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Conditional expectation of weak random elements

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Abstract

We prove that the limit of a sequence of Pettis integrable bounded scalarly measurable weak random elements, of finite weak norm, with values in the dual of a non-separable Banach space is Pettis integrable. Then we provide basic properties for the Pettis conditional expectation, and prove that it is continuous. Calculus of Pettis conditional expectations in general is very different from the calculus of Bochner conditional expectations due to the lack of strong measurability and separability. In two examples, we derive the Pettis conditional expectations.

Keywords: Pettis integral; Pettis conditional expectation; non-separable Banach spaces; weak p-th order random elements

1. Introduction

Recently, studies of infinite dimensional processes have increased dramatically due to the progress of technologies which allow us to store more and more information while modern instruments are able to collect data much more effectively due to their increasingly sophisticated design [1]. Although the strong second order processes are well developed and widely known [2], the weak second order processes are not rich enough in theory. This, strongly motivates us to define the conditional expectation of weak second order random processes which play a crucial role in the study of different subjects such as martingale theory.

A few authors tried to develop the theory of Pettis conditional expectation of p-th order random elements. Brooks [3] gives a formal representation of the conditional expectation of strong measurable and Pettis integrable random elements for a given σ-field. Uhl [4] provides sufficient conditions for the existence of the Pettis conditional expectation. There are also counter examples which show that the Pettis conditional expectations do not exist in general, (see Raybakov [5]). Also, Heinich [6] provides an example of a Banach space valued Pettis integrable function on $[0,1]^2$ which does not have conditional expectation with respect to the σ-field of Lebesgue measurable sets. Riddle and Saab [7] prove sufficient conditions for the existence of Pettis conditional expectation for scalarly measurable bounded random elements for all σ-fields inside the σ-field of the underlying probability space. In this paper, we go for the Riddle and Saab settings, and establish basic ingredients for the calculus of the Pettis conditional expectation of weak first-order scalarly measurable random elements with values in the dual space of a non-separable Banach space. The upshot is the continuity property, established here. This article is organized as follows. In Section 2 we provide notations and preliminaries. Certain new results are also provided, Lemma 2.2 and Lemma 2.3. Section 3 is devoted to the main results, basic properties of the conditional expectation are given in Theorem 3.1, and the continuity is given in Theorem 3.2.

2. Preliminaries

Let C be the space of complex numbers. Suppose that X and Y be complex Banach spaces, $\mathrm{B}(X)$ stands for the Borel σ-field: the smallest σ-field generated by open subsets of X. The notation $\langle x^*, x\rangle$ is used to denote $x^*(x)$ when $x \in X$ and $x^* \in X^*$, where X^* is the dual space of X. Let $(\Omega, \mathrm{F}, \mu)$ stand for a probability space.

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A Y-valued function (random element) $\vartheta : \Omega \to Y$ is called strongly measurable when it is F/B(Y) measurable. A random element ξ from Ω into Y is called scalarly measurable if the complex-valued random variable $\langle y^*, \xi \rangle$ is measurable, i.e. F/B(C) is measurable for every $y^* \in Y^*$. Let us introduce some classical function spaces which will be used throughout this paper.

• $Ł^p(Y, \mu)$ stands for the space of all strongly measurable random elements ϑ on Y equipped with the norm,

$$\|\vartheta\|_p = (E\|\vartheta\|_Y^p)^{1/p},$$

which are called strong random elements of order p.

• $L_w^p(Y, \mu)$ $1 \le p < \infty$ stands for the space of scalarly measurable random elements ξ in Y for which,

$$\|\xi\|_p^w = \sup_{\|y^*\| \le 1} (E |\langle y^*, \xi \rangle|^p)^{1/p},$$

is finite. Such a random element ξ is called weak scalarly measurable random element of order p. Evidently, the weak p-th order property is weaker than the strong one.

