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The modified Exp-function method and its applications to the generalized K(n,n) and BBM equations with variable coefficients

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Abstract

In this article, the modified exp-function method is used to construct many exact solutions to the nonlinear generalized K(n,n) and BBM equations with variable coefficients. Under different parameter conditions, explicit formulas for some new exact solutions are successfully obtained. The proposed solutions are found to be important for the explanation of some practical physical problems.

Keywords: Generalized K(n,n) equation with variable coefficients; generalized BBM equation with variable coefficients; exact traveling wave solutions; Exp-function method

1. Introduction

In the nonlinear science, many important phenomena in various fields can be described by the nonlinear evolution equations (NLEEs). Searching for exact soliton solutions of NLEEs plays an important and significant role in the study on the dynamics of those phenomena. With the development of soliton theory, many powerful methods for obtaining the exact solutions of NLEEs have been presented, such as the extended tanh-function method [1-5], the tanh-sech method [6-8], the sine-cosine method [9-11], the homogeneous balance method [12, 13], the exp-function method [14-17], the Jacobi elliptic function method [18-21], the F-expansion method [22], the homotopy perturbation method [23, 24], the variational iteration method [25], the inverse scattering transformation method [26], the Bäcklund transformation method [27], the Hirota bilinear method [28, 29] and so on. To our knowledge, most of the aforementioned methods are related to constant coefficients models. Recently, much attention has been paid to the variable-coefficient nonlinear equations which can describe many nonlinear phenomena more realistically than their constant-coefficient ones.

The objective of this article is to apply the modified exp-function method using a generalized wave transformation to find the exact solutions of the following two nonlinear dispersive equations with variable coefficients:

1. The generalized K(n,n) equation with variable coefficients [30]

$$u_t + a(t)u_x + b(t)(u^n)_x + k(t)(u^n)_{xxx} = 0, \quad n \neq 0,1 \tag{1}$$

where *a(t), b(t)* and *k(t)* are nonzero functions of *t*. As a model that characterizes long waves in nonlinear dispersive media, Eq. (1) was formally derived to describe the propagation of surface water waves in a uniform channel. Now it has been established that the equation provides a model for not only the surface waves of long wavelength in liquids, but also hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals, and acoustic gravity waves in compressible fluids. Eq. (1) has been discussed by Wazwaz [31] using sine-cosine method and tanh-method when the functions *a(t), b(t)* and *k(t)* are nonzero constants.

2. The generalized Benjamin-Bona-Mahony (BBM) equation with variable coefficients [30]

$$u_t + a(t)u_x + b(t)(u^n)_x + k(t)(u^n)_{xxt} = 0, \quad n \neq 0,1 \tag{2}$$

where *a(t), b(t)* and *k(t)* all are nonzero functions of *t*. The case $n = 1$ with constant coefficients when $a(t)=b(t)=k(t)=1$ corresponds to the BBM equation, which was first proposed by Benjamin et al. [32]. Eq. (2) is an alternative to Eq. (1), and also describes the unidirectional propagation of small-amplitude long waves on the surface of water in a channel. This equation is not only convenient for shallow water waves but also for hydromagnetic

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and acoustic waves, and therefore it has some advantages compared with the KdV equation. When $n = 2$, Eq. (1) is called the modified BBM equation. Lv et al [30] have discussed Eqs. (1) and (2) using the auxiliary differential equation method and obtained various exact traveling wave solutions.

2. Description of the modified exp-function method

To illustrate the basic idea of exp-function method the following nonlinear evolution equations are considered:

$$\phi(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \ldots) = 0. \tag{3}$$

Where *u=u(x,t)* is an unknown function, ϕ is a polynomial in u and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method:

Step 1. Using the generalized wave transformation

$$u(x,t) = u(\xi), \qquad \xi = hx + \int \tau(t)dt, \tag{4}$$

where h is a constant, while $\tau(t)$ is an integrable function of t to be determined. Then, Eq. (3) is reduced to the following ODE:

$$\begin{aligned} &p(u, \tau(t)u', hu', \tau(t)^2 u'', \\ &h^2 u'', h\tau(t)u'', \ldots) = 0, \end{aligned} \tag{5}$$

where $' = \frac{d}{d\xi}$ and P is a polynomial in u and its total derivatives.

