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Subdıvısıons of the spectra for cesaro, rhaly and weighted mean operators on c_0 ,c and ℓ^p

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Abstract

There are many different ways to subdivide the spectrum of a bounded linear operator; some of them are motivated by applications to physics (in particular, quantum mechanics). In this study, the relationship between the subdivisions of spectrum which are not required to be disjoint and Goldberg's classification are given. Moreover, these subdivisions for some summability methods are studied.

***Keywords**:* Spectrum, fine spectrum, approximate point spectrum, defect spectrum, compression spectrum, weighted mean operators, Rhaly operators, Cesáro operators.

1. Introduction

Let X be a Banach space and $B(X)$ denote the linear space of all bounded linear operators on X. Given an operator $L \in B(X)$, the set

$$\rho(L) := \{\lambda \in K : \lambda I - L \text{ bijection}\} \quad (1.1)$$

is called the resolvent set of L (where $\mathrm{K} = \mathrm{R}$ or $\mathrm{K} = \mathrm{C}$), its complement

$$\sigma(L) := \mathrm{K} \setminus \rho(L) \quad (1.2)$$

the spectrum of L. We denote the operator $R(\lambda; L)$ as follows:

$$R(\lambda; L) := (\lambda I - L)^{-1}. \quad (1.3)$$

By the closed graph theorem, the inverse operator

$$R(\lambda; L) := (\lambda I - L)^{-1} \quad (\lambda \in \rho(\mathrm{L}))$$

is always bounded; this operator is usually called resolvent operator of L at λ.

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1.1. Subdivision of the spectrum: The point spectrum, continuous spectrum and residual spectrum

Let X be a Banach space over K and $L \in B(X)$. Recall that a number $\lambda \in \mathrm{K}$ is called an eigenvalue of L if the equation $Lx = \lambda x$ has a nontrivial solution $x \in X$. Any such x is then called eigenvector, and the set of all eigenvectors is a subspace of X called eigenspace.

Throughout the following, we will call the set of eigenvalues

$$\sigma_p(L) := \{\lambda \in \mathrm{K} : Lx = \lambda x \text{ for some } x \neq 0\}. \quad (1.4)$$

We say that $\lambda \in \mathrm{K}$ belongs to the continuous spectrum $\sigma_c(L)$ of L if the resolvent operator (1.3) is defined on a dense subspace of X and is unbounded. Furthermore, we say that $\lambda \in \mathrm{K}$ belongs to the residual spectrum $\sigma_r(L)$ of L if the resolvent operator (1.3) exists, but its domain of definition (i.e. the range $R(\lambda I - L)$ of $(\lambda I - L)$ is not dense in X; in this case $R(\lambda; L)$ may be bounded or unbounded. Together with the point spectrum (1.4), these two subspectra form a disjoint subdivision

$$\sigma(L) = \sigma_p(L) \cup \sigma_r(L) \cup \sigma_c(L) \quad (1.5)$$

of the spectrum of L. Loosely speaking, the elements λ in the subspectrum $\sigma_p(L)$ characterize some lack of injectivity, those in $\sigma_r(L)$ lack of surjectivity, and those in $\sigma_c(L)$ lack of stability of the operator $\lambda I - L$. We illustrate the subdivision (1.5) in Table 1.

Observe that the case in the first row and second column cannot occur in a Banach space X by the closed graph theorem. If we are not in the third column, i.e., if λ is not an eigenvalue of L, we may always consider the resolvent operator (1.3) (on a possibly "thin" domain of definition) as "algebraic" inverse of $\lambda I - L$.

Table 1. Disjoint subdivision of spectrum

	$R(\lambda;L)$ exists and is bounded	$R(\lambda;L)$ exists and is unbounded	$R(\lambda;L)$ does not exists
$R(\lambda I-L)=X$	$\lambda\in\rho(L)$	–	$\lambda\in\sigma_p(L)$
$\overline{R(\lambda I-L)}=X$	$\lambda\in\rho(L)$	$\lambda\in\sigma_c(L)$	$\lambda\in\sigma_p(L)$
$\overline{R(\lambda I-L)}\neq X$	$\lambda\in\sigma_r(L)$	$\lambda\in\sigma_r(L)$	$\lambda\in\sigma_p(L)$

1.2. The approximate point spectrum, defect spectrum and compression spectrum

Given a bounded linear operator L in a Banach space X, we call a sequence $(x_k)_k$ in X a Weyl sequence for L if $\|x_k\|=1$ and $\|Lx_k\|\to 0$ as $k\to\infty$.

In what follows, we call the set

$$\sigma_{ap}(L):=\left\{\begin{array}{l}\lambda\in \mathrm{K} : \text{there is a Weyl}\\ \text{sequence for } \lambda I - L\end{array}\right\} \quad (1.6)$$

the approximate point spectrum of L. Moreover, the subspectrum

$$\sigma_\delta(L):=\left\{\begin{array}{l}\lambda\in \mathrm{K} : \lambda I - L\\ \text{is not surjective}\end{array}\right\} \quad (1.7)$$

is called defect spectrum of L.

