
Numerical solution of nonlinear optimal control problems based on state parametrization

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Abstract

In this paper, solution of nonlinear optimal control problems and the controlled Duffing oscillator, as a special class of optimal control problems, are considered and an efficient algorithm is proposed. This algorithm is based on state parametrization as a polynomial with unknown coefficients. By this method, the control and state variables can be approximated as a function of time. Also, the numerical value of the performance index is obtained readily. The convergence of the algorithm is proved. To demonstrate reliability and efficiency of the proposed algorithm, the scheme is tested on some numerical examples.

Keywords: Optimal control problems; state parametrization; control linear oscillator and duffing oscillator; weierstrass approximation theorem

1. Introduction

Perhaps the true birth of optimal control theory was in the 17th century with Johann Bernoulli's famous *brachistochrone* problem. In fact, solutions of several reputable mathematicians to this problem, marks the beginnings of optimal control theory. With over 300 years of research in this area, many significant advancements have been made, like calculus of variations, which was first elaborated by Euler in 1733. Optimal control theory, in its modern sense, began in the 1950's with the formulation of two design optimization techniques: Dynamic Programming and the Pontryagin Maximum Principle. In the 1950's, Bellman pioneered work in dynamic programming which led to sufficient conditions for optimality using the Hamilton-Jacobi-Bellman equation [1]. Pontryagin's development of the maximum (minimum) principle in 1962 provided a method to determine the optimal control for constrained problems [2]. The optimal control problem has been studied by many researchers [3-8]. As analytical solutions for problems of optimal control are not always available, finding a numerical solution for solving optimal control problems is at least the most logical way to treat them and has provided an attractive field for researchers of mathematical sciences. Numerical methods for solving optimal control problems are plentiful and vary greatly in

their approach and complexity. In [9, 10], the authors presented a numerical technique for solving nonlinear constrained optimal control problems. Gindy presented a numerical solution for solving optimal control problems and the controlled Duffing oscillator, using a new Chebyshev spectral procedure [11]. Jaddu, presented numerical methods to solve unconstrained and constrained optimal control problems [12]. In [13], the authors presented a spectral method of solving the controlled Duffing oscillator. In [14], a numerical technique is shown for solving the controlled Duffing oscillator, in which the control and state variables are approximated by Chebyshev series. In [15], an algorithm for solving optimal control problems and the controlled Duffing oscillator is presented; in the algorithm the solution is based on state parametrization such that the state variable can be considered as a linear combination of Chebyshev polynomials with unknown coefficients. State parametrization converts the problem to a nonlinear optimization problem and finds $(n + 1)$ unknown polynomial coefficients of degree at most n in the form of $\sum_{k=0}^n a_k t^k$ for optimal solution [16, 17].

In recent years, different numerical computational methods and efficient algorithms have been used to solve the optimal control problems [18-21]. An optimal control software package, *MISER3*, has been developed by Jennings et al. [22] to solve the optimal control problems. *MISER3* has been used

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in solving various kinds of control problems in different aspects [23].

This paper is organized into the following sections of which this introduction is the first. In Section 2, we introduce mathematical formulation. Section 3 is about State Parametrization. The modified algorithm is derived in Section 4. In Section 5 convergence analysis is obtained. In Section 6 we present a numerical example to illustrate the efficiency and reliability of the presented method. The solution of control linear oscillator and Duffing oscillator with the new algorithm are presented in section 7. Finally, the paper is concluded.

2. Mathematical formulation for optimal control problems

The kind of information available to the controller at each instant of time plays an important role in describing a control model. Two cases exist; either the controller has no information during the system operation, known as open loop, or the controller knows the state of the system at each instant of time t , known as feedback. In optimal control problem, we have to determine one of these presented controls. The control function u is assumed to be piecewise from class of admissible controls, U . Each choice of control $u(\cdot) \in U \subset \mathcal{R}^m$ yields a process $x(t) \in \mathcal{R}^m$ which is the unique solution of

$$\dot{x}(t) = f(t, x(t), u(t)), \quad (1)$$

which is called the *equation of motion*, on a fixed interval $[t_0, t_1]$ with initial condition

$$x(t_0) = x_0. \quad (2)$$

Along with this controlled process, a cost functional of the form:

$$J(x_0, u) = \phi(t_1, x(t_1)) + \int_{t_0}^{t_1} L(t, x(t), u(t)) dt, \quad (3)$$

is defined. Here, $L(t, x, u)$ is the running cost, and $\phi(t, x)$ is the terminal cost. This cost functional depends on the initial position (t_0, x_0) and the choice of control $u(\cdot)$. The optimization problem is therefore to minimize $J(x_0, u)$, for each (x_0, u) , over all controls $u(t) \in U$. The pair (x_0, u) which achieves this minimum is called an optimal control. In fact, the optimization problem with performance index as defined in equation (3)

is called a *Bolza* problem. There are two other equivalent optimization problems, which are called *Lagrange* and *Mayer* problems [4].

