounded. Set $a_n(t) = \|t x_n + (1-t)p_1 - p_2\|$ so that $\lim_{n\to\infty} a_n(0) = \| p_1 - p_2\|$ and $\lim_{n\to\infty} a_n(1) = \| x_n - p\|$ exist. For each $n \in N$, define a mapping $W_n : C \to C$ by

$$W_n x = (1-a_n)\, x + a_n S^n A_n x,$$

where $A_n x = (1-b_n)\, x + b_n T^n x,$ for all $x \in C.$ It is easy to verify that

$$\|W_n x - W_n y\| \le k_n^2 \,\|x-y\|, \quad \forall\, x, y \in C.$$

Set

$$R_{n,m} = W_{n+m-1} W_{n+m-2} ... W_n,$$

and

$$b_{n,m} = \| R_{n,m}\ (tx_n + (1-t)\, p_1) - tR_{n,m}\, x_n\ + (1-t)\ \ p_1)\ \|,$$
$\forall\, n, m \in N$.

Then it follows that $R_{n,m}\, x_n = x_{n+m,}\ R_{n,m} p = p$ for all $p \in F(S) \cap F(T)$ and $\|R_{n,m}\, x - R_{n,m}\, y\| \le K_n\ \|x-y\|$ for all $x, y \in C$ where $K_n = \prod_{j=n}^{\infty} k_j^2$. Since $k_n \to 1,$ therefore $K_n \to 1.$
Now

$$\begin{aligned} a_{n+m}(t) &= \| tx_{n+m} + (1-t)\ p_1 - p_2 \| \\ &\le b_{n,m} + \| R_{n,m}\ (tx_n + (1-t)\, p_1) - p_2)\| \\ &= b_{n,m} + \| R_{n,m}\ (tx_n + (1-t)\, p_1) - R_{n,m}\ p_2)\| \\ &\le b_{n,m} + k_n \| (tx_n + (1-t)\, p_1) - p_2)\| \\ &\le b_{n,m} + k_n\, a_n(t) \end{aligned}$$

That is

$$a_{n+m}\ (t) \le b_{n,m} + k_n\, a_n(t). \tag{11}$$

By Lemma 2, there exists a strictly increasing continuous function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\begin{aligned} b_{n,m} &\le K_n\, g^{-1}(\| x_n - p_1)\| - K_n^{-1} \| R_{n,m}\, x_n - R_{n,m} p_1)\|) \\ &= K_n\, g^{-1}(\| x_n - p_1)\| - K_n^{-1} \| x_{n+m} - p_1)\|). \end{aligned}$$

Combining it with (11), we get

$$a_{n+m}\ (t) \le K_n\, g^{-1}(\| x_n - p_1)\| - K_n^{-1} \| x_{n+m} - p_1)\|) + k_n\, a_n(t).$$

Since $\lim_{n\to\infty} \| x_n - p\|$ exists $\lim_{n\to\infty} K_n = 1,$ and $g^{-1}(0) = 0,$ keeping n fixed and letting $m \to \infty$, it follows that

$$\limsup_{n\to\infty} a_n(t) \le \liminf_{n\to\infty} a_n(t)\ .$$

Hence $\lim_{n\to\infty} \|(tx_n + (1-t)\ p_1 - p_2\|$ exists for all $t \in [0,1]$.

Theorem 1. Let E be a uniformly convex Banach space satisfying Opial's condition and let C, S, T and $\{x_n\}$ be as taken in Lemma 8. If $F(T) \ne \phi,$ then $\{x_n\}$ converges weakly to a common fixed point of S and T.

Proof: Let $p \in F(T)$. Then $\lim_{n\to\infty}\|x_n - p\|$ exists as proved in Lemma 8. We prove that $\{x_n\}$ has a unique weak subsequential limit in *F*.

So, let u and v be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 8, $\lim_{n\to\infty}\|x_n - Tx\| = 0$ and $I - T$ is demiclosed with respect to zero by Lemma 4, therefore we obtain $Tu = u$. Thus $u \in F$. Again in the same fashion, we can prove that $v \in F$. Next, we prove the uniqueness. To this end, if u and v are distinct, then by Opial's condition,

$$\begin{aligned}\lim_{n\to\infty}\| x_n - u\| &= \lim_{n_i\to\infty}\|x_{n_i} - u\| \\ &< \lim_{n_i\to\infty}\|x_{n_i} - v\| \\ &= \lim_{n\to\infty}\|x_n - v\| \\ &= \lim_{n_i\to\infty}\|x_{n_i} - v\| \\ &< \lim_{n_i\to\infty}\|x_{n_i} - u\| \\ &= \lim_{n\to\infty}\| x_n - u\|.\end{aligned}$$

This contradiction completes the proof.

Theorem 2. Let *E* be a uniformly convex Banach space such that its dual E^* satisfies the Kadec-Klee property. Let *C*, *S*, *T* and $\{x_n\}$ be as taken in Lemma 8. If $F(T) \neq \phi$, then $\{x_n\}$ converges weakly to a common fixed point of *S* and *T*.