By definition, we then have $\|\lambda x - Lx\|\geq c\|x\|$ for all $x\in X$ if $\lambda\notin\sigma_{ap}(L)$; equivalently, this may be stated as

$$\inf\{\|\lambda e - Le\| : e\in S(X)\}>0 \quad (\lambda\notin\sigma_{ap}(L)) \quad (1.8)$$

where $S(X):=\{x\in X : \|x\|=r\}$. The two subspectra (1.6) and (1.7) form a (not necessarily disjoint) subdivision

$$\sigma(L)=\sigma_{ap}(L)\cup\sigma_\delta(L) \quad (1.9)$$

of the spectrum. There is another subspectrum,

$$\sigma_{co}(L):=\left\{\lambda\in\mathrm{K} : \overline{R(\lambda I - L)}\neq \mathrm{X}\right\} \quad (1.10)$$

which is often called compression spectrum in the literature and which gives rise to another (not necessarily disjoint) decomposition

$$\sigma(L)=\sigma_{ap}(L)\cup\sigma_{co}(L) \quad (1.11)$$

of the spectrum. Clearly, $\sigma_p(L)\subseteq\sigma_{ap}(L)$ and $\sigma_{co}(L)\subseteq\sigma_\delta(L)$. Moreover, comparing these subspectra with those in (1.5) we note that

$$\sigma_r(L)=\sigma_{co}(L)\setminus\sigma_p(L) \quad (1.12)$$

and

$$\sigma_c(L)=\sigma(L)\setminus\left[\sigma_p(L)\cup\sigma_{co}(L)\right] \quad (1.13)$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint, building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints.

Proposition 1.1. ([1], Proposition 1.3). The spectra and subspectra of an operator $L\in B(X)$ and its adjoint $L^*\in B(X^*)$ are related by the following relations:

(a) $\sigma\left(L^{*}\right)=\sigma\left(L\right)$, (b) $\sigma_{c}\left(L^{*}\right)\subseteq\sigma_{ap}\left(L\right)$,

(c) $\sigma_{ap}\left(L^{*}\right)=\sigma_{\delta}\left(L\right)$,

(d) $\sigma_{\delta}\left(L^{*}\right)=\sigma_{ap}\left(L\right)$, (e) $\sigma_{p}\left(L^{*}\right)=\sigma_{co}\left(L\right)$,

(f) $\sigma_{co}\left(L^{*}\right)\supseteq\sigma_{p}\left(L\right)$,

(g) $\sigma\left(L\right)=\sigma_{ap}\left(L\right)\cup\sigma_{p}\left(L^{*}\right)=\sigma_{p}\left(L\right)\cup\sigma_{ap}\left(L^{*}\right)$.

The relations (c)-(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum.

The last equation (g) implies, in particular, that $\sigma(L)=\sigma_{ap}(L)$ if X is a Hilbert space and L is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (see [1]).

1.3. Goldberg's classification of spectrum

If X is a Banach space, $B(X)$ denotes the collection of all bounded linear operators on X and $T\in B(X)$, so there are three possibilities for $R(T)$, the range of T :

(I) $R(T)=X$, (II) $\overline{R(T)}=X$, but $R(T)\neq X$, (III) $\overline{R(T)}\neq X$.

and three possibilities for T^{-1} :

(1) T^{-1} exists and is continuous, (2) T^{-1} exists but is discontinuous, (3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3 If an operator is in state III_2 for example, then $\overline{R(T)}\neq X$ and T^{-1} exist but are discontinuous (see [2]).

The relationship between the fine spectrum of bounded linear operator and fine spectrum of its adjoint is given by Fig. 1.

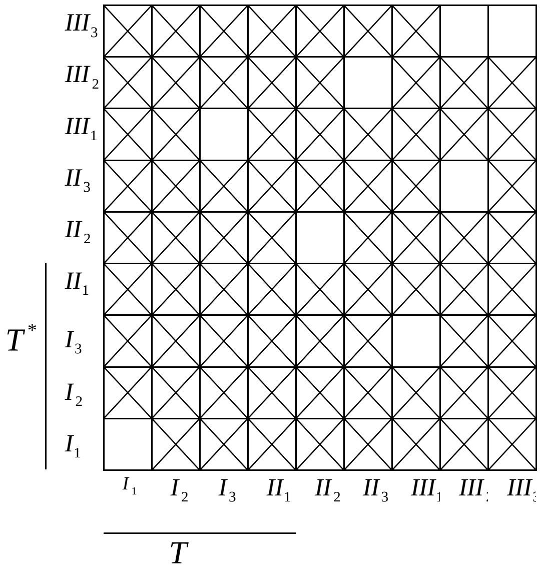

Fig. 1. State diagram for $B(X)$ and $B(X^{*})$ for a non-reflective Banach space X

If λ is a complex number such that $T=\lambda I-L\in I_1$ or $T=\lambda I-L\in II_1$, then $\lambda\in\rho(L,X)$. All scalar values of λ not in $\rho(L,X)$ comprise the spectrum of L. The further classification of $\sigma(L,X)$ gives rise to the fine spectrum of L .That is, $\sigma(L,X)$ can be divided into the subsets $I_2\sigma(L,X)=\emptyset, I_3\sigma(L,X), II_2\sigma(L,X), II_3\sigma(L,X)$, $III_1\sigma(L,X), III_2\sigma(L,X), III_3\sigma(L,X)$. For example, if $T=\lambda I-L$ is in a given state, III_2 (say), then we write $\lambda\in III_2\sigma(L,X)$.