To clarify these definitions, the following example is described:

Example: Suppose

$$\dot{x}(t) = u(t), x(0) = 1, [t_0, t_1] = [0, 2],$$

$$L \equiv 0, \phi(t, x) = x^2, U = L^\infty([0, 2]; [-1, 1])$$

Since we have no running cost, our best course of action is to steer as fast as possible toward zero (where $\phi(t, x)$ is minimized). We are limited to velocities in the range $[-1, 1]$ but we have 2 time units to get to 0, so we can easily reach 0. In fact, there are many optimal controls here such as:

$$u(t) \equiv -\frac{1}{2},$$

and

$$u(t) = \begin{cases} -1, & 0 \leq t \leq 1, \\ 0, & t \geq 1, \end{cases}$$

If instead, $x(0) = -2$, then the only optimal control would be $u(t) \equiv 1$.

An optimal control problem can be solved by one of the following methods [24]:

- Bellman's dynamic programming method which is based on the principle of optimality (Hamilton-Jacobi-Bellman equation).
- Pontryagin's minimum principle and variational method (Euler-Lagrange equations).
- Direct methods using parametrization or discretization (nonlinear mathematical programming).

3. State parametrization

The optimal control problems can be converted into a mathematical programming problem by using the parametrization techniques. One such technique is known as state parametrization. The idea of state parametrization is to approximate only the system state variable by a sequence of given functions with unknown parameters

$$x_n(t) = \sum_{k=0}^N a_k^{(n)} \phi_k(t), \quad n = 1, 2, 3, \dots$$

and then the control variable are obtained from the state equations. The state parametrization can be employed using different basis functions [25]. In this work, polynomial basis will be applied to

introduce a new algorithm for solving optimal control problems numerically.

4. Deriving the modified algorithm

In this section, we use a state parametrization method to derive a robust method for solving optimal control problems numerically. The problem is to find the optimal control $u(t)$ that minimizes the performance index

$$J = \int_0^1 L(t, x(t), u(t)) dt \quad (4)$$

subject to the following constraints.

(a) State equation described by the following nonlinear differential equation on the fixed time interval $[0, 1]$:

$$u(t) = f(t, x(t), \dot{x}(t), \ddot{x}(t)) \quad (5)$$

where $x(\cdot): [0, 1] \rightarrow R$ is the state variable, and $u(\cdot): [0, 1] \rightarrow R$ is the control variable, f is a real-valued continuously differentiable function.

(b) Initial and final conditions:

$$x(0) = x_0, \dot{x}(0) = \dot{x}_0, x(1) = x_1, \dot{x}(1) = \dot{x}_1, \quad (6)$$

where x_0, \dot{x}_0, x_1 and \dot{x}_1 are given states in R and $x(\cdot)$ belongs to the class of all continuously differentiable functions on $[0, 1]$; $C^1([0, 1])$. Let Q be set of all functions satisfying initial and final conditions, which $Q \subset C^1([0, 1])$. Substituting (5) into (4) shows that performance index (4) can be explained as a function of x . Then the optimal control problem (4)-(6) may be considered as minimization of J on the set Q . Now, we assume that Q_n is a subset of Q consisting of all polynomials of degree at most n as

$$x_n(t) = \sum_{k=0}^n a_k t^k, \quad n = 1, 2, 3, \dots \quad (7)$$

And consider the minimization of J on Q_n with $\{a_k\}_{k=0}^n$ as unknowns. This is an optimization problem in $(n+1)$ -dimensional space and $J(x_n)$ may be considered as $J(a_0, a_1, \dots, a_n)$. Here the algorithm in [16] is generalized to fit for problem (4)-(6) on the interval $[0, 1]$:

Algorithm 1. Choose a small number ε for accuracy of the solution.

Step 1. Set $n = 1$, and calculate

$$x_1(t) = x_0 + \dot{x}_0 t + (-2\dot{x}_0 - 3x_0 + 3x_1 - \dot{x}_1)t^2 + (2x_0 + \dot{x}_0 - 2x_1 + \dot{x}_1)t^3$$

then set $\alpha_1 = J(x_1(\cdot))$.

Step 2. Set $n \rightarrow n+1$ and find $\alpha_n = \inf_{Q_n} J$.