Step 2. The exp-function method is based on the assumption that traveling wave solutions for Eq. (5) can be expressed in the following form

$$u(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)}, \tag{6}$$

where c, d, p and q are positive integers which are unknown to be determined later, a_n and b_m are unknown constants. Eq. (6) can be written in the form

$$u(\xi) = \frac{a_c \exp(c\xi) + \ldots\ldots + a_{-d} \exp(-d\xi)}{a_p \exp(p\xi) + \ldots\ldots + a_{-q} \exp(-d\xi)}, \tag{7}$$

Step 3. Determine the values of c and p by balancing the linear term of highest order in Eq. (5) with the highest order nonlinear term. Similarly, determine the values of d and q by balancing the linear term of lowest order in Eq. (5) with the lowest order nonlinear term.

Step 4. Substitute (7) into Eq. (5) and collect all terms with the same order of $e^{n\xi}$ together, the left-hand side of Eq. (5) is converted into a polynomial in $e^{n\xi}$. Then set each coefficient of this polynomial to zero to derive a set of algebraic equations for a_i, b_j.

Step 5. Solve the algebraic system obtained in Step4 by the use of *Maple* or *Mathematica*. Then we can obtaine the exact solutions of Eq. (3).

3. Application

In this section, the modified exp-function method is applied to construct the exact solutions of two nonlinear evolution equations with variable coefficients via the nonlinear generalized K(n,n) equation with variable coefficients (1) and the nonlinear generalized Benjamin-Bona-Mahony (BBM) equation with variable coefficients (2).

3.1. Example1. The generalized K(n,n) equation with variable coefficients

In order to obtain the exact solutions of Eq. (1), it is assumed that the solution of this equation can be written in the form

$$u(x,t) = \left[\upsilon(x,t)\right]^{\frac{-2}{n-1}}, \tag{8}$$

Substituting (8) into (1), we have

$$\begin{aligned} &(n-1)^2\left[\upsilon^4(\xi)\upsilon_t(\xi) + a(t)\upsilon^4(\xi)\upsilon_x(\xi)\right] \\ &+2n\left(6n^2 - 5n + 1\right)k(t)\left[\upsilon_x(\xi)\right]^3 \\ &+n(n-1)^2 k(t)\upsilon^2(\xi)\upsilon_{xxx}(\xi) + n(n-1)^2 b(t)\upsilon^2(\xi)\upsilon_x(\xi) \\ &-3n(n-1)(3n-1)k(t)\upsilon(\xi)\upsilon_x(\xi)\upsilon_{xx}(\xi) = 0. \end{aligned} \tag{9}$$

Using the generalized wave transformation (4), Eq. (9) converts to the nonlinear ODE:

$$\begin{aligned} &(n-1)^2\left[\tau(t) + ha(t)\right]\upsilon^4(\xi)\upsilon'(\xi) \\ &+2n(6n^2 - 5n + 1)h^3 k(t)\left[\upsilon'(\xi)\right]^3 \\ &-3n(n-1)(3n-1)h^3 k(t)\upsilon(\xi)\upsilon'(\xi)\upsilon''(\xi) \\ &+n(n-1)^2 h^3 k(t)\upsilon^2(\xi)\upsilon'''(\xi) \\ &+n(n-1)^2 hb(t)\upsilon^2(\xi)\upsilon'(\xi) = 0. \end{aligned} \tag{10}$$

Using the ansatz (7), since there is no linear term in Eq. (10), in order to determine the values of c, d, p and q, we balance the nonlinear term of highest order $\upsilon^2(\xi)\upsilon'''(\xi)$ with the nonlinear term $\upsilon^4(\xi)\upsilon'(\xi)$ as follows:

$$\upsilon^2(\xi)\upsilon'''(\xi) = \frac{c_1 \exp[(3c+7p)\xi] + ...}{c_2 \exp[10p\xi] + ...}, \tag{11}$$

$$\upsilon^4(\xi)\upsilon'(\xi) = \frac{c_3 \exp[(5c+5p)\xi] + ...}{c_4 \exp[10p\xi] + ...}, \tag{12}$$

where c_i are coefficients for simplicity. By balancing the highest order of exp-function in Eqs. (11) and (12), we have $3c+7p = 5c+5p$, which leads to