• Let $L_{w*}^\infty(X^*, \mu)$ denote the space of scalarly measurable random elements ζ in X^* equipped with the norm,

$$\|\zeta\|_\infty^{w^*} = \sup_{\|x\| \le 1} \operatorname*{esssup}_{\omega \in \Omega} |\langle \zeta, x \rangle|, x \in X, \zeta \in X^*.$$

• $L_\infty(\mu)$ stands for the space of all mappings from Y into C that are bounded $\mu - a.e.$

Definition 2.1. A μ-measurable function $f : \Omega \to X$ is called Bochner integrable if there exists a sequence of simple functions $\{f_n\}$ such that

$$lim_{n \to \infty} \int_\Omega \|f_n - f\| = 0.$$

In this case $(\mathrm{B}) - \int_E f \, \mathrm{d}\mu$ is defined for each $E \in \mathsf{F}$ by

$$(\mathrm{B}) - \int_E f \, \mathrm{d}\mu = lim_{n \to \infty} \int_E f_n \, \mathrm{d}\mu$$

where $\int_E f_n \, \mathrm{d}\mu$ is defined in the obvious way and $(\mathrm{B}) - \int_E f \, \mathrm{d}\mu$ is called Bochner integral of f with respect to μ.

Let $\vartheta \in Ł^p(Y, \mu)$, then $\mathsf{E}_\mathrm{B}\vartheta := (\mathrm{B}) - \int_\Omega \vartheta \mathrm{d}\mu$ is defined in the sense of Bochner integral which defines bounded linear transformation of $Ł^1(Y, \mu)$ into Y. Also, the conditional expectation of ϑ given a σ-field Γ in F is defined as being a random element in $Ł^1(Y, \mu)$, denoted by $\mathsf{E}_\mathrm{B}[\vartheta \mid \Gamma]$, which is Γ-measurable and satisfies the condition that

$$(\mathrm{B}) - \int_A \vartheta \mathrm{d}\mu = (\mathrm{B}) - \int_A \mathsf{E}_\mathrm{B}[\vartheta \mid \mathrm{G}] \mathrm{d}\mu, \qquad \text{for all } A \in \Gamma.$$

From now on, we use the term "random element" for scalarly measurable random elements only.

Definition 2.2. A random element $\xi : \Omega \to Y$ is called Pettis integrable with respect to μ, if

(i) $\xi \in L_w^1(Y, \mu)$,

(ii) For every $E \in \mathsf{F}$, there exists an element ξ_E in Y such that,

$$\langle y^*, \xi_E \rangle = \int_E \langle y^*, \xi \rangle \mathrm{d}\mu, \quad \text{for every } y^* \in Y^*. \quad (1)$$

The element ξ_E is called the Pettis integral of ξ over E with respect to the measure μ and it is denoted by $(\mathrm{P}) - \int_E \xi \mathrm{d}\mu$. In particular, $\mathsf{E}_\mathrm{P}\xi$ stands for ξ_Ω.

If Y is a reflexive Banach space, then every separably-valued random element of a weak order one is Pettis integrable, [8].

Lemma 2.1. Let ξ and η be two random elements in Y. Then,

(i) $\xi + \eta$ is a random element in Y.

(ii) If $A : Y \to Y$ is a bounded linear operator, then $A\xi$ is a random element in Y.

(iii) Let $\{\xi_n\}$ be a sequence of random elements in Y such that $\xi_n \mapsto \xi \quad \mu - a.e.$ (i.e., $\langle y^*, \xi_n - \xi \rangle \to 0, \mu - a.e.$ for every $y^* \in Y^*$), then ξ is a random element in Y.

Proof: The proof is straightforward.

Definition 2.3. A sequence $\{y_n\}$ in a Banach space Y is called weakly Cauchy if for any positive number ε and every $y^* \in Y^*$, there exists a positive integer N, depending on ε and y^*, such that for all $m,n > N$, we have $|\langle y^*, y_n - y_m\rangle| < \varepsilon$.

Definition 2.4. A subset K of the Banach space Y is called weakly precompact if each sequence in K has a weakly Cauchy subsequence.

Definition 2.5. Pettis integrable random elements ξ and η in Y, are called weakly equivalent if $\langle y^*,\xi\rangle = \langle y^*,\eta\rangle \ \mu - a.e.$, for all $y^* \in Y^*$, $\xi \overset{w}{=} \eta \ \ \mu - a.e.$

Definition 2.6. Let ξ be a Pettis integrable random element in Y and let Γ be a σ-field in F. A Pettis integrable random element $\mathsf{E}_{\mathrm{P}}[\xi \mid \Gamma]$ in Y is said to be Pettis conditional expectation of ξ with respect to Γ if,

(i) $\mathsf{E}_{\mathrm{P}}[\xi \mid \Gamma]$ is scalarly Γ-measurable and Pettis integrable,

(ii) $(\mathrm{P})-\int_G \xi \mathrm{d}\mu = (\mathrm{P})-\int_G \mathsf{E}_{\mathrm{P}}[\xi \mid \Gamma]\mathrm{d}\mu,$ for every $G \in \Gamma$.