Step 3. If $|\alpha_{n-1} - \alpha_n| < \varepsilon$ then stop, otherwise return to Step 2.

In the above algorithm, the solution of an optimization problem in all iterations is required and the solution of each iteration is not used to construct the next one. This seems too expensive from a computational view point. Here, this algorithm is improved. Now, a state parametrization method is used to derive a robust method for solving optimal control problems numerically. In comparison with other numerical methods, the number of unknowns of the proposed method is lower than those in control state parametrization. In state parametrization, the solution of the approximation is considered as in equation (7), by using $(n+1)$ terms of $1, t, t^2, \dots, t^n$ as a basis for Q_n . Algorithm 1 yields a solution of the optimization problem in all iterations, but the solution of each step is obtained independently from previous steps, so it is costly. Algorithm 1 is made more efficient in this section. The following approximation is considered for $x(\cdot)$ to start with:

$$x_1(t) = \sum_{i=0}^4 a_i t^i. \quad (8)$$

By using boundary conditions (6) we obtain a linear system of equations. Variables a_0, a_1, a_2 and a_3 are obtained as a function of a_4 by solving the linear system of equations given by:

$$a_0 = x_0, \quad a_2 = -3x_0 - 2\dot{x}_0 + 3x_1 - \dot{x}_1 + a_4, \\ a_1 = \dot{x}_0, \quad a_3 = 2x_0 + \dot{x}_0 - 2x_1 + \dot{x}_1 - 2a_4, \quad (9)$$

Now substituting relations (9) into (8) yields:

$$x_1(t) = x_0 + \dot{x}_0 t + (-3x_0 - 2\dot{x}_0 + 3x_1 - \dot{x}_1 + a_4)t^2 + (2x_0 + \dot{x}_0 - 2x_1 + \dot{x}_1 - 2a_4)t^3 + a_4 t^4, \quad (10)$$

and then $u(t)$ is obtained from equation (5). Now we obtain J as a function of a_4 by calculating

$$J = \int_0^1 L(t, x(t), u(t)) dt \text{ and refer to it as } J(a_4)$$

. Let $J(a^*)$ be the value which minimizes $J(a_4)$, then $J(a^*)$ is the solution of optimal control problem (4)-(6). Also, the state and control variables can be calculated from a^* approximately. In the next step, $x_2(t)$ is approximated as below:

$$x_2(t) = x_1(t) + a_0 + a_1 t + \sum_{i=3}^5 a_i t^i. \quad (11)$$

Then using boundary conditions we have:

$$\begin{aligned} a_0 &= x_0, & a_2 &= -4x_0 - 3\dot{x}_0 + 4x_1 - \dot{x}_1 + a_5, \\ a_1 &= \dot{x}_0, & a_3 &= 3x_0 + 2\dot{x}_0 - 3x_1 + \dot{x}_1 - 2a_5, \end{aligned} \quad (12)$$

Substituting relations (12) into equation (11) yields $x_2(t)$ and then $u(t)$ can be obtained from equation (5). Now we obtain J as a function of a_5

by calculating $J = \int_0^1 L(t, x(t), u(t)) dt$ and refer

to it as $J(a_5)$. If a^* is the value which minimizes $J(a_5)$, then $J(a^*)$ is the solution of optimal control problem in equations (4)-(6). Also, we can calculate state and control variables from a^* approximately. By continuing this procedure we obtain a favourable accuracy, for example in the $(n+1)^{th}$ step, the approximate solution is given by:

$$x_{n+1}(t) = x_n(t) + a_0 + a_1 t + \sum_{i=n+2}^{n+4} a_i t^i. \quad (13)$$

By using boundary conditions (6) we have:

$$x_{n+1} = x_n + a_{n+4}(1 - 2t + t^2)t^{n+2}, \quad (14)$$

Now only one unknown variable, a_{n+4} remains. The above results lead to the following algorithm which obtains the optimal performance index $J(\cdot)$.

Algorithm 2. Choose a small number ε for accuracy of the solution.

Step1: For $n = 1$, calculate

$$\begin{aligned} x_1(t) &= x_0 + \dot{x}_0 t + (-3x_0 - 2\dot{x}_0 + 3x_1 - \dot{x}_1 + a_4)t^2 \\ &\quad + (2x_0 + \dot{x}_0 - 2x_1 + \dot{x}_1 - 2a_4)t^3 + a_4 t^4 \end{aligned}$$

and then calculate $a_1^* \in \text{Argmin}\{J(a) : a \in \mathcal{R}\}$ and set $\rho_1 = J(a_1^*)$.