By the definitions given above, Table 2 can be generalized as shown below.

Table 2. The relationship between subdivisions of the spectrum and Golberg's classification

		1	2	3
		$R(\lambda;L)$ exists and is bounded	$R(\lambda;L)$ exists and is unbounded	$R(\lambda;L)$ does not exists
I	$R(\lambda I-L)=X$	$\lambda\in\rho(L)$	–	$\lambda\in\sigma_p(L)$ $\lambda\in\sigma_{ap}(L)$
II	$\overline{R(\lambda I-L)}=X$	$\lambda\in\rho(L)$	$\lambda\in\sigma_c(L)$ $\lambda\in\sigma_{ap}(L)$ $\lambda\in\sigma_\delta(L)$	$\lambda\in\sigma_p(L)$ $\lambda\in\sigma_{ap}(L)$ $\lambda\in\sigma_\delta(L)$
III	$\overline{R(\lambda I-L)}\neq X$	$\lambda\in\sigma_r(L)$ $\lambda\in\sigma_\delta$	$\lambda\in\sigma_r(L)$ $\lambda\in\sigma_{ap}(L)$ $\lambda\in\sigma_\delta(L)$ $\lambda\in\sigma_{co}(L$	$\lambda\in\sigma_p(L)$ $\lambda\in\sigma_{ap}(L)$ $\lambda\in\sigma_\delta(L)$ $\lambda\in\sigma_{co}(L)$

Let $w; c_0; c; \ell^p$ denote the set of all sequences; the space of all null sequences; convergent sequences; sequences such that $\sum_k |x_k|^p<\infty$, respectively.

An infinite matrix A is said to be conservative if it is a selfmap of c, the space of convergent sequences. Necessary and sufficient conditions for A to be conservative are the well-known Kojima-Schur conditions; i.e.,

(i). $\|A\|:=\sup_n \sum_{k=0}^{\infty}|a_{nk}|<\infty;$

(ii). $\lim_n a_{nk}=:\alpha_k$, exists for each k, and

(iii). $t:=\lim_n \sum a_{nk}<\infty$ exists.

Associated with each conservative matrix A is a function χ defined by $\chi(A)=t-\sum\alpha_k$. If $\chi(A)\neq 0$, A is called coregular, and, if $\chi(A)=0$ then A is called conull. A matrix $A=(a_{nk})$ is said to be regular if $\lim_A x=\lim x$ for each $x\in c$. If $\alpha_k=0$ for each k and $t=1$ in (iii), then the operator A is called regular (see [3]).

2. The approximate point spectrum, defect spectrum and compression spectrum of C_1

In this section we devoloped the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $C_1=(c_{nk})$, where otherwise $c_{nk}=1/(n+1)$, $k\le n$ and $c_{nk}=0$, n,k. Reade [4] and in 1975, Wenger [5] determined spectra and the fine spectra of Cesaro operator C_1 on c, the space of convergent sequences, respectively.

2.1. Subdivision of the spectrum of C_1 on c.

Theorem 2.1.

(a) $\sigma_{ap}(C_1,c)=\{\lambda\in C:|\lambda-1/2|=1/2\}$,

(b) $\sigma_\delta(C_1,c)=\{\lambda\in C:|\lambda-1/2|\le 1/2\}$,

(c) $\sigma_\delta(C_1,c)=\{\lambda\in C:|\lambda-1/2|<1/2\}\cup\{1\}$.

Proof:

(a) $\sigma_{ap}(C_1,c)=\sigma(C_1,c)\setminus III_1\sigma(C_1,c)$ is obtained from Table 2. By [[5], Theorem 1-3], we have

$$\begin{aligned}\sigma_{ap}(C_1,c) &=\left\{\lambda\in C:\mathrm{Re}\frac{1}{\lambda}\ge 1\right\}\setminus\left\{\lambda\in C:\mathrm{Re}\frac{1}{\lambda}>1\right\}\\ &=\left\{\lambda\in C:\mathrm{Re}\frac{1}{\lambda}=1\right\}\\ &=\{\lambda\in C:|\lambda-1/2|=1/2\}.\end{aligned}$$

(b) $\sigma_\delta(C_1,c)=\sigma(C_1,c)\setminus I_3\sigma(C_1,c)$ is obtained from Table 2. Moreover, since the equality $\sigma(C_1,c)=III_1\sigma(C_1,c)\cup II_2\sigma(C_1,c)\cup III_3\sigma(C_1,c)$ holds by [[5], Theorem 1-5] and the subdivisions in Goldberg's classification are disjoint, then the equalitises $I_3\sigma(C_1,c)=\varnothing$, $II_3\sigma(C_1,c)=\varnothing$, $III_2\sigma(C_1,c)=\varnothing$ are valid. Hence $\sigma_\delta(C_1,c)=\sigma(C_1,c)$.