Definition (2.6) agrees with the one given in [4] if the random element ξ is strongly measurable.

From now on, we assume $Y = X^*$, i.e. the desired random element takes its values in the dual space of a non-separable Banach space. This enables us to use certain weak* properties. We note that this assumption is satisfied whenever Y is a Hilbert space or $Y = \ell_\infty$.

The following theorem is given by L. H. Riddle and E. Saab [7]. It gives sufficient conditions for bounded Pettis integrable random elements to have Pettis conditional expectation.

Theorem 2.1. Let $\xi : (\Omega,\mathsf{F},\mu) \to X^*$ be a bounded Pettis integrable random element. If the set $\{\langle \xi, x\rangle : \|x\| \le 1\}$ is weakly precompact in $L_\infty(\mu)$, then ξ has Pettis conditional expectation with respect to all sub-σ-fields in F.

To establish the main properties of Pettis conditional expectation, we first prove the following result.

Lemma 2.2. Let $\xi_n \to \xi$ in $\mathsf{L}^1_w(X^*,\mu)$, and $\{\xi_n\}$ be a sequence of Pettis integrable random elements. Then ξ is a Pettis integrable random element in X^*.

Proof: It is enough to show that for all $A \in \mathsf{F}$, there exists $\xi_A \in X^*$ such that $\int_A \langle x^{**},\xi\rangle \mathrm{d}\mu = \langle x^{**},\xi_A\rangle$. Since $\xi_n \to \xi$ in $\mathsf{L}^1_w(X^*,\mu)$,

$$\sup_{\|x^{**}\|\le 1} \int_A |\langle x^{**},\xi_n - \xi\rangle| \mathrm{d}\mu \to 0,$$

and since $\int_A \langle x^{**},\xi_n\rangle \mathrm{d}\mu = \langle x^{**},\xi_{nA}\rangle$ and $\sup_{\|x^{**}\|\le 1} \int_\Omega |\langle x^{**},\xi_n - \xi_m\rangle| \mathrm{d}\mu \to 0$, so,

$$\sup_{\|x^{**}\|\le 1} |\langle x^{**},\xi_{nA} - \xi_{mA}\rangle| \to 0.$$

Therefore, $\{\xi_{nA}\}$ is a weak Cauchy sequence in X^* and converges weakly to some element $\xi_A \in X^*$ and $\langle x^{**},\xi_A\rangle = \int_A \langle x^{**},\xi\rangle \mathrm{d}\mu$.

Lemma 2.3. Let ξ be a random element in the Banach space X^* for which $\mathsf{E}_{\mathrm{P}}[\xi \mid \Gamma]$ exists, then

$$\langle x^{**},\mathsf{E}_{\mathrm{P}}[\xi \mid \Gamma]\rangle = E[\langle x^{**},\xi\rangle \mid \Gamma] \quad \forall x^{**} \in X^{**}.$$

Proof:
Since $(\mathrm{P})-\int_G \mathsf{E}_{\mathrm{P}}[\xi \mid \Gamma]\mathrm{d}\mu = (\mathrm{P})-\int_G \xi \mathrm{d}\mu$ for all $G \in \Gamma$, then

$$\int_G \langle x^{**},\mathsf{E}_{\mathrm{P}}[\xi \mid \Gamma]\rangle \mathrm{d}\mu = \int_G \langle x^{**},\xi\rangle \mathrm{d}\mu$$
$$= \int_G E[\langle x^{**},\xi\rangle \mid \Gamma]\mathrm{d}\mu, \textit{for all } G \in \Gamma.$$

Since $\langle x^{**},\mathsf{E}_{\mathrm{P}}[\xi \mid \Gamma]\rangle$ and $E[\langle x^{**},\xi\rangle \mid \Gamma]$ are Γ-measurable functions, we conclude that $\langle x^{**},\mathsf{E}_{\mathrm{P}}[\xi \mid \Gamma]\rangle = E[\langle x^{**},\xi\rangle \mid \Gamma], \mu - a.e.$

Corollary 2.1. Let ξ be a random element in the Banach space X^* for which $E_P[\xi \mid \Gamma]$ exists, then

$$\langle E_P[\xi \mid \Gamma], x\rangle = E[\langle \xi, x\rangle \mid \Gamma] \quad for\, all\ x \in X.$$

Proof: Apply Lemma (2.3) and the fact that X can be embeded in X^{**}.