Step2. Let $n \rightarrow n+1$ and calculate

$$x_{n+1} = x_n + a_{n+4}(1 - 2t + t^2)t^{n+2}$$

Step3. Calculate $a_{n+1}^* \in \text{Argmin}\{J(a) : a \in \mathcal{R}\}$ and set $\rho_{n+1} = J(a_{n+1}^*)$.

Step4. If $|\rho_{n+1} - \rho_n| < \varepsilon$ then stop, otherwise return to Step 3.

5. Convergence analysis

This section covers the convergence analysis of the proposed algorithms. As a well-known powerful tool, for convergence of this method we have Weierstrass approximation theorem (1885).

Theorem 1. Let $f \in C([a, b], \mathcal{R})$. Then there is a sequence of polynomials $P_n(x)$ that converges uniformly to $f(x)$ on $[a, b]$.

Proof: See [26].

In the following theorems the convergence of the presented algorithms are proved. In fact, these results lead to the algorithms 1 and 2, which obtain the optimal performance index $J(\cdot)$. In the following theorems the convergence of the algorithm 1 is proved.

Theorem 2. If $\alpha_n = \inf_{\mathcal{Q}_n} J$, for $n = 1, 2, \dots$, then

$$\lim_{n \rightarrow \infty} \alpha_n = \alpha \text{ where } \alpha = \inf_{\mathcal{Q}} J.$$

Proof:

Let $\hat{\alpha}$ be the limit of non-increasing sequence

$$\{\alpha_n\}. \text{ If } \hat{\alpha} > \alpha \text{ then } \varepsilon = \frac{\hat{\alpha} - \alpha}{2} > 0 \text{ so}$$

$$\exists x(\cdot) \in \mathcal{Q}, J(x(\cdot)) < \varepsilon + \alpha = \frac{\hat{\alpha} - \alpha}{2} + \alpha = \frac{\hat{\alpha} + \alpha}{2} < \hat{\alpha}$$

such that $\hat{\alpha} > J(x(\cdot))$, which contradicts the continuity of J and Theorem 1.

The following theorem proves the convergence of the algorithm 2.

Theorem 3. If J has continuous first derivatives, then $\lim_{n \rightarrow \infty} \rho_n = \alpha$, where $\alpha = \inf_Q J$.

Proof:

If we define $\rho_n = \min_{a_n \in \mathcal{R}} J(a_n)$, then

$$\rho_n = J(a_n^*),$$

such that

$$a_n^* \in \text{Arg min}\{J(a_n) : a_n \in \mathcal{R}\},$$

Let $x_n^*(t) \in \text{Arg min}\{J(x(t)) : x(t) \in \mathcal{Q}_n'\}$ then

$$J(x_n^*(t)) = \min_{x(t) \in \mathcal{Q}_n'} J(x(t)), \text{ in which } \mathcal{Q}_n' \text{ is a}$$

class of polynomials in t of degree n . It is obvious that $\rho_n = J(x_n^*(t))$, furthermore, we have:

$$\min_{x(t) \in \mathcal{Q}_{n+1}'} J(x(t)) \leq \min_{x(t) \in \mathcal{Q}_n'} J(x(t))$$

Thus, we will have $\rho_{n+1} \leq \rho_n$, which means ρ_n is a non-increasing sequence, now according to Theorem 1, the proof is complete, that is:

$$\lim_{n \rightarrow \infty} \rho_n = \min_{x(t) \in \mathcal{Q}} J(x(t)).$$

6. An application example

To illustrate the efficiency of the presented algorithm, we consider the following example. This problem has continuous optimal controls and can be solved analytically. This allows verification and validation of the method by comparing with the results of exact solution. Besides, the problems of controlled linear and Duffing oscillators are solved here.

Example 1. The object is to find the optimal control which minimizes

$$J = \frac{1}{2} \int_0^2 u^2(t) dt, \quad 0 \leq t \leq 2, \quad (15)$$

when

$$u(t) = \dot{x}(t) + \ddot{x}(t), \quad (16)$$

and

$$x(0) = 0, \dot{x}(0) = 0, x(2) = 5, \dot{x}(2) = 2, \quad (17)$$

are satisfied [5]. Where analytical solution is

$$x(t) = -6.103 + 7.289t + 6.696e^{-t} - 0.593e^t,$$

and

$$u(t) = 7.289 - 1.186e^t.$$

Therefore, the exact value of performance index is $J = 16.7454386$. By using Step 2 of the algorithm 2 we consider an approximation of $x_1(t)$ to start with as:

$$x_1(t) = \left(\frac{11}{4} + 4a_4\right)t^2 - \left(\frac{3}{4} + 4a_4\right)t^3 + a_4t^4, \quad (18)$$