(c) From Table 2,

$$\sigma_{co}(C_1,c)=\sigma(C_1,c)\setminus\left\{\begin{matrix}II_2\sigma(C_1,c)\cup II_3\sigma\\(C_1,c)\cup I_3\sigma(C_1,c)\end{matrix}\right\}$$

and by
$I_3\sigma(C_1,c)=\varnothing,\ II_3\sigma(C_1,c)=\varnothing,$ we have

$$\begin{aligned}\sigma_{co}(C_1,c)&=\left\{\lambda\in C:\operatorname{Re}\frac{1}{\lambda}\geq 1\right\}\setminus\left\{\lambda\in C:\operatorname{Re}\frac{1}{\lambda}=1,\lambda\neq 1\right\}\\&=\left\{\lambda\in C:\operatorname{Re}\frac{1}{\lambda}>1\right\}\cup\{1\}\\&=\{\lambda\in C:|\lambda-1/2|<1/2\}\cup\{1\}.\end{aligned}$$

The following corollary can be obtained by Proposition 1.1.

Corollary 2.1.
(a) $\sigma_{ap}(C_1^*,\ell^1)=\{\lambda\in C:|\lambda-1/2|\leq 1/2\}$,
(b) $\sigma_\delta(C_1^*,\ell^1)=\{\lambda\in C:|\lambda-1/2|=1/2\}$.

2.2. Subdivision of the spectrum of C_1 on ℓ^p.

In 1985, M. Gonzales [6] determined the fine spectra of Cesaro operator C_1 on ℓ^p.

Theorem 2.2. [6] Let $1<p<\infty,\ p^{-1}+q^{-1}=1,$ and C_1 acting on ℓ^p.
(a) For each $z\in\operatorname{int}\sigma(C_1,\ell^p)=\{\lambda:|\lambda-q/2|<q/2\}$, $zI-C_1\in III_1$.
(b) For each $z\in\partial\sigma(C_1,\ell^p)=\{\lambda:|\lambda-q/2|=q/2\}$, $zI-C_1$ is injective with dense range, that is, $zI-C_1\in II_2$.

Theorem 2.3. Let $p>1$ and $p^{-1}+q^{-1}=1,$ then
(a) $\sigma_{ap}(C_1,\ell^p)=\{\lambda\in C:|\lambda-q/2|=q/2\}$,
(b) $\sigma_\delta(C_1,\ell^p)=\{\lambda\in C:|\lambda-q/2|\leq q/2\}$,
(c) $\sigma_{co}(C_1,\ell^p)=\{\lambda\in C:|\lambda-q/2|<q/2\}$.

Proof: The equality $I_3\sigma(C_1,\ell^p)=\varnothing$ is clear with Theorem 2.2. Therefore, the proof is taken by Theorem 2.2.
The following corollary can be obtained by Proposition 1.1.

Corollary 2.2. Let $p>1$ and $p^{-1}+q^{-1}=1,$ then
(a) $\sigma_{ap}(C_1^*,\ell^q)=\{\lambda\in C:|\lambda-q/2|\leq q/2\}$,
(b) $\sigma_\delta(C_1^*,\ell^q)=\{\lambda\in C:|\lambda-q/2|=q/2\}$.

3. The approximate point spectrum, defect spectrum and compression spectrum of rhaly operator

We assume that, given a scalar sequence of $a=(a_n)$, a Rhaly matrix $R_a=(a_{nk})$ is the lower triangular matrix where $a_{nk}=a_n,\ k\leq n$ and $a_{nk}=0$ otherwise.
(a) $L=\lim_n(n+1)a_n$ exists, finite,
(b) $a_n>0$ for all n, and
(c) $a_i\neq a_j$ for $i\neq j$. Let S denote the set $\{a_n:n=0,1,2,\ldots\}$.
(d) $a=(a_n)$ is monotone decreasing.

In [7], the spectrum of the Rhaly operators on c_0 and c, under the assumption that $\lim_n(n+1)a_n=L\neq 0$ has been determined. Also, in [8-12] the spectrum of the Rhaly operator over some kinds of spaces has been determined.

3.1. Subdivision of the spectrum of R_a on c_0 for $L=0$.

Theorem 3.1. If $L=\lim_n(n+1)a_n=0,$ then
(a) $\sigma_{ap}(R_a,c_0)=S\cup\{0\}$,
(b) $\sigma_\delta(R_a,c_0)=S\cup\{0\}$, (c) $\sigma_{co}(R_a,c_0)=S$.

Proof: The proof is taken by [[9], Theorem 5-7].
The following corollary can be obtained by Proposition 1.1.

Corollary 3.1. (a) $\sigma_{ap}(R_a^*,\ell^1)=S\cup\{0\}$, (b) $\sigma_\delta(R_a^*,\ell^1)=S\cup\{0\}$.