$$p = c. \tag{13}$$

Similarly, from the ansatz (7), we have

$$\upsilon^2(\xi)\upsilon'''(\xi) = \frac{... + d_1 \exp[-(3d+7q)\xi]}{... + d_2 \exp[-10q\xi]}, \tag{14}$$

$$\upsilon^4(\xi)\upsilon'(\xi) = \frac{... + d_3 \exp[-(5d+5q)\xi]}{... + d_4 \exp[-10q\xi]}, \tag{15}$$

where d_i are coefficients for simplicity. By balancing the lowest order of exp-function in Eqs. (14) and (15), we have $-(3d+7q) = -(5d+5q)$, which leads to

$$d = q. \tag{16}$$

Choosing $p = c = 1$ and $q = d = 1$, Eq. (7) becomes

$$\upsilon(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1}\exp(-\xi)}. \tag{17}$$

Substituting Eq. (17) into Eq. (10), and by the help of *Maple*, we get

$$\frac{1}{A}\{\alpha_{10}e^{10\xi} + \alpha_9 e^{9\xi} + \alpha_8 e^{8\xi} + \alpha_7 e^{7\xi} + \alpha_6 e^{6\xi} + \alpha_5 e^{5\xi} + \alpha_4 e^{4\xi} + \alpha_3 e^{3\xi} + \alpha_2 e^{2\xi} + \alpha_1 e^{\xi} + \alpha_0\} = 0, \tag{18}$$

where

$$A = e^{5\xi}\left(b_1 e^{\xi} + b_0 + b_{-1}e^{-\xi}\right)^6, \tag{19}$$

and α_n are coefficients of $e^{n\xi}$. Equating all the coefficients of $e^{n\xi}$ to zero, we obtain a system of algebraic equations which can be solved by the *Maple*, to obtain the following cases of solutions

Case 1.

$$a_0 = a_0,\ a_1 = 0,\ a_{-1} = 0,\ b_0 = 0,\ b_1 = 0,\ b_{-1} = b_{-1},$$
$$\tau(t) = -h\,a(t),\ k(t) = \frac{-(n-1)^2\, b(t)}{4h^2 n^2},$$
$$a(t) = a(t),\ b(t) = b(t), \tag{20}$$

In this case, the exact solution of Eq. (1) has the form:

$$u(\xi) = \left(\frac{a_0 e^{\xi}}{b_{-1}}\right)^{\frac{-2}{n-1}}, \tag{21}$$

Case 2.

$$a_0 = a_0,\ a_1 = 0,\ a_{-1} = 0,\ b_0 = 0,\ b_1 = b_1,\ b_{-1} = 0,$$
$$\tau(t) = -h\,a(t),\ k(t) = \frac{-(n-1)^2\, b(t)}{4h^2 n^2},$$
$$a(t) = a(t),\ b(t) = b(t), \tag{22}$$

In this case, the exact solution of Eq. (1) has the form:

$$u(\xi) = \left(\frac{a_0}{b_1 e^{\xi}}\right)^{\frac{-2}{n-1}}, \tag{23}$$

Case 3.

$$a_0 = 0,\ a_1 = 0,\ a_{-1} = a_{-1},\ b_0 = b_0,\ b_1 = 0,\ b_{-1} = 0,$$
$$\tau(t) = -h\,a(t),\ k(t) = \frac{-(n-1)^2\, b(t)}{4h^2 n^2},$$
$$a(t) = a(t),\ b(t) = b(t), \tag{24}$$

In this case, the exact solution of Eq. (1) has the form:

$$u(\xi) = \left(\frac{a_{-1} e^{-\xi}}{b_0}\right)^{\frac{-2}{n-1}}, \tag{25}$$

Case 4.

$$a_0 = 0,\ a_1 = 0,\ a_{-1} = a_{-1},\ b_0 = 0,\ b_1 = b_1,\ b_{-1} = 0,$$
$$\tau(t) = -h\,a(t),\ k(t) = \frac{-(n-1)^2\, b(t)}{16h^2 n^2},$$
$$a(t) = a(t),\ b(t) = b(t), \tag{26}$$

In this case, the exact solution of Eq. (1) has the form:

$$u(\xi) = \left(\frac{a_{-1}}{b_1 e^{2\xi}}\right)^{\frac{-2}{n-1}}, \tag{27}$$

Case 5.

