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The measurement of returns to scale under a simultaneous occurrence of multiple solutions in a reference set and a supporting hyperplane with weight restrictions

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Abstract

Sueyoshi and Sekitani in 2007 presented a paper that explores the measurement of Returns to Scale under a possible occurrence of multiple solution in a reference set and a supporting hyperplane. The occurrence of multiple solutions is classified into Type I and Type II. Type I is an occurrence of multiple solutions in a reference set. Type II is an occurrence of multiple solutions on a supporting hyperplane passing on the reference set. In this paper we want to apply their method for estimating RTS under weight restrictions. For this goal we use tone's method for estimating RTS under weight restrictions.

Keywords: Data envelopment analysis; returns to scale; weight restrictions

1. Introduction

Data envelopment analysis is a nonparametric technique for measuring and evaluating the relative efficiencies of decision making units with multiple inputs and multiple outputs. Specifically, it determines a set of weights such that the efficiency of a target DMU relative to the other DMUs is maximized [1]. As we know, the imposition of weight restrictions has been recognized as one of the important factors when applying DEA to actual situations, and several models have been developed for this purpose, [2, 3]. Therefore, determining the returns to scale status (constant, increasing, or decreasing returns to scale) under weight restrictions is important. The occurrence of multiple solutions is classified into Type I and Type II. Type I is an occurrence of multiple solutions in a reference set. Type II is an occurrence of multiple solutions on a supporting hyperplane passing on the reference set. The two types of multiple solutions influence a degree of RTS in the DEA measurement, [4]. We want to explore this issue under weight restrictions.

2. Preliminaries

In this paper, we focus on the input-oriented weighted DEA models. Suppose that we have n DMUs, every DMU_j, $j=1,..,n$ producing the same s outputs in (possibly) different amounts, y_{rj}, $r=1,..,s$ using the same m inputs, x_{ij}, $i=1,..,m$ also in (possibly) different amounts. All inputs and outputs are assumed to be nonnegative, but at least one input and one output are positive. We define $X=[x_1,..,x_m]$ as the m × n matrix of inputs and $Y=[y_1,..,y_n]$ as the s × n matrix of outputs. There are many alternative ways to characterize production technology. The most general representation is production possibility set T which is defined as a set of semipositive (x,y) as T={(x, y) | x can produce y}.

3. Tone's method

Consider the following linear program weighted BCC model as presented in Tone's paper:

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$$
\begin{aligned}
&\mathrm{DWR_0}: \\
&\max \quad uy_0 + u_0 \\
&vx_0 = 1 \\
&uy - vx - u_0 \le 0 \\
&vp \le 0 \qquad (a) \\
&uQ \le 0 \qquad (b) \\
&u, v \ge 0 \ , u_0 \ \text{free}
\end{aligned}
\qquad (1)
$$

DWR_0 where p and Q are associated with weight restrictions. The dual of the model represented in (1) is obtained from the same data which are then used in the following form:

$$
\begin{aligned}
&\mathrm{WR_0}: \\
&\theta_w^* = \min \theta_w \\
&x\lambda - p\pi \le \theta_w x_0 \\
&y\lambda + Q\tau \ge y_0 \\
&1\lambda = 1 \\
&\lambda, \pi, \tau \ge 0
\end{aligned}
\qquad (2)
$$

where π and τ are dual variables corresponding to constraints (a) and (b), respectively.