3.2. Subdivision of the spectrum of R_a on c for $L=0$.

Theorem 3.2. If $L=\lim_n(n+1)a_n=0,$ then
(a) $\sigma_{ap}(R_a,c)=S\cup\{0\}$,

(b) $\sigma_\delta\left(R_a, c\right) = S \cup \{0\}$,

(c) $\sigma_{co}\left(R_a, c\right) = S \cup \{0\}$.

Proof: The proof is taken by [[9], Theorem 12, 14, 15].

The following corollary can be obtained by Proposition 1.1.

Corollary 3.2. (a) $\sigma_{ap}\left(R_a^*, \ell^1\right) = S \cup \{0\}$, (b) $\sigma_\delta\left(R_a^*, \ell^1\right) = S \cup \{0\}$.

3.3. Subdivision of the spectrum of R_a on ℓ^p for $L = 0$.

Leibowitz [[13], Proposition 3.1] shows that

(a) If $\{(n+1)a_n\}$ is bounded, then R_a acts boundedly on ℓ^p for $p > 1$, and $\|R_a\| \le (p/(p-1)) \sup_n |(n+1)a_n|$.

(b) If $\lim_n |(n+1)a_n| = 0$, then R_a is compact operator on ℓ^p for every $p > 1$.

(c) If $\lim_n |(n+1)a_n| = \infty$, then R_a is not bounded on ℓ^p for every $p > 1$.

Theorem 3.3. If $L = \lim_n (n+1)a_n = 0$, then (a) $\sigma_{ap}\left(R_a, \ell^p\right) = S \cup \{0\}$ for $p \ge 2$,

(b) $\sigma_\delta\left(R_a, \ell^p\right) = S \cup \{0\}$ for $p \ge 2$, (c) $\sigma_{co}\left(R_a, \ell^p\right) = S$ for $p \ge 2$.

Proof: The proof is taken by [[10], Theorem 2.3-2.5].

The following corollary can be obtained by Proposition 1.1.

Corollary 3.3. If $L = \lim_n (n+1)a_n = 0$, then

(a) $\sigma_{ap}\left(R_a^*, \ell^q\right) = S \cup \{0\}$ for $p \ge 2$, $p^{-1} + q^{-1} = 1$,

(b) $\sigma_\delta\left(R_a^*, \ell^q\right) = S \cup \{0\}$ for $p \ge 2$, $p^{-1} + q^{-1} = 1$.

3.4. Subdivision of the spectrum of R_a on c for $0 < L < \infty$.

Theorem 3.4. Let $0 < L < \infty$, then

(a) $\sigma_{ap}\left(R_a, c\right) = \left\{\begin{matrix}\lambda : |\lambda - L/2| \\ = L/2\end{matrix}\right\} \cup \{a_i \in S : a_i \ge L\}$,

(b) $\sigma_\delta\left(R_a, c\right) = \{\lambda : |\lambda - L/2| \le L/2\} \cup S$,

(c) $\sigma_{co}\left(R_a, c\right) = \{\lambda : |\lambda - L/2| < L/2\} \cup S \cup \{L\}$.

Proof:

(a) Since $\sigma_{ap}\left(R_a, c\right) = \sigma\left(R_a, c\right) \setminus III_1\sigma\left(R_a, c\right)$,

$$\sigma_{ap}\left(R_a, c\right) = \left[\{\lambda : |\lambda - L/2| = L/2\} \cup S\right] \setminus \left[\begin{matrix}\{\{\lambda : L\,\mathrm{Re}(1/\lambda) > 1\} \setminus S\} \\ \cup \{a_i \in S : a_i \ge L\}\end{matrix}\right]$$

$$= \left[\{\lambda : |\lambda - L/2| \le L/2\} \setminus \{\lambda : |\lambda - L/2| < L/2\}\right] \cup \{a_i \in S : a_i \ge L\}$$

$$= \{\lambda : |\lambda - L/2| = L/2\} \cup \{a_i \in S : a_i \ge L\}$$

is obtained by [[7], Theorem 3.3 and [14] Theorem 2.1].

(b) Since $\sigma_\delta\left(R_a, c\right) = \sigma\left(R_a, c\right) \setminus I_3\sigma\left(R_a, c\right)$ from Table 2 and $I_3\sigma\left(R_a, c\right) = \varnothing$ is obtained by [[7], Theorem 3.3 and [14] Theorem 2.1-2.4], then $\sigma_\delta\left(R_a, c\right) = \sigma\left(R_a, c\right)$.

(c) Since the equality $\sigma_{co}\left(R_a, c\right) = III_1\sigma\left(R_a, c\right) \cup III_2\sigma\left(R_a, c\right) \cup III_3\sigma\left(R_a, c\right)$ from Table 2, it can be easily seen by [[14], Theorem 2.1-2.4].

The following corollary can be obtained by Proposition 1.1.

Corollary 3.4.

