IJST (2013) 37A3 (Special issue-Mathematics): 403-410
Iranian Journal of Science & Technology
http://ijsts.shirazu.ac.ir

On characterization of spacelike dual biharmonic curves in dual Lorentzian Heisenberg group $\mathbb{D}^3_{\mathbf{Heis}^3}$

T. Korpinar[1], E. Turhan[2] and V. Asil[3]*

[1]*Department of Mathematics, Mus Alparslan University, 49250, Mus, Turkey*
[2, 3]*Department of Mathematics, Firat University, 23119, Elazig, Turkey*
E-mails: talatkorpinar@gmail.com, essin.turhan@gmail.com & vasil@firat.edu.tr

Abstract

In this paper, we study spacelike dual biharmonic curves. We characterize spacelike dual biharmonic curves in terms of their curvature and torsion in the Lorentzian dual Heisenberg group $\mathbb{D}^3_{\mathbf{Heis}^3}$. We give necessary and sufficient conditions for spacelike dual biharmonic curves in the Lorentzian dual Heisenberg group $\mathbb{D}^3_{\mathbf{Heis}^3}$. Therefore, we prove that all spacelike dual biharmonic curves are spacelike dual helix. Moreover, we give their explicit parametrizations of spacelike dual biharmonic curves. Finally, we illustrate our main results in Figs. 1 and 2.

Keywords: Bienergy; Biharmonic curve; helix; Heisenberg group

1. Introduction

Dual numbers had been introduced by W.K. Clifford (1849-1879) as a tool for his geometrical investigations. After him, E. Study used dual numbers and dual vectors in his research on line geometry and kinematics. He devoted special attention to the representation of oriented lines by dual unit vectors and defined the famous mapping: The set of oriented lines in an Euclidean three-dimension space $\mathbb{E}^3$ is one to one correspondence with the points of a dual space $\mathbb{D}^3$ of triples of dual numbers [1-9].

The theory of relativity opened a door to the use of degenerate submanifolds, and researchers have treated some topics of classical differential geometry extended to Lorentz manifolds [6, 10]. In light of the existing literature, we study dual biharmonic curves in Lorentzian Heisenberg group Heis^3.

Let (M^m, g) and (N^n, h) be two Riemannian manifolds, the energy functional of a map $\phi \in C^1(M^m, N^n)$ is defined by

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g, \tag{1}$$

where $|d\phi|$ is the Hilbert--Schmidt norm of the differential $d\phi$ and dv_g is the volume element on M. A map $\phi \in C^2(M^m, N^n)$ is called harmonic if it is a critical point of the energy functional, that is, if it is a solution of the Euler--Lagrange equation associated to (1)

$$\tau(\phi) = \mathrm{Tr}_g \nabla d\phi = 0, \tag{2}$$

$\tau(\phi)$ is called the tension field of ϕ. Harmonic maps are solutions of a second order nonlinear elliptic system and they play a very important rôle in many branches of mathematics and physics where they may serve as a model for liquid crystal. One can refer to [11] for background on harmonic maps.

A natural generalization of harmonic maps is given by integrating the square of the norm of the tension field. More precisely, the bi-energy functional of a map $\phi \in C^2(M^m, N^n)$ is defined by

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 dv_g, \tag{3}$$

a map $\phi \in C^4(M^m, N^n)$ is called biharmonic if it is a critical point of the bi-energy functional, that

*Corresponding author
Received: 13 June 2011 / Accepted: 18 October 2011

is, if it is a solution of the Euler-Lagrange equation associated to (3)

$$\tau_2(\phi) = -\mathrm{Tr}_g\left(\nabla^{\phi}\right)^2\tau(\phi) - \mathrm{Tr}_g R^N\left(\tau(\phi), d\phi\right)d\phi = 0, \qquad (4)$$

where

$$\mathrm{r}_g\left(\nabla^{\phi}\right)^2 = \sum_{i=1}^{m}\left(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla^M_{e_i}e_i}\right) \qquad (5)$$

for an orthonormal frame $\{e_1, e_2, .., e_m\}$, is the Laplacian on sections of the pull-back bundle $\phi^{-1}TN$ and R^N is the curvature operator on N. Clearly, harmonic maps are biharmonic. Jiang [12, 13] proved that if M is compact without boundary and the sectional curvature Riem $_N \leq 0$, then any biharmonic map $\phi : (M^m, g) \to (N^n, h)$ is harmonic. So it is interesting to construct non-harmonic biharmonic curves [14-22].