Lemma 2.4. Let X be a Banach space, then $\{x \in X : \|x\| \le 1\}$ is weakly precompact if and only if $\{x \in X : \|x\| \le M\}$ is weakly precompact.

Proof: The result follows from the fact that

$$\{\langle x^*, x\rangle : \|x\| \le M\} = \{\langle x^*, \frac{x}{M}\rangle : \left\|\frac{x}{M}\right\| \le 1\},$$

Lemma 2.5. Let ξ be a bounded random element in the Banach space X^*. If $\{\langle \xi, x\rangle : \|x\| \le 1\}$ is weakly precompact in $L_\infty(\mu)$, then $E_P[\xi \mid \Gamma]$ is bounded and $\{\langle E_P[\xi \mid \Gamma], x\rangle : \|x\| \le 1\}$ is weakly precompact in $L_\infty(\mu)$.

Proof: According to the assumptions of the Lemma (2.5) and Theorem (2.1), $E_P[\xi \mid \Gamma]$ exists. Since $\{\langle \xi, x\rangle : \|x\| \le 1\}$ is weakly precompact set, so for each sequence $\{\langle \xi, x_k\rangle\}$ there is a subsequence $\{\langle \xi, x_{k(i)}\rangle\}$ which is weakly Cauchy in $L_\infty(\mu)$. Also, by Lemma (2.3) and the fact that X is embedded in X^{**} [9], we have
$\langle E_P[\xi \mid \Gamma], x_{k(i)} - x_{k(j)}\rangle = E[\langle \xi, x_{k(i)} - x_{k(j)}\rangle \mid \Gamma]$, and so $E[\langle \xi, x_{k(i)} - x_{k(j)}\rangle \mid \Gamma] \to 0$, since $\langle \xi, x_{k(i)} - x_{k(j)}\rangle \to 0$ in $L_\infty(\mu)$.

3. Main Results

We first derive some important properties of Pettis conditional expectation, given in the following theorem.

Theorem 3.1. Suppose that ξ and η are bounded scalarly measurable and Pettis integrable random elements such that $\{\langle \xi, x\rangle : \|x\| \le 1\}$ and $\{\langle \eta, x\rangle : \|x\| \le 1\}$ are weakly precompact sets in $L_\infty(\mu)$. Then:

(i) If $\xi = c$, $\mu - a.e.$, then $E_P[\xi \mid \Gamma] \overset{w}{=} c$, $\mu - a.e.$

(ii) If k is a scalar, then $E_P[k\xi + \eta \mid \Gamma] \overset{w}{=} kE_P[\xi \mid \Gamma] + E_P[\eta \mid \Gamma]$, $\mu - a.e.$

(iii) If $\Gamma = \{\Omega, \varnothing\}$ then $E_P[\xi \mid \Gamma] \overset{w}{=} E_P \xi$, $\mu - a.e.$

(iv) $E_P[\xi \mid F] \overset{w}{=} \xi$, $\mu - a.e.$

(v) If $\Gamma_1 \subset \Gamma_2$ then $E_P[E_P[\xi \mid \Gamma_2] \mid \Gamma_1] \overset{w}{=} E_P[\xi \mid \Gamma_1]$, $\mu - a.e.$

(vi) If A is a bounded linear operator on X^*, then $E_P[A\xi \mid \Gamma] \overset{w}{=} AE_P[\xi \mid \Gamma]$, $\mu - a.e.$

Proof: (i), (ii) and (iii)are immediate from the linearity and basic properties of Pettis integrals, and by Definition (2.6).
For (iv) we have

$$\int_G \langle x^{**}, E_P[\xi \mid F]\rangle d\mu = \int_G \langle x^{**}, \xi\rangle d\mu, \quad for\, all\ G \in F.$$