Then $u(t)$ is obtained from equations (16) and (18) as:

$$u(t) = \frac{11}{2} + 8a_4 + (1 - 16a_4)t - \frac{9}{4}t^2 + 4a_4t^3. \quad (19)$$

then substituting equations (19) into equation (15) gives:

$$J(a_4) = \frac{1472}{105}a_4^2 - \frac{16}{15}a_4 + \frac{1007}{60},$$

where $a^* = \frac{7}{184}$ is the value which minimizes

$J(a_4)$ and $J(a^*) = 16.7630435$ is the solution of optimal control problem (15)-(17). By substituting a^* into equations (18) and (19) we can calculate state and control variables approximately as:

$$x_1(t) = \frac{267}{92}t^2 - \frac{83}{92}t^3 + \frac{7}{184}t^4,$$

and

$$u(t) = \frac{267}{46} + \frac{9}{23}t - \frac{9}{4}t^2 - \frac{7}{46}t^3,$$

The solution is obtained and the analytical solutions are plotted in Fig. 1.

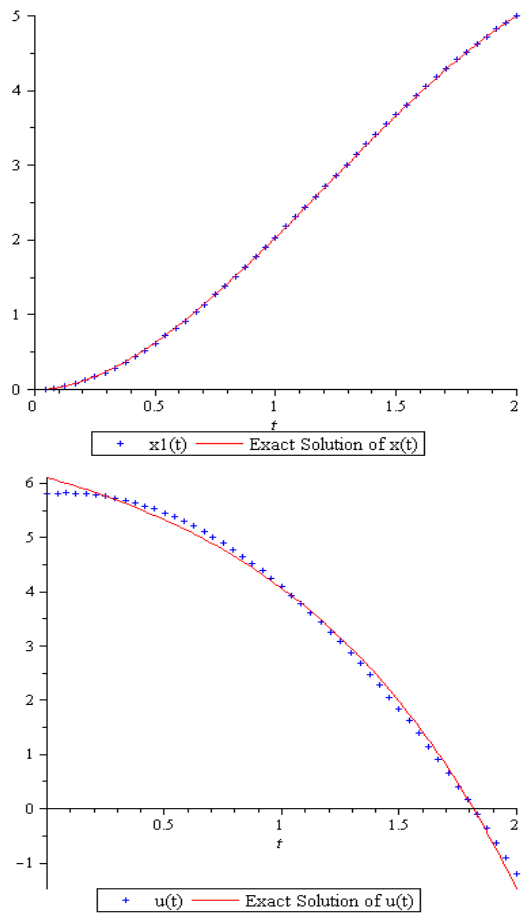


Fig. 1. Solution of Example 1. The solution in the first iteration is compared with the actual analytical solution.

In the next step $x_2(t)$ approximates the solution as follows:

$$x_2(t) = x_1(t) + a_0 + a_1t + \sum_{i=3}^5 a_i t^i,$$

Now, the above procedure is repeated. In the second iteration, $a^* = -\frac{27}{1856}$ is the value which minimizes $J(a_5) = \frac{7424}{315}a_5^2 + \frac{24}{35}a_5 + \frac{7711}{460}$ and $J(a^*) = 16.74543860$ is the solution of optimal control problem (15)-(17). By using a^* , we can calculate state and control variables approximately as:

$$x_2(t) = \frac{267}{92}t^2 - \frac{10249}{10672}t^3 + \frac{1027}{10672}t^4 - \frac{27}{1856}t^5,$$

and

$$u(t) = \frac{267}{46} + \frac{225}{5336}t - \frac{801}{464}t^2 + \frac{1003}{10672}t^3 - \frac{135}{1856}t^4,$$

The solution is obtained and the analytical solutions are plotted in Fig. 2.