We define WR-efficiency as Definition 1 (WR-efficiency). A DMUo is WR-efficient if and only if $\theta_w^{**} = 1$ satisfies:

All optimal slack values in model (2) are zero. Otherwise it is called WR-inefficient. Suppose DMUo is WR-efficient. Let the inf and sup uo, u_0^- *and* u_0^+ for the optimal solution of DWRo be

$$
\begin{aligned}
&u_0^+ = \sup \ u_0 \\
&u_0^- = \inf \ u_0 \\
&vx_0 = 1 \\
&uy_j - vx_j - u_0 \le 0 \ , \ j \ne 0 \\
&uy_0 - vx_0 - u_0 = 0 \\
&vp \le 0 \\
&uQ \le 0 \\
&u, v \ge 0.
\end{aligned}
$$

We apply Tone's method for testing returns to scale as

$$
\begin{aligned}
&u_0^+ < 0 \Rightarrow IRS \\
&u_0^- > 0 \Rightarrow DRS \\
&u_0^- < 0 < u_0^+ \ \text{or} \ u_0^- = 0 = u_0^+ \Rightarrow CRS
\end{aligned}
$$

Proposition 1 (scsc conditions): There exists a pair of optimal solutions (1) and (2) that satisfies the following conditions:

$$
\begin{aligned}
&\bar{\lambda}_j^* + (u^* y_j - v^* x_j - u_0^*) > 0 && j = 1,..,n \\
&\pi_j^* - v^* p_j > 0 && j = 1,..,n \\
&\tau_j^* - u^* Q_j > 0 && j = 1,..,n \\
&v_i^* + (\bar{\theta}_w^* x_{i0} - \sum_{j=1}^{n} \bar{\lambda}_j x_{ij} + \sum_{j=1}^{n} \pi_j p_{ij}) > 0 && i = 1,..,m \\
&u_r^* + (\sum_{j=1}^{n} \bar{\lambda}_j y_{rj} + \sum_{j=1}^{n} \tau_j Q_{ij} - y_{r0}) > 0 && r = 1,..,s
\end{aligned}
$$

Proof: See the proof of SCSC in [5, 6] by expanding it, the desired result is yielded.

Definition 1. (Supporting hyperplane): supporting hyperplane can be generally defined as follows: A hyperplane

$\begin{pmatrix} x^* \\ y^* \end{pmatrix}$ Is referred to as a supporting hyperplane of P on $H = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : uy - vx - u_0 = 0 \right\}$

If the hyperplane satisfies:

1) $\begin{pmatrix} x^* \\ y^* \end{pmatrix} \in \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : uy - vx - u_0 = 0 \right\} \cap P$ *and*

2) $uy - vx \le u_0$, *for all* $\begin{pmatrix} x \\ y \end{pmatrix} \in P$

Definition 2. (Face): For a convex set (S), Face is called a face of S if there exists a supporting hyperplane (H) such that Face = S ∩ H.

Proposition 2. let $\begin{pmatrix} \theta_w^* x_0 \\ y_0 \end{pmatrix}$ be the minimum face containing F_{ace}^*.

Then the minimum face is expressed by

$$
F_{ace}^* = \left\{ \begin{aligned} &\begin{pmatrix} x \\ y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} = \sum_{j \in A} \begin{pmatrix} x_j \\ y_j \end{pmatrix} \lambda_j + \sum_{j \in B} \begin{pmatrix} -P_j \\ 0 \end{pmatrix} \pi_j + \sum_{j \in C} \begin{pmatrix} 0 \\ Q_j \end{pmatrix} \tau_j \\ &+ \sum_{i \in I_v} \begin{pmatrix} s_i^- \\ 0 \end{pmatrix} - \sum_{r \in I_u} \begin{pmatrix} 0 \\ s_r^+ \end{pmatrix} \ : \ \sum_{j \in A} \bar{\lambda}^*_j = 1 \ , s_i^- \ge 0 \ , s_r^+ \ge 0 \end{aligned} \right\}
$$

$(\bar{v}^*, \bar{u}^*, \bar{u}_0^*)$ *and* $(\bar{\lambda}^*, \bar{\theta}_w^*, \bar{\pi}^*, \bar{\tau}^*)$ that optimal solutions (1) and (2):

Satisfied in SCSC conditions and

$A=\{j:\overline{\lambda}_j^*>0\}$, $B=\{j:\overline{\pi}_j^*>0\}$, $C=\{j:\overline{\tau}_j^*>0\}$
$\bar{I}_v=\{i:\overline{v}_i^*=0\}$, $\bar{I}_u=\{r:\overline{u}_r^*=0\}$