(a) $\sigma_{ap}\left(R_a^*, \ell^1\right) = \{\lambda : |\lambda - L/2| = L/2\} \cup S$,

(b) $\sigma_\delta\left(R_a^*, \ell^1\right) = \{\lambda \in \mathrm{R} : |\lambda - L/2| \le L/2\} \cup S$.

3.5. Subdivision of the spectrum of R_a on ℓ^p for $0 < L < \infty$.

Theorem 3.5.

Let $0 < L < \infty$, $p > 1$ and $p^{-1} + q^{-1} = 1$, then

(a) $\sigma_{ap}\left(R_a, \ell^p\right) = \{\lambda : |\lambda - qL/2| = qL/2\} \cup S$,

(b) $\sigma_\delta\left(R_a, \ell^p\right) = \{\lambda : |\lambda - qL/2| \le qL/2\} \cup S$,

(c) $\sigma_{co}\left(R_a, \ell^p\right) = \{\lambda : |\lambda - qL/2| < qL/2\} \cup S$.

Proof: The proof is taken by [[10], Theorem 3.3] and [[15], Theorem 6-8].

The following corollary can be obtained by Proposition 1.1.

Corollary 3.5.
Let $0 < L < \infty$, $p > 1$ and $p^{-1} + q^{-1} = 1$, then
(a) $\sigma_{ap}\left(R_a^*, \ell^q\right) = \left\{\lambda : |\lambda - qL/2| = qL/2\right\} \cup S$,
(b) $\sigma_\delta\left(R_a^*, \ell^q\right) = \left\{\lambda \in R : |\lambda - qL/2| \le qL/2\right\} \cup S.$

4. The approximate point spectrum, defect spectrum and compression spectrum of weıght mean operator

A weight mean matrix A is a lower triangular matrix with entries $a_{nk} = p_k / P_n$, where $p_0 > 0,\ p_n \ge 0$ for $n > 0$, and $P_n = \sum_{k=0}^{n} p_k$. The necessary and sufficient condition for the regularity of A is that $\lim P_n = \infty$.

In [16-20] the spectrum and fine spectrum of weight mean matrix over some kinds of spaces has been determined.

Let

$$\delta = \overline{\lim}_n\, p_n / P_n,\ \gamma = \underline{\lim}_n\, p_n / P_n,$$
$$S = \overline{\left\{p_n / P_n : n \ge 0\right\}}$$

and $E = \overline{\left\{p_n / P_n : p_n / P_n < \gamma / (2-\gamma)\right\}}$ in the followings.

4.1. Subdivision of the spectrum of A on c.

We shall consider those regular weighted mean methods for which $\delta = \gamma$, i.e., for which the main diagonal entries converge.

Theorem 4.1. Let A be a regular weighted mean method such that $\lim_n p_n / P_n = \gamma > 0,\ p_n / P_n \ge \gamma$ for all n sufficently large and suppose no diagonal entry of A occurs an infinite number of times, then

(a) $\sigma_{ap}(A, c) = \left\{\begin{matrix} \lambda : |\lambda - 1/(2-\gamma)| \\ = (1-\gamma)/(2-\gamma) \end{matrix}\right\} \cup E$,

(b) $\sigma_\delta(A, c) = \left\{\lambda : |\lambda - 1/(2-\gamma)| \le (1-\gamma)/(2-\gamma)\right\} \cup S$,

(c) $\sigma_{co}(A, c) = \left\{\lambda : |\lambda - 1/(2-\gamma)| < (1-\gamma)/(2-\gamma)\right\} \cup S.$

Proof: (a) Since the relation

$$III_1\sigma(A,c) = \left[\left\{\lambda : |\lambda - 1/(2-\gamma)| < (1-\gamma)/(2-\gamma)\right\} \setminus S\right] \cup \left\{\lambda = a_{nn} : \lambda/(2-\gamma) < a_{nn} < 1\right\}$$

holds by [[18] Theorem 1-2], use [[17] Corollary 2] to get

$$\begin{aligned}\sigma_{ap}(A,c) &= \sigma(A,c) \setminus III_1\sigma(A,c) \\ &= \left\{\lambda : |\lambda - 1/(2-\gamma)| = (1-\gamma)/(2-\gamma)\right\} \cup E.\end{aligned}$$

(b) $\sigma(A,c) = III_1\sigma(A,c) \cup II_2\sigma(A,c) \cup III_3\sigma(A,c)$ is easily seen by [[17] Corollary 2] and [[18] Theorem 1-4]. Therefore, $III_2\sigma(A,c) = I_3\sigma(A,c) = II_3\sigma(A,c) = \varnothing$ and hence,

$$\sigma_\delta(A,c) = \sigma(A,c) \setminus I_3\sigma(A,c) = \sigma(A,c).$$

(c) Since, $\sigma_{co}(A,c) = III_1\sigma(A,c) \cup III_2\sigma(A,c) \cup III_3\sigma(A,c)$ the result is taken by [[17] Corollary 2] and [[18] Theorem 1-4].

The following corollary can be obtained by Proposition 1.1.