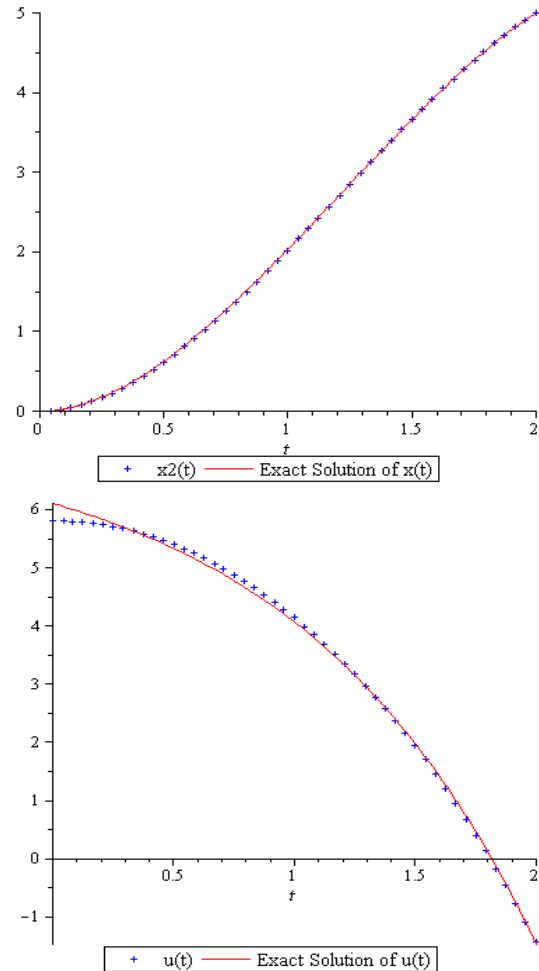


Fig. 2. Solution of Example 1. The solution in the second iteration compared with the actual analytical solution.

The optimal cost functional J , obtained by algorithm 2, is shown in Table 1.

Table 1. optimal cost functional J for Example 1

Iteration	Present method	Error
1	16.7630435	$1.8 e^{-2}$
2	16.74417688	$1.3 e^{-3}$
3	16.74531717	$1.2 e^{-4}$

7. The control linear oscillator

We will consider the optimal control of a linear oscillator governed by the differential equation

$$u(t) = \ddot{x}(t) + \omega^2 x(t), \quad t \in [-T, 0], \quad (20)$$

in which T is specified. Equation (20) is equivalent to the dynamic state equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -\omega^2 x_1(t) + u(t), \end{aligned}$$

with the boundary conditions

$$\begin{aligned} x_1(-T) &= x_0, & x_2(-T) &= \dot{x}_0 \\ x_1(0) &= 0, & x_2(0) &= 0. \end{aligned} \quad (21)$$

It is desired to control the state of this plant such that the performance index

$$J = \frac{1}{2} \int_{-T}^0 u^2(t) dt, \quad (22)$$

is minimized over all admissible control functions $u(t)$. Pontryagin's maximum principle method [2] applied to this optimal control problem yields the following exact analytical solution [10]:

$$\begin{aligned} x_1(t) &= \frac{1}{2\omega^2} [A\omega t \sin \omega t + B(\sin \omega t - \omega t \cos \omega t)], \\ x_2(t) &= \frac{1}{2\omega} [A(\omega t \sin \omega t + \omega t \cos \omega t) + B\omega t \sin \omega t], \\ u(t) &= A \cos \omega t + B \sin \omega t, \\ J &= \frac{1}{8\omega} [2\omega T(A^2 + B^2) + (A^2 - B^2) \sin 2\omega T - 4AB \sin^2 \omega T], \end{aligned}$$

where

$$\begin{aligned} A &= \frac{2\omega[x_0\omega^2 T \sin \omega T - \dot{x}_0(\omega T \cos \omega T - \sin \omega T)]}{\omega^2 T^2 - \sin^2 \omega T}, \\ B &= \frac{2\omega^2[\dot{x}_0 T \sin \omega T + x_0(\omega T \cos \omega T + \sin \omega T)]}{\omega^2 T^2 - \sin^2 \omega T}. \end{aligned}$$

7.1. Solution of the problem using Algorithm 2

We apply algorithm 2 in interval $[-T, 0]$. So, the approximation of $x(\cdot)$ is considered as follows:

$$x_1(t) = \sum_{i=0}^4 a_i t^i, \quad (23)$$

by using boundary conditions (21) we have:

$$\begin{cases} a_0 - a_1 T + a_2 T^2 - a_3 T^3 + a_4 T^4 = x_0, \\ a_0 = x_1, \\ a_1 - 2T a_2 + 3T^2 a_3 - 4T^3 a_4 = \dot{x}_0, \\ a_1 = \dot{x}_1, \end{cases} \quad (24)$$

then a_1 , a_2 , a_3 and a_4 are obtained as a function of a_4 by solving the linear system of equations given by:

$$\begin{aligned} a_0 &= x_0, & a_2 &= \frac{2\dot{x}_1 T - 3x_1 + 3x_0 + \dot{x}_0 T + a_4 T^4}{T^2}, \\ a_1 &= \dot{x}_0, & a_3 &= \frac{2\dot{x}_1 T - 3x_1 + 3x_0 + \dot{x}_0 T + a_4 T^4}{T^2}, \end{aligned} \quad (25)$$