Proof: see [1].

$$F_{ace}^*=H(\overline{V}^*,\overline{U}^*,\overline{U}_0^*)\cap P=\left\{\begin{pmatrix}X\\Y\end{pmatrix}\Big|\overline{U}^*Y-\overline{V}^*X-\overline{U}_0^*=0\right\}\cap P$$

$$CSC\ \ conditions:\ \overline{\lambda}_j^*(\overline{U}^*Y-\overline{V}^*X-\overline{U}_0^*)=0 \qquad (1)$$

$$SCSC\ \ conditions:\ \overline{\lambda}_j^*+(\overline{U}^*Y-\overline{V}^*X-\overline{U}_0^*)>0 \qquad (2)$$

$$(1),(2)\ \Rightarrow \{j\ |\overline{U}_j^*Y-\overline{V}_j^*X-\overline{U}_0^*=0\}=\{\overline{\lambda}_j^*>0\}=A$$

$$\Rightarrow F_{ace}^*=\left\{\begin{matrix}\begin{pmatrix}x\\y\end{pmatrix}:\begin{pmatrix}x\\y\end{pmatrix}=\sum_{j\in A}\begin{pmatrix}x_j\\y_j\end{pmatrix}\lambda_j+\sum_{j\in B}\begin{pmatrix}-P_j\\0\end{pmatrix}\pi_j+\sum_{j\in C}\begin{pmatrix}0\\Q_j\end{pmatrix}\tau_j\\ +\sum_{i\in\bar{I}_v}\begin{pmatrix}s_i^-\\0\end{pmatrix}-\sum_{r\in\bar{I}_u}\begin{pmatrix}0\\s_r^+\end{pmatrix}\ :\ \sum_{j\in A}\overline{\lambda}^*_j=1\ ,\ s_i^-\geq 0\ ,\ s_r^+\geq 0\end{matrix}\right\}$$

4. Sueyoshi and Sekitani's method

let $\alpha_k=m+s-(\,dimension\ \ of\ \ (F_{ace}^*))-1\ then$

Proposition 3.
Even under simultaneous occurrence of types (I and II) the supporting hyperplane H of P on $\begin{pmatrix}\theta_w^*x_0\\y_0\end{pmatrix}$
Is satisfied into the following three cases:

$(a)\ if\ \ \alpha_k>0\ \ then\ H\ is\ not\ uniquely\ determined$

$(b)\ if\ \ \alpha_k=0\ \ then\ H\ is\ uniquely\ determined$

$(c)\ if\ \ \alpha_k<0$ never occurs

Definition 4. Let

$$\mathrm{n(A)=the\ number\ of}\ \{\overline{\lambda}_j^*>0\}$$

$$\mathrm{n(B)=the\ number\ of}\ \{\overline{\pi}_j^*>0\}$$

$$\mathrm{n(C)=the\ number\ of}\ \{\overline{\tau}_j^*>0\}$$

$$\mathrm{n}(\bar{\mathrm{I}}_v)=\mathrm{the\ number\ of}\ \{\overline{\mathrm{u}}_j^*>0\}$$

$$\mathrm{n}(\bar{\mathrm{I}}_u)=\mathrm{the\ number\ of}\ \{\overline{\mathrm{v}}_j^*>0\}$$

and matrix $[\mathrm{m+n+s}]\times[\mathrm{n(A)}$ $+\mathrm{n(B)+n(C)}+\mathrm{n}(\bar{\mathrm{I}}_u)$ $+\mathrm{n}(\bar{\mathrm{I}}_v)]$ of M as shown below

$$\mathrm{M}(\mathrm{A,B,C},\bar{\mathrm{I}}_u,\bar{\mathrm{I}}_v)=\underbrace{\begin{pmatrix}x_j\\y_j\\1\end{pmatrix}}_{j\in A}\cup\underbrace{\begin{pmatrix}p_j\\0\\0\end{pmatrix}}_{j\in B}\cup\underbrace{\begin{pmatrix}0\\Q_j\\0\end{pmatrix}}_{j\in C}\cup\underbrace{\begin{pmatrix}e_i\\0\\0\end{pmatrix}}_{i\in\bar{I}_v}\cup\underbrace{\begin{pmatrix}0\\e_r\\0\end{pmatrix}}_{r\in\bar{I}_u}$$