Proof: By the boundedness of $\{x_n\}$ and reflexivity of *E*, we have a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some *p* in *C*. By Lemma 8, we have $\lim_{i\to\infty}\|S\, x_{n_i} - x_{n_i}\| = 0 = \lim_{i\to\infty}\|T\, x_{n_i} - x_{n_i}\|$. This gives $p \in F$. To prove that $\{x_n\}$ converges weakly to some *p*, suppose that $\{x_{n_k}\}$ is another subsequence of $\{x_n\}$ in *C* that converges weakly to some *q* in *C*. Then by Lemmas 8 and 4, $p, q \in W \cap F$ where $W = \omega_W(\{x_n\})$. Since $\lim_{n\to\infty}\|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$ by Lemma 9, $p = q$ by Lemma 3. Consequently, $\{x_n\}$ converges weakly to $p \in F$.

By putting $S = T$ in the above two theorems, we have the following corollaries.

Corollary 1. Let *E* be a uniformly convex Banach space satisfying the Opial's condition and *C*, *T* be as taken in Lemma 8 and $\{x_n\}$ as in (4). If $F(T) \neq \phi$, then $\{x_n\}$ converges weakly to a common fixed point of *T*.

We now turn to strong convergence theorems. Our first result is in this direction, in a general real Banach space and goes as follows:

Theorem 3. Let *E* be a real Banach space and $C, \{x_n\}, S, T$ be as taken in Lemma 8. If $F(T) \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of *T* if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$ where $d(x,F) = \inf\{\|x - p\| : p \in F\}$.

Proof: Necessity is above. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. As proved in Lemma 8, we have

$$\|x_{n+1} - p\| \leq k_n \|x_n - p\|.$$

This gives

$$d(x_{n+1}, F) \leq k_n d(x_n, F),$$

So that $\lim_{n\to\infty} d(x_n, F)$ exists. But by hypothesis $\liminf_{n\to\infty} d(x_n, F)$, therefore we must have $\lim_{n\to\infty} d(x_n, F) = 0.$

Next we show that $\{x_n\}$ is a Cauchy sequence in *C*. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists n_0 in *N* such that for all $n \geq n_0$, we have

$$d(x_n, F) < \frac{\varepsilon}{4}.$$

In particular, $inf\left\{\|x_{n_0} - p\| : p \in F\right\} < \frac{\varepsilon}{4}$. There must exist $p^* \in F$ such that $\|x_{n_0} - p^*\| < \frac{\varepsilon}{4}$.

Now for $n, m \geq n_0$ we have

$$\|n_{n+m} - x_n\| \le \|x_{n+m} - P^*\| + \|x_n - P^*\|$$
$$\le 2\|x_n - P^*\|$$
$$\le 2\left(\frac{\varepsilon}{4}\right) = \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset C of a Banach space E, therefore it must converge in C. Let $\lim_{n\to\infty} x_n = q$. Now $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(x, F) = 0$. It is well-known that F is closed and so $q \in F$.

We now modify the so-called Condition (A') given by Fukhar-ud-din and Khan [11] and call it Condition (A^*) as follows:

Two mappings $S, T : C \to C$ are said to satisfy Condition (A^*) if there exists a nondecreasing function $f : (0, \infty] \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|Sx - Tx\| \ge f(d(x, F))$ for all $x \in C$.

Note that Condition (A^*) reduces to Condition (A') when one of S and T is identity mapping.

Our next theorem is an application of Theorem 3 and makes use of the Condition (A^*).

Theorem 4. Let E be a uniformly convex Banach space and $C, \{x_n\}$ be as taken in Lemma 8. Let $S, T : C \to C$ be jointly asymptotically nonexpansive mappings satisfying Condition (A^*). If $F(T) \ne \phi$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.

Proof: By Lemma 8, $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F$. Let it be c for some $c \ge 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$. Now $\|x_{n+1} - x^*\| \le k_n \|x_n - x^*\|$ gives that $d(x_{n+1}, F) \le k_n d(x_n.F)$ and so $\lim_{n\to\infty} \|Sx_n - Tx_n\| = 0.$ exists. Moreover, by Lemma 7,

Using condition (A^*)

$$\lim_{n\to\infty} \|Sx_n - Tx_n\| = 0.$$
$$\lim_{n\to\infty} f(d(x_n, F)) \ge \lim_{n\to\infty} \|Sx_n - Tx_n\| = 0.$$

That is,

$$\lim_{n\to\infty} f(d(x_n, F)) = 0$$

Since f is a nondecreasing function and $f(0) = 0$, $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Now Theorem 3 gives the result.

Corollary 3. Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E. Let $S, T : C \to C$ satisfying condition (A') and

$$\|S^n x - S^n y\| \le k_n \|x - y\|,$$
$$\|T^n x - T^n y\| \le k_n \|x - y\|$$

for all $n \in N$, where $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Construct an iterative process $\{x_n\}$ as in (3) with $0 < 1 - \delta \le a_n, b_n \le \delta < 1$ for all $n \in N$. If $F = F(S) \cap F(T) \ne \emptyset$ then $\{x_n\}$ converges strongly to a common fixed point of S and T.

Remark: Theorems 1-4 set analogues of the corresponding results in [12] and [13] for two jointly asymptotically nonexpansive mappings on a bounded domain.

Open Question: Can Theorems 1-4 be proved on an unbounded domain in a uniformly convex Banach space E?

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