Corollary 4.1. Let A be a regular weighted mean method such that $\lim_n p_n / P_n = \gamma > 0,\ p_n / P_n \ge \gamma$ for all n sufficently large and suppose no diogonal entry of A occurs an infinite number of times, then

(a) $\sigma_{ap}(A^*, \ell^1) = \left\{\lambda : |\lambda - 1/(2-\gamma)| \le (1-\gamma)/(2-\gamma)\right\} \cup S$,

(b) $\sigma_\delta(A^*, \ell^1) = \left\{\lambda : |\lambda - 1/(2-\gamma)| = (1-\gamma)/(2-\gamma)\right\} \cup E$,

(c) $\sigma_p(A^*, \ell^1) = \left\{\lambda : |\lambda - 1/(2-\gamma)| < (1-\gamma)/(2-\gamma)\right\} \cup S.$

4.2. Subdivision of the spectrum of A on ℓ^p.

In [16] it was shown that, if

$$\lim_n p_n / P_n = \gamma > 0, \quad (4.1)$$

then $A \in B(\ell^p)$ and

$$\sigma(A, \ell^p) = \left\{\lambda : |\lambda - 1/(2-\gamma)| \le (1-\gamma)/(2-\gamma)\right\} \cup S. \quad (4.2)$$

In Theorem 4.2 and Corollary 4.2 A is a weighted mean matrix satisfying $\gamma < 1$, since $\gamma = 1$ implies $\sigma(A, \ell^p) = S$.

Theorem 4.2. Let A be a regular weighted mean method such that $\gamma = \lim_n p_n / P_n$ exists and $p_n / P_n \ge \gamma$ for all n sufficently large. Suppose that no main diogonal entry of A occurs an infinite number of times, then

(a) $\sigma_{ap}\left(A,\ell^{P}\right)=\left\{\lambda:\left|\lambda-1/(2-\gamma)\right|=(1-\gamma)/(2-\gamma)\right\}\cup S,$

(b) $\sigma_{\delta}\left(A,\ell^{P}\right)=\left\{\lambda:\left|\lambda-1/(2-\gamma)\right|\leq(1-\gamma)/(2-\gamma)\right\}\cup S,$

(c) $\sigma_{co}\left(A,\ell^{P}\right)=\left\{\lambda:\left|\lambda-1/(2-\gamma)\right|<(1-\gamma)/(2-\gamma)\right\}\cup E.$

Proof: (a) Since the relation

$$III_1\sigma\left(A,\ell^{P}\right) = \left[\left\{\lambda:\left|\lambda-1/(2-\gamma)\right|<(1-\gamma)/(2-\gamma)\right\}\setminus S\right] \cup\left\{\lambda=p_n/P_n:\lambda/(2-\gamma)<p_n/P_n<1\right\}$$

holds by [[20] Theorem 1-3], then from (4.2) we have

$$\sigma_{ap}\left(A,\ell^{P}\right) =\sigma\left(A,\ell^{P}\right)\setminus III_1\sigma\left(A,\ell^{P}\right) =\left\{\lambda:\left|\lambda-1/(2-\gamma)\right|=(1-\gamma)/(2-\gamma)\right\}\cup E.$$

(b) From [[20] Theorem 1.5] and (4.2), we have $\sigma\left(A,\ell^{P}\right)=III_1\sigma\left(A,\ell^{P}\right)\cup II_2\sigma\left(A,\ell^{P}\right)\cup III_3\sigma\left(A,\ell^{P}\right).$ Therefore,

$$III_2\sigma\left(A,\ell^{P}\right)=I_3\sigma\left(A,\ell^{P}\right)=II_3\sigma\left(A,\ell^{P}\right)=\varnothing$$

Hence we get

$$\sigma_{\delta}\left(A,\ell^{P}\right)=\sigma\left(A,\ell^{P}\right)\setminus I_3\sigma\left(A,\ell^{P}\right)=\sigma\left(A,\ell^{P}\right).$$

(c) Since $\sigma_{co}\left(A,\ell^{P}\right)=III_1\sigma\left(A,\ell^{P}\right)\cup III_2\sigma\left(A,\ell^{P}\right)\cup III_3\sigma\left(A,\ell^{P}\right),$ the result is taken by [[20] Theorem1-5].

The following corollary can be obtained by Proposition 1.1.

Corollary 4.2. Let A be a regular weighted mean method such that $\gamma=\lim_n p_n/P_n$ exists and $p_n/P_n\geq\gamma$ for all n sufficently large. Suppose that no main diogonal entry of A occurs an infinite number of times, then

(a) $\sigma_{ap}\left(A^*,\ell^{q}\right)=\left\{\lambda:\left|\lambda-1/(2-\gamma)\right|\leq(1-\gamma)/(2-\gamma)\right\}\cup S,$

(b) $\sigma_{\delta}\left(A^*,\ell^{q}\right)=\left\{\lambda:\left|\lambda-1/(2-\gamma)\right|=(1-\gamma)/(2-\gamma)\right\}\cup E,$