Since $\langle x^{**}, E_P[\xi \mid F]\rangle$ and $\langle x^{**}, \xi\rangle$ are measurable F, we obtain

$$\langle x^{**}, E_P[\xi \mid F]\rangle = \langle x^{**}, \xi\rangle,$$

which means $E_P[\xi \mid F] \overset{w}{=} \xi$, $\mu - a.e.$

For (v) we note that scalar measurability and weak integrability of $E_P[\xi \mid \Gamma_1]$ are immediately followed by Definition (2.6),

$$\int_G \langle x^{**}, E_P[E_P[\xi \mid \Gamma_2] \mid \Gamma_1]\rangle d\mu$$
$$= \int_G \langle x^{**}, E_P[\xi \mid \Gamma_2]\rangle d\mu, \quad for\, all\ G \in \Gamma_1.$$

Since $\Gamma_1 \subset \Gamma_2$, we obtain that

$$\begin{aligned}\int_G \langle x^{**}, E_P[E_P[\xi \mid \Gamma_2] \mid \Gamma_1]\rangle d\mu &= \int_G \langle x^{**}, E_P[\xi \mid \Gamma_2]\rangle d\mu \\ &= \int_G \langle x^{**}, \xi\rangle d\mu, \\ &= \int_G \langle x^{**}, E_P[\xi \mid \Gamma_1]\rangle d\mu, \\ &for\ every\ \ G \in \Gamma_1.\end{aligned}$$

So $E_P[E_P[\xi \mid \Gamma_2] \mid G_1] \overset{w}{=} E_P[\xi \mid \Gamma_1]$, $\mu - a.e.$

For (vi), the existence of $E_P[A\xi \mid \Gamma]$ is guaranteed by Lemma (2.4). By Definition (2.6):

$$\int_G \langle x^{**}, \mathsf{E}_P[A\xi \mid \Gamma]\rangle d\mu = \int_G \langle x^{**}, A\xi\rangle d\mu$$
$$= \int_G \langle A^* x^{**}, \xi\rangle d\mu$$
$$= \int_G \langle A^* x^{**}, \mathsf{E}_P[\xi \mid \Gamma]\rangle d\mu$$
$$= \int_G \langle x^{**}, A\mathsf{E}_P[\xi \mid \Gamma]\rangle d\mu,$$
$$\textit{for every } G \in \Gamma;$$

hence

$$\mathsf{E}_P[A\xi \mid \Gamma] \overset{w}{=} A\mathsf{E}_P[\xi \mid \Gamma], \mu - a.e.$$

The following theorem is the main result of this article. It gives the continuity of the Pettis conditional expectation under the Riddle and Saab assumptions given in [7].

Theorem 3.2. Let $\{\xi_n\}$ be a sequence of bounded random elements of weak order one in the Banach space X^*. Let $\{\langle \xi_n, x\rangle, \|x\| \le 1\}$ be a weakly precompact set in $L_\infty(\mu)$ and $\xi_n \to \xi$ in $\mathsf{L}^1_w(X^*,\mu) \cap \mathsf{L}^\infty_{w*}(X^*,\mu)$, i.e, $\|\xi_n - \xi\|^{w^*}_\infty + \|\xi_n - \xi\|^w_1 \to 0$ as $n \to \infty$, where ξ is a bounded random element on X^*, then $\mathsf{E}_P[\xi \mid \Gamma]$ exists and $\mathsf{E}_P[\xi_n \mid \Gamma] \to \mathsf{E}_P[\xi \mid \Gamma]$ in $\mathsf{L}^\infty_{w*}(X^*,\mu)$.

Proof: To prove the existence, since ξ_n and ξ are in $\mathsf{L}^1_w(X^*,\mu)$, it will be enough to show that $\{\langle \xi, x\rangle, \|x\| \le 1\}$ is a weakly precompact set in $L_\infty(\mu)$. Let $\{\langle \xi, x_i\rangle\}_i$ be an arbitrary subsequence of $\{\langle \xi, x\rangle, \|x\| \le 1\}$. Since for every ξ_m, $\{\langle \xi_m, x\rangle, \|x\| \le 1\}$ is weakly precompact, then for $x_1, x_2, \ldots$ in X such that $\|x_i\| \le 1$, the sequence $\{\langle \xi_m, x_i\rangle\}_i$ has a subsequence $\{\langle \xi_m, x_{k_{mi}}\rangle\}_i$ which is weakly Cauchy, $m = 1, 2, \ldots$. We prove by induction that there exist a sequence $\{x_{k_{(m)i}}\}_i$ for which $\{\langle \xi_m, x_{k_{mi}}\rangle\}_i$ is weakly Cauchy. Thus by the diagonalization method $\{\langle \xi_m, x_{k_{ii}}\rangle\}_i$ is weakly Cauchy for each m.