$$a_0=0,\ a_1=a_1,\ a_{-1}=0,\ b_0=0,\ b_1=0,\ b_{-1}=b_{-1},$$
$$\tau(t)=-ha(t),\ k(t)=\frac{-(n-1)^2\,b(t)}{16h^2n^2},$$
$$a(t)=a(t),\ b(t)=b(t), \tag{28}$$

In this case, the exact solution of Eq. (1) has the form:

$$u(\xi)=\left(\frac{a_1e^{2\xi}}{b_{-1}}\right)^{\frac{-2}{n-1}}, \tag{29}$$

Case 6.

$$a_0=0,\ a_1=a_1,\ a_{-1}=0,\ b_0=b_0,\ b_1=0,\ b_{-1}=0,$$
$$\tau(t)=-ha(t),\ k(t)=\frac{-(n-1)^2\,b(t)}{4h^2n^2},$$
$$a(t)=a(t),\ b(t)=b(t), \tag{30}$$

In this case, the exact solution of Eq. (1) has the form:

$$u(\xi)=\left(\frac{a_1e^{\xi}}{b_0}\right)^{\frac{-2}{n-1}}, \tag{31}$$

Case 7.

$$a_0=a_0,\ a_1=0,\ a_{-1}=a_{-1},\ b_0=b_0,\ b_1=\frac{a_0b_0}{a_{-1}},\ b_{-1}=0,$$
$$\tau(t)=-ha(t),\ k(t)=\frac{-(n-1)^2\,b(t)}{4h^2n^2},$$
$$a(t)=a(t),\ b(t)=b(t), \tag{32}$$

In this case, the exact solution of Eq. (1) has the form:

$$u(\xi)=\left[\frac{a_{-1}\left(a_0+a_{-1}e^{-\xi}\right)}{b_0\left(a_{-1}+a_0e^{\xi}\right)}\right]^{\frac{-2}{n-1}}, \tag{33}$$

Case 8.

$$a_0=a_0,\ a_1=a_1,\ a_{-1}=0,\ b_0=b_0,\ b_1=0,\ b_{-1}=\frac{a_0b_0}{a_1},$$
$$\tau(t)=-ha(t),\ k(t)=\frac{-(n-1)^2\,b(t)}{4h^2n^2},$$
$$a(t)=a(t),\ b(t)=b(t), \tag{34}$$

In this case, the exact solution of Eq. (1) has the form:

$$u(\xi)=\left[\frac{a_1\left(a_0+a_1e^{\xi}\right)}{b_0\left(a_1+a_0e^{-\xi}\right)}\right]^{\frac{-2}{n-1}}, \tag{35}$$

where ξ in the above cases 1-8 has the form

$$\xi=h\left[x-\int a(t)dt\right]. \tag{36}$$

Case 9.

$$a_0=a_0,\ a_1=0,\ a_{-1}=0,\ b_0=0,\ b_1=b_1,\ b_{-1}=b_{-1},$$
$$\tau(t)=\frac{-h\left[2b_1b_{-1}(n+1)b(t)+na_0^2a(t)\right]}{na_0^2},$$
$$k(t)=\frac{-(n-1)^2\,b(t)}{4h^2n^2},\ a(t)=a(t),\ b(t)=b(t), \tag{37}$$

In this case, the exact solution of Eq. (1) has the form:

$$u(\xi)=\left(\frac{a_0}{b_1e^{\xi}+b_{-1}e^{-\xi}}\right)^{\frac{-2}{n-1}}, \tag{38}$$

where

$$\xi=h\left\{x-\int\frac{\left[2b_1b_{-1}(n+1)b(t)+na_0^2a(t)\right]}{na_0^2}dt\right\}. \tag{39}$$