(c) $\sigma_{p}\left(A^*,\ell^{q}\right)=\left\{\lambda:\left|\lambda-1/(2-\gamma)\right|<(1-\gamma)/(2-\gamma)\right\}\cup S,$

where $p^{-1}+q^{-1}=1.$

In [16] it was shown that, if

$$\lim_n np_n/P_n=\alpha>1/p, \tag{4.3}$$

then $A\in B\left(\ell^{p}\right)$ and

$$\sigma\left(A,\ell^{p}\right)=\left\{\begin{array}{l}\lambda:\left|\lambda-\alpha p/2(\alpha p-1)\right|\\ \leq\alpha p/2(\alpha p-1)\end{array}\right\}\cup S. \tag{4.4}$$

Theorem 4.3. Let A be a weighted mean method such that $\lim_n np_n/P_n=\alpha>1/p$. Suppose no diogonal entry of A occurs an infinite number of times, if

$$\lim_n n\Delta\left(np_n/P_n\right)=\lim_n n\left(nc_n-(n+1)c_{n+1}\right)=0,$$

then

(a) $\sigma_{ap}\left(A,\ell^{P}\right)=\left\{\lambda:\left|\lambda-\alpha p/2(\alpha p-1)\right|=\alpha p/2(\alpha p-1)\right\},$

(b) $\sigma_{\delta}\left(A,\ell^{P}\right)=\left\{\lambda:\left|\lambda-\alpha p/2(\alpha p-1)\right|\leq\alpha p/2(\alpha p-1)\right\}\cup S,$

(c) If A is a triangle, then

$$\sigma_{co}\left(A,\ell^{P}\right)=\left\{\begin{array}{l}\lambda:\left|\lambda-\alpha p/2(\alpha p-1)\right|\\ <\alpha p/2(\alpha p-1)\end{array}\right\}\cup\left[S\setminus\{0\}\right],$$

If $p_k=0$ for some $k>0$, then

$$\sigma_{co}\left(A,\ell^{P}\right)=\left\{\begin{array}{l}\lambda:\left|\lambda-\alpha p/2(\alpha p-1)\right|\\ <\alpha p/2(\alpha p-1)\end{array}\right\}\cup S\cup\{0\},$$

where $c_n=p_n/P_n.$

Proof: (a) The proof is taken by (4.4) and [[20] Theorem 6,8].

(b) From [[20] Theorem 6-9] and (4.4), $\sigma\left(A,\ell^{P}\right)=III_1\sigma\left(A,\ell^{P}\right)\cup II_2\sigma\left(A,\ell^{P}\right).$ Therefore,

$$III_2\sigma\left(A,\ell^{P}\right)=I_3\sigma\left(A,\ell^{P}\right) =II_3\sigma\left(A,\ell^{P}\right)=III_3\sigma\left(A,\ell^{P}\right)=\varnothing .$$

Hence we get

$$\sigma_{\delta}\left(A,\ell^{P}\right)=\sigma\left(A,\ell^{P}\right)\setminus I_3\sigma\left(A,\ell^{P}\right)=\sigma\left(A,\ell^{P}\right).$$

(c) Since,

$$\sigma_{co}\left(A,\ell^{P}\right)=III_1\sigma\left(A,\ell^{P}\right)\cup III_2\sigma\left(A,\ell^{P}\right)\cup III_3\sigma\left(A,\ell^{P}\right)$$

from [[20] Theorem 6-9], the result is taken.

The following corollary can be obtained by Proposition 1.1, such that $p^{-1}+q^{-1}=1.$

Corollary 4.3. Let A be a weighted mean method such that $\lim_n np_n/P_n=\alpha>1/p$. Suppose no diogonal entry of A occurs an infinite number of times, if

$$\lim_n n\Delta\left(np_n / P_n\right) = \lim_n n\left(nc_n - \left(n+1\right)c_{n+1}\right) = 0,$$

then

(a) $\sigma_{ap}\left(A^*, \ell^q\right) = \left\{\lambda : \left|\lambda - \alpha p / 2\left(\alpha p - 1\right)\right| \le \alpha p / 2\left(\alpha p - 1\right)\right\} \cup S,$

(b) $\sigma_{\delta}\left(A^*, \ell^q\right) = \left\{\lambda : \left|\lambda - \alpha p / 2\left(\alpha p - 1\right)\right| = \alpha p / 2\left(\alpha p - 1\right)\right\},$

(c) If A is a triangle, then

$$\sigma_p\left(A^*, \ell^q\right) = \left\{\begin{matrix}\lambda : \left|\lambda - \alpha p / 2\left(\alpha p - 1\right)\right| \\ < \alpha p / 2\left(\alpha p - 1\right)\end{matrix}\right\} \cup \left[S \setminus \{0\}\right],$$

If $p_k = 0$ for some $k > 0$, then

$$\sigma_p\left(A^*, \ell^q\right) = \left\{\begin{matrix}\lambda : \left|\lambda - \alpha p / 2\left(\alpha p - 1\right)\right| \\ < \alpha p / 2\left(\alpha p - 1\right)\end{matrix}\right\} \cup S \cup \{0\},$$

where $c_n = p_n / P_n$.

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