$F_{ace}^*=rank\ \ M(A,B,C,\bar{I}_v,\bar{I}_u)-1$ By definition minimum Face: dimension of

$$\alpha_k=m+s-\dim ension\ \ of\ \ (F_{ace}^*)-1$$

$$\Rightarrow\alpha_k=m+s-rank\ \ M(A,B,C,\bar{I}_v,\bar{I}_u)$$

5. An occurrence of multiple solutions (type II) in RTS measurement under type I with weight restrictions

Let $\alpha_k=\mathrm{m+s-(dimension\ \ of\ \ (F_{ace}^*))-1}$

then measurement RTS on $\begin{pmatrix}x_0\\y_0\end{pmatrix}\in\mathrm{P}$

Is classified into the following cases:

$(a)\quad if\ \ \alpha_k>0\ \ then\ \ H\ \ is\ \ not\ \ uniquely\ \ determined\ ,\ \ let$

$$\begin{aligned}&u_0^+=sup\ \ u_0\\&u_0^-=inf\ \ \ u_0\\&vx_0=1\\&uy_j-vx_j-u_0\leq 0\ ,\ j\neq 0\\&uy_0-vx_0-u_0=0\\&vp\leq 0\\&uQ\leq 0\\&u,v\geq 0\end{aligned}$$

is determined as: $\begin{pmatrix}x_0\\y_0\end{pmatrix}$ then the type of RTS in

$u_0^+<0\ \Rightarrow IRS$

$u_0^->0\ \Rightarrow DRS$

$u_0^-<0<u_0^+\ \ or\ \ u_0^-=0=u_0^+\ \Rightarrow CRS$

$(b)\quad if\ \ \alpha_k=0\ \ then\ \ H\ \ is\ \ uniquely\ \ determined\ ,\ \ let$
$(v^*,u^*,u_0^*)\ \ be\ \ optimal\ \ solution\ \ for:$

$$\begin{aligned}&max\quad uy_0+u_0\\&vx_0=1\\&uy-vx-u_0\leq 0\\&vp\leq 0\\&uQ\leq 0\\&u,v\geq 0\ \ ,u_0\ free\end{aligned}$$

is determined as: $\begin{pmatrix}x_0\\y_0\end{pmatrix}$ then the type of RTS in

$u_0^*<0\ \Rightarrow IRS$

$u_0^*>0\ \Rightarrow DRS$

$u_0^*=0\Rightarrow CRS$

References

[1] Tone, K. (2001).On returns to scale under weight restrictions in data envelopment analysis. *Journal of Productivity Analysis, 16,* 31-47.

[2] Podinovski, V. V. (1999). Side effects of absolute weight bounds in DEA models. *European Journal of Operational Research, 115,* 583-595

[3] F. Pedraja-Chaparo, J. Salinas-Jiminez & Smith, P. (1997). On the role of weight restrictions in data envelopment analysis. *Journal of Productivity Analysis 8,* 215-230.

[4] Sueyoshi, T. & K. Sekitani. (2007). The measurement of returns to scale under a simultaneous occurrence of multiple solutions in a reference set and a supporting hyperplane. *European Journal of Operational Research, 181,* 549-570.

[5] Cooper, W. W., Park, K. S. & Pastor, J. T. R. (2000a). A range adjusted measure of efficiency. *Journal of Productivity Analysis 11,* 5-42.

[6] Cooper, W. W., Seiford, L. M. & Tone, K. (2000b). *Data Envelopment Analysis.* Boston, USA: Kluwer Academic Publishers.