3.2. Example2. The generalized BBM equation with variable coefficients

In order to obtain the exact solutions of Eq. (2), we assume that the solution of this equation has the same from (8). Substituting (8) into (2), we have

$$\begin{aligned}&(n-1)^2\left[\upsilon^4(\xi)\upsilon_t(\xi)+a(t)\upsilon^4(\xi)\upsilon_x(\xi)\right]\\&+2n\left(6n^2-5n+1\right)k(t)\upsilon_t(\xi)\left[\upsilon_x(\xi)\right]^2\\&-2n(n-1)(3n-1)k(t)\upsilon(\xi)\upsilon_x(\xi)\upsilon_{xt}(\xi)\\&+n(n-1)^2k(t)\upsilon^2(\xi)\upsilon_{xxt}(\xi)+n(n-1)^2b(t)\upsilon^2(\xi)\upsilon_x(\xi)\\&-n(n-1)(3n-1)k(t)\upsilon(\xi)\upsilon_t(\xi)\upsilon_{xx}(\xi)=0.\end{aligned} \tag{40}$$

Using the generalized wave transformation (4), Eq. (40) converts to the nonlinear ODE:

$$\begin{aligned}&(n-1)^2\left[\tau(t)+ha(t)\right]\upsilon^4(\xi)\upsilon'(\xi)\\&+2n(6n^2-5n+1)h^2k(t)\tau(t)\left[\upsilon'(\xi)\right]^3\\&-3n(n-1)(3n-1)h^2k(t)\tau(t)\upsilon(\xi)\upsilon'(\xi)\upsilon''(\xi)\\&+n(n-1)^2h^2k(t)\tau(t)\upsilon^2(\xi)\upsilon'''(\xi)\\&+n(n-1)^2hb(t)\upsilon^2(\xi)\upsilon'(\xi)=0.\end{aligned} \tag{41}$$

Substituting Eq. (17) into Eq. (41), and by the help of *Maple*, we get

$$\frac{1}{B}\left\{\beta_{10}e^{10\xi}+\beta_9e^{9\xi}+\beta_8e^{8\xi}+\beta_7e^{7\xi}+\beta_6e^{6\xi}+\beta_5e^{5\xi}+\beta_4e^{4\xi}+\beta_3e^{3\xi}+\beta_2e^{2\xi}+\beta_1e^{\xi}+\beta_0\right\}=0, \quad (42)$$

where

$$B=e^{5\xi}\left(b_1e^{\xi}+b_0+b_{-1}e^{-\xi}\right)^6, \quad (43)$$

and β_n are coefficients of $e^{n\xi}$. Equating all the coefficients of $e^{n\xi}$ to zero, we obtain a system of algebraic equations which can be solved by the *Maple*, and get the following cases of solutions

Case 1.

$$a_0=0,\ a_1=a_1,\ a_{-1}=0,\ b_0=b_0,\ b_1=0,\ b_{-1}=0,$$
$$\tau(t)=-h\,a(t),\ k(t)=\frac{(n-1)^2\,b(t)}{4h^2n^2a(t)},$$
$$a(t)=a(t),\ b(t)=b(t), \quad (44)$$

In this case, the exact solution of Eq. (2) has the form:

$$u(\xi)=\left(\frac{a_1e^{\xi}}{b_0}\right)^{\frac{-2}{n-1}}, \quad (45)$$

Case 2.

$$a_0=a_0,\ a_1=0,\ a_{-1}=0,\ b_0=0,\ b_1=b_1,\ b_{-1}=0,$$
$$\tau(t)=-h\,a(t),\ k(t)=\frac{(n-1)^2\,b(t)}{4h^2n^2a(t)},$$
$$a(t)=a(t),\ b(t)=b(t), \quad (46)$$

In this case, the exact solution of Eq. (2) has the form:

$$u(\xi)=\left(\frac{a_0}{b_1e^{\xi}}\right)^{\frac{-2}{n-1}}, \quad (47)$$

Case 3.

$$a_0=a_0,\ a_1=0,\ a_{-1}=0,\ b_0=0,\ b_1=0,\ b_{-1}=b_{-1},$$
$$\tau(t)=-h\,a(t),\ k(t)=\frac{(n-1)^2\,b(t)}{4h^2n^2a(t)},$$
$$a(t)=a(t),\ b(t)=b(t), \quad (48)$$

In this case, the exact solution of Eq. (2) has the form:

$$u(\xi)=\left(\frac{a_0e^{\xi}}{b_{-1}}\right)^{\frac{-2}{n-1}}, \quad (49)$$

Case 4.

$$a_0=0,\ a_1=0,\ a_{-1}=a_{-1},\ b_0=b_0,\ b_1=0,\ b_{-1}=0,$$
$$\tau(t)=-h\,a(t),\ k(t)=\frac{(n-1)^2\,b(t)}{4h^2n^2a(t)},$$
$$a(t)=a(t),\ b(t)=b(t), \quad (50)$$