Now substituting relations (25) into (23) yields:

$$\begin{aligned} x_1(t) &= x_0 + \dot{x}_0 t + \frac{2\dot{x}_1 T - 3x_1 + 3x_0 + \dot{x}_0 T + a_4 T^4}{T^2} t^2 \\ &\quad + \frac{2\dot{x}_1 T - 3x_1 + 3x_0 + \dot{x}_0 T + a_4 T^4}{T^2} t^3, \end{aligned} \quad (26)$$

Then $u(t)$ is obtained from equation (20). Now we obtain J as a function of a_4 by calculating $\frac{1}{2} \int_{-T}^0 u^2(t) dt$ and denote it by $J(a_4)$. So, the value which minimizes $J(a_4)$ is shown by a^* . In fact, $J(a^*)$ is the solution of optimal control problem (20)-(22). Also we can calculate state and control variables from a^* approximately. Now we report the approximation of the state and control variables of the controlled linear oscillator problem with the following choice of the numerical values of the parameters in the standard case:

$$\omega = 1, T = 2, x_0 = 0.5, \dot{x}_0 = -0.5, \quad (27)$$

Substituting the values in (27) into equation (26) yields:

$$x_1(t) = \left(\frac{1}{8} + 4a_4\right)t^2 + 4a_4 t^3 + a_4 t^4, \quad (28)$$

then $u(t)$ is obtained from equation (20) as follows:

$$\begin{aligned} u(t) &= \frac{1}{4} + 8a_4 + 24a_4 t + \left(\frac{1}{8} + 16a_4\right)t^2 \\ &\quad + 4a_4 t^3 + a_4 t^4, \end{aligned} \quad (29)$$

and also by equation (22) we have,

$$J(a_4) = \frac{47}{240} + \frac{24}{35}a_4 + \frac{3392}{315}a_4^2,$$

where $a^* = \frac{27}{848}$ is the value which minimizes

$J(a_4)$ and $J(a^*) = 0.184916886$ is the solution of optimal control problem (20)-(22). By substituting a^* into equations (28) and (29) we can calculate state and control variables approximately as:

$$x_1(t) = -\frac{1}{424}t^2 - \frac{27}{212}t^3 - \frac{27}{848}t^4,$$

and

$$u(t) = \frac{1}{212} - \frac{81}{106}t - \frac{163}{424}t^2 - \frac{27}{212}t^3 - \frac{27}{848}t^4,$$

The solution is obtained and the analytical solutions are plotted in Fig. 3.

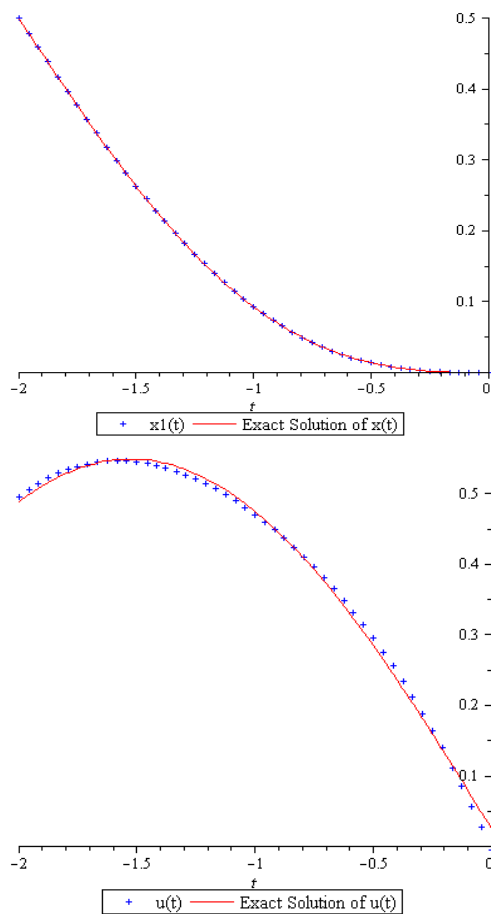


Fig. 3. Solution of the controlled linear oscillator problem. The solution in the first iteration compared with the actual analytical solution.