Now we need to show that $\{\langle \xi, x_{k_{ii}}\rangle\}_i$ is also weakly Cauchy. For simplicity we let $x_{k_{ii}} = y_i$. Let $\varepsilon > 0$ and $f^* \in L^*_\infty(\mu)$ be given, there exists n such that $\|\xi_n - \xi\|^{w^*}_\infty < \varepsilon/(4\|f^*\|)$, so

$$| f^*(\langle \xi_n - \xi, y_i - y_j\rangle) | < \frac{\varepsilon}{2}, \quad \text{for every } i, j.$$

Since $\{\langle \xi_n, y_i\rangle\}_i$ is weakly Cauchy for this choice of n, there exists a scalar M such that for every $i, j > M$, $| f^*(\langle \xi_n, y_i - y_j\rangle) | < \frac{\varepsilon}{2}$. Therefore,

$$| f^*(\langle \xi, y_i - y_j\rangle) | \le | f^*(\langle \xi_n - \xi, y_i - y_j\rangle) |$$
$$+ | f^*(\langle \xi_n, y_i - y_j, \rangle) | < \varepsilon, \quad \forall\, i, j > M;$$

This leads us to the weak precompactness of $\{\langle \xi, x\rangle, \|x\| \le 1\}$. Hence $\mathsf{E}_P[\xi \mid \Gamma]$ exists.

To prove the convergence, using Lemma (2.3) we have

$$\sup_{\|x^{**}\| \le 1} \operatorname*{esssup}_{\omega \in \Omega} | \langle x^{**}, \mathsf{E}_P[\xi_n \mid \Gamma] - \mathsf{E}_P[\xi \mid \Gamma]\rangle |$$
$$= \sup_{\|x^{**}\| \le 1} \operatorname*{esssup}_{\omega \in \Omega} | \langle x^{**}, \mathsf{E}_P[\xi_n - \xi \mid \Gamma]\rangle |$$
$$= \sup_{\|x^{**}\| \le 1} \operatorname*{esssup}_{\omega \in \Omega} | E[\langle x^{**}, \xi_n - \xi\rangle \mid \Gamma] |$$
$$\le \sup_{\|x^{**}\| \le 1} \operatorname*{esssup}_{\omega \in \Omega} | \langle x^{**}, \xi_n - \xi\rangle |$$
$$\to 0, \quad \text{as } n \to \infty,$$

The last two assertions follow from the fact that $\xi_n \to \xi$ in $\mathsf{L}^\infty_{w*}(X^*,\mu) \cap \mathsf{L}^1_w(X^*,\mu)$, and the contraction of the conditional expectation [10]. Hence $\mathsf{E}_P[\xi_n \mid \Gamma] \to \mathsf{E}_P[\xi \mid \Gamma]$ in $\mathsf{L}^\infty_{w*}(X^*,\mu)$.

Let us conclude this section by presenting two examples which are not strongly measurable [11], pages 44-45 and [12]. We obtain the Pettis conditional expectation of the corresponding random elements.

Example 1. Let $([0,1], \mathrm{B}([0,1]), \lambda)$ be a probability space where λ is the Lebesgue measure on $[0,1]$ and B([0,1]) is the Borel σ-field on $[0,1]$. Also, let $\{e_t, t \in [0,1]\}$ be an orthonormal basis for the non-separable Hilbert space $l_2[0,1]$, (the space of all complex valued functions on $[0,1]$ which disappear everywhere except at the most countable points of $[0,1]$, and the sequence of values at those points is square summable.)