In this case, the exact solution of Eq. (2) has the form:

$$u(\xi)=\left(\frac{a_{-1}e^{-\xi}}{b_0}\right)^{\frac{-2}{n-1}}, \quad (51)$$

Case 5.

$$a_0=0,\ a_1=0,\ a_{-1}=a_{-1},\ b_0=0,\ b_1=b_1,\ b_{-1}=0,$$
$$\tau(t)=-h\,a(t),\ k(t)=\frac{(n-1)^2\,b(t)}{4h^2n^2a(t)},$$
$$a(t)=a(t),\ b(t)=b(t), \quad (52)$$

In this case, the exact solution of Eq. (2) has the form:

$$u(\xi)=\left(\frac{a_{-1}e^{-2\xi}}{b_1}\right)^{\frac{-2}{n-1}}, \quad (53)$$

Case 6.

$$a_0=a_0,\ a_1=a_1,\ a_{-1}=0,\ b_0=b_0,\ b_1=0,\ b_{-1}=\frac{a_0b_0}{a_1},$$
$$\tau(t)=-h\,a(t),\ k(t)=\frac{(n-1)^2\,b(t)}{16h^2n^2a(t)},$$
$$a(t)=a(t),\ b(t)=b(t), \quad (54)$$

In this case, the exact solution of Eq. (2) has the form:

$$u(\xi)=\left[\frac{a_1\left(a_0+a_1e^{\xi}\right)}{b_0\left(a_1+a_0e^{-\xi}\right)}\right]^{\frac{-2}{n-1}}, \quad (55)$$

Case 7.

$$a_0=a_0,\ a_1=0,\ a_{-1}=a_{-1},\ b_0=b_0,\ b_1=\frac{a_0b_0}{a_{-1}},\ b_{-1}=0,$$
$$\tau(t)=-h\,a(t),\ k(t)=\frac{(n-1)^2\,b(t)}{4h^2n^2a(t)},$$
$$a(t)=a(t),\ b(t)=b(t), \quad (56)$$

In this case, the exact solution of Eq. (2) has the form:

$$u(\xi)=\left[\frac{a_{-1}\left(a_0+a_{-1}e^{\xi}\right)}{b_0\left(a_{-1}+a_0e^{\xi}\right)}\right]^{\frac{-2}{n-1}}, \tag{57}$$

where ξ in above cases 1-7 has the same form (36).

Case 8.

$$a_0=a_0,\ a_1=0,\ a_{-1}=0,\ b_0=0,\ b_1=b_1,\ b_{-1}=b_{-1},$$

$$\tau(t)=\frac{ha_0^2\left(n-1\right)^2a(t)}{-a_0^2\left(n-1\right)^2+8h^2n\left(n+1\right)b_1b_{-1}k(t)},$$

$$a(t)=a(t),\quad b(t)=b(t),$$

$$k(t)=\frac{\left[-a_0^2\left(n-1\right)^2+8h^2n\left(n+1\right)b_1b_{-1}k(t)\right]b(t)}{-4h^2a_0^2n^2a(t)}. \tag{58}$$

In this case, the exact solution of Eq. (2) has the form:

$$u(\xi)=\left[\frac{a_0}{b_1e^{\xi}+b_{-1}e^{-\xi}}\right]^{\frac{-2}{n-1}}, \tag{59}$$

where ξ in the above cases 1-8 has the form

$$\xi=h\left[x-\int\frac{a_0^2\left(n-1\right)^2a(t)}{a_0^2\left(n-1\right)^2-8h^2n\left(n+1\right)b_1b_{-1}k(t)}dt\right]. \tag{60}$$

Remark: All solutions of this article have been checked with *Maple* by putting them back into the original equations (1) and (2).

4. Conclusions

In this article, the modified exp-function method was applied in order to find the exact solutions of the generalized K(n,n) and BBM equations with variable coefficients. The exp-function method is a very powerful and effective technique in finding the exact solutions for a wide range of problems. The solutions so obtained have also been verified to satisfy the original equation. This method is important because the solutions obtained can be applied to a wide range of problems in science and engineering.

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