In the second iteration, $a^* = 0.0009954$ is the value which minimizes

$$J(a_4) = 0.1849 - 0.03809a_5 + 19.1354a_5^2$$

and $J(a^*) = 0.184897931$ is the solution of optimal control problem (20)-(22). By using a^* state and control variables are calculated as follow:

$$x_2(t) = -0.002358t^2 - 0.1234t^3 - 0.02786t^4 + 0.0009954t^5,$$

and

$$u(t) = -0.004717 - 0.7403t - 0.3366t^2 - 0.1035t^3 - 0.02786t^4 + 0.0009954t^5,$$

The solution is obtained and the analytical solutions are plotted in Fig. 4.

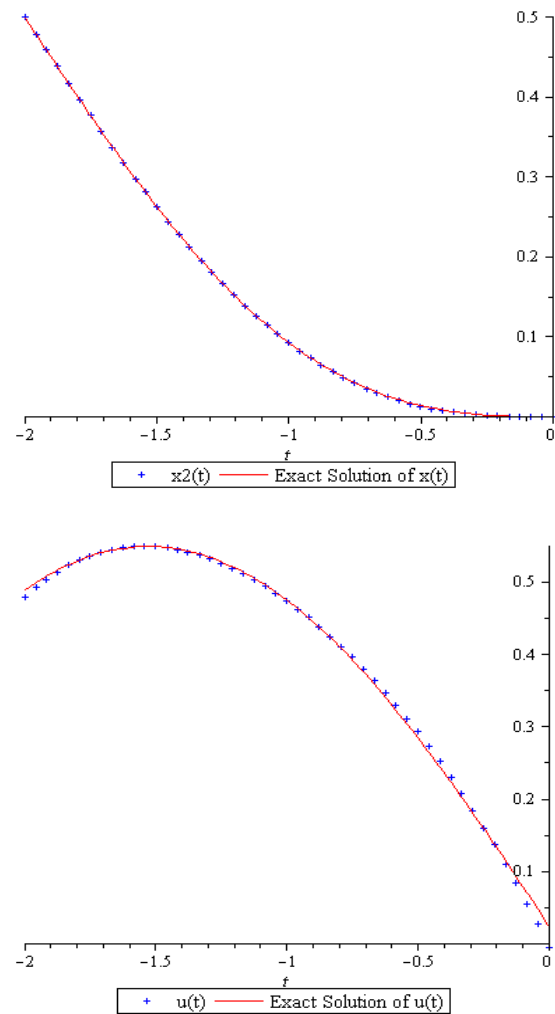


Fig. 4. Solution of the controlled linear oscillator problem. The solution in the second iteration compared with the actual analytical solution.

The approximate solution for the performance index as given in [10] is $J = 0.1848585422$. The optimal cost functional J , obtained by algorithm 2, is shown in Table 2.

Table 2. optimal cost functional J for for the controlled linear oscillator problem.

Iteration	Present method	Error
1	0.184916891	$5.8 e^{-5}$
2	0.184897931	$3.9 e^{-5}$
3	0.184897926	$0.4 e^{-5}$

7.2. The controlled Duffing Oscillator

Let us now investigate the optimal control of the Duffing oscillator, described by the nonlinear differential equation

$$u(t) = \ddot{x}(t) + \omega^2 x(t) + \varepsilon x^3(t), \quad t \in [-T, 0], \quad (30)$$

Subject to the boundary conditions and with the performance index pointed out as in the previously linear case. The exact solution in this case is not known. Table 3 lists the optimal values of the cost functional J for various values of ε in three iterations.

Table 3. optimal cost functional J for Duffing oscillator problem for various values of ε .

Iteration	Present method		
	$\varepsilon = 0.15$	$\varepsilon = 0.5$	$\varepsilon = 0.75$
1	0.187529231	0.193708065	0.198192324
2	0.187497301	0.193632589	0.198073948
3	0.187497044	0.193630281	0.198066860

8. Conclusion

This paper presents a numerical technique for solving nonlinear optimal control problems and the controlled Duffing oscillator as a special class of optimal control problems. The solution is based on state parametrization as a polynomial with unknown coefficients. Here an optimization problem in $(n + 1)$ -dimensional space is changed into one-dimensional optimization problem which can then be solved easily. In fact, a new parametrization is introduced, which can accurately represent continuous control and state variables as a function of time. This method provides a simple way to adjust and obtain an optimal control which can easily be applied to complex problems as well. The merit of our method in comparison with other numerical methods, is that the number of unknowns of the proposed method is lower than those in

existing methods and also its fast convergence. Here, the state and control variables can be calculated as functions of time and thus, the numerical value of the performance index is obtained readily. The convergence of the algorithms is proved. Some illustrative examples are solved by this method, the results show that the presented method is more powerful, requiring less computational work and hence reduce the computational costs significantly, which is an important factor to choose the method in engineering applications.

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