To find the conditional expectation of a random element $f:[0,1]\to l_2[0,1]$ defined by $f(t)=e_t$, we use Riesz Representation Theorem to obtain $\langle f,x\rangle=0 \quad ,\mu-a.e.$, for every $x\in l_2[0,1]$. Thus f is a random element in $l_2[0,1]$. It is obvious that its conditional expectation with respect to any sub-σ-field Γ of B([0,1]) is zero, which is Γ -measurable and satisfies the integral equation of conditional expectation.

Example 2. Suppose that $\{A_n\}$ is a sequence of subintervals of $[0,1]$, which has the following properties:

(i) $A_1=[0,1]$,

(ii) each A_n is a nonempty subinterval of $[0,1]$,

(iii) $\lim\lambda(A_n)=0$, where λ is the Lebesgue measure,

(iv) $A_n=A_{2n}\cup A_{2n+1}$ for all n,

(v) $A_m\cap A_j=\varnothing$ for each pair (m,j) with $2^i\le m<j\le 2^{i+1}-1$, for some i.

Let $f:[0,1]\to l_\infty\subset l_1^*$ be defined by $f(t)=(\chi_{A_n}(t))_n, t\in[0,1]$ and let $\phi\in l_\infty^*$. By Yosida-Hewitt Theorem, there exist a unique ϕ_1 and ϕ_2 in l_∞^* such that $\phi=\phi_1+\phi_2$, where ϕ_1 is countably additive part of ϕ and ϕ_2 is the purely finitely additive part of ϕ. It is shown in [10] that $f(t)$ is a non-measurable scalarly measurable function. Now we find the Pettis conditional expectation of f with respect to the σ-field Γ, generated by a countable partition $\{G_1,G_2,\cdots\}$, of $[0,1]$. Indeed $(E[\chi_{A_n}\mid\Gamma])_n$ is a version of Pettis conditional expectation of f, where $E[\chi_{A_n}\mid\Gamma]$ is the usual conditional expectation of random variable χ_{A_n} with respect to Γ. To verify this, it is easy to show that

$$E[\chi_{A_n}\mid\Gamma]=\sum_m\{\frac{1}{\lambda(G_m)}\int_{G_m}\chi_{A_n}(t)dt\}\chi_{G_m}.$$

The measurability of $\langle\phi,\mathrm{E_P}[f(t)\mid\Gamma]\rangle$ is obvious because of the structure of Γ. For the second condition of the Definition (2.6), it is enough to check the equality for one G_i;

$$\begin{aligned}&\int_{G_i}\langle\varphi,(E[\chi_{A_n}\mid\Gamma](t))_n\rangle dt\\&=\int_{G_i}\langle\varphi,(\frac{1}{\lambda(G_i)}\int_{G_i}\chi_{A_n}(t)dt)_n\rangle dt\\&=\lambda(G_i)\cdot\frac{1}{\lambda(G_i)}\langle\varphi,(\lambda(A_n\cap G_i))_n\rangle\\&=\langle\varphi,(\lambda(A_n\cap G_i))_n\rangle.\end{aligned}$$

Also, we need to show that

$$\begin{aligned}&\int_{G_i}\langle\varphi,(\chi_{A_n}(t))_n\rangle dt\\&=\int_{G_i}\langle\varphi_1,(\chi_{A_n}(t))_n\rangle dt\\&+\int_{G_i}\langle\varphi_2,(\chi_{A_n}(t))_n\rangle dt\\&=\langle\varphi,(\lambda(A_n\cap G_i))_n\rangle.\end{aligned}$$

Since $\lambda(A_n)\to 0$ as $n\to\infty$, so $\lambda(A_n\cap G_i)\to 0$ as $n\to\infty$, hence $\langle\phi_2,(\lambda(A_n\cap G_i))_n\rangle=0$. Therefore,

$$\begin{aligned}\int_{G_i}\langle\phi,(\chi_{A_n}(t))_n\rangle dt &=\int_{G_i}\langle\phi_1,(\chi_{A_n}(t))_n\rangle dt\\&=\langle\phi_1,(\int_{G_i}(\chi_{A_n}(t)dt)_n\rangle\\&=\langle\phi_1,(\lambda(A_n\cap G_i))_n\rangle\\&=\langle\phi,(\lambda(A_n\cap G_i))_n\rangle.\end{aligned}$$

Thus $(E[\chi_{A_n}\mid\Gamma])_n$ is a version of $\mathrm{E_P}[f(t)\mid\Gamma]$.

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