Biharmonic functions are utilized in many physical situations, particularly in fluid dynamics and elasticity problems. Most important applications of the theory of functions of a complex variable were obtained in the plane theory of elasticity and in the approximate theory of plates subject to normal loading. That is, in cases when the solutions are biharmonic functions or functions associated with them. In linear elasticity, if the equations are formulated in terms of displacements for two-dimensional problems then the introduction of a stress function leads to a fourth-order equation of biharmonic type. For instance, the stress function is proved to be biharmonic for an elastically isotropic crystal undergoing phase transition, which follows spontaneous dilatation. Biharmonic functions also arise when dealing with transverse displacements of plates and shells. They can describe the deflection of a thin plate subjected to uniform loading over its surface with fixed edges. Biharmonic functions arise in fluid dynamics, particularly in Stokes flow problems (i.e., low-Reynolds-number flows). There are many applications for Stokes flow such as in engineering and biological transport phenomena (for details, see [23]). Fluid flow through a narrow pipe or channel, such as that used in micro-fluidics, involves low Reynolds number. Seepage flow through cracks and pulmonary alveolar blood flow can also be approximated by Stokes flow. Stokes flow also arises in flow through porous media, which have been long applied by civil engineers to groundwater movement. The industrial applications include the fabrication of microelectronic components, the effect of surface roughness on lubrication, the design of polymer dies and the development of peristaltic pumps for sensitive viscous materials. In natural systems, creeping flows are important in biomedical applications and studies of animal locomotion.

In this paper, we study spacelike dual biharmonic curves. We characterize spacelike dual biharmonic curves in terms of their curvature and torsion in the Lorentzian dual Heisenberg group $\mathbb{D}^3_{\mathbf{Heis}^3}$. We give necessary and sufficient conditions for spacelike dual biharmonic curves in the Lorentzian dual Heisenberg group $\mathbb{D}^3_{\mathbf{Heis}^3}$. Therefore, we prove that all spacelike dual biharmonic curves are spacelike dual helix. Moreover, we give their explicit parametrizations of spacelike dual biharmonic curves. Finally, we illustrate our main results in Figs. 1 & 2.

2. Lorentzian dual heisenberg group $\mathbb{D}^3_{\mathbf{Heis}^3}$

The Heisenberg group Heis ^{3} is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x,y,z)*(\bar{x},\bar{y},\bar{z}) = (x+\bar{x}, y+\bar{y}, z+\bar{z}-\frac{1}{2}\bar{x}y+\frac{1}{2}x\bar{y}).$$

The identity of the group is $(0,0,0)$ and the inverse of (x,y,z) is given by $(-x,-y,-z)$. The left-invariant Lorentz metric on Heis ^{3} is

$$g = -dx^2 + dy^2 + (xdy + dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra [24]:

$$\left\{\mathbf{e}_1 = \frac{\partial}{\partial z}, \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \mathbf{e}_3 = \frac{\partial}{\partial x}\right\}. \qquad (6)$$

The characterising properties of this algebra are the following commutation relations:

$$[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1, [\mathbf{e}_3, \mathbf{e}_1] = 0, [\mathbf{e}_2, \mathbf{e}_1] = 0,$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, g(\mathbf{e}_3, \mathbf{e}_3) = -1. \qquad (7)$$

Proposition 2.1. For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above, the following is true:

$$\nabla = \frac{1}{2}\begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{8}$$

where the (i, j)-element in the table above equals $\nabla_{e_i} e_j$ for our basis

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator [25-27]:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = g(R(X,Y)W,Z).$$

Moreover, we put

$$R_{abc} = R(\mathbf{e}_a, \mathbf{e}_b)\mathbf{e}_c, R_{abcd} = R(\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c, \mathbf{e}_d),$$

where the indices a, b, c and d take the values $1,2$ and 3.

Then the non-zero components of the Riemannian curvature tensor field and of the Riemannian curvature tensor are, respectively,

$$R_{121} = \frac{1}{4}\mathbf{e}_2, R_{131} = \frac{1}{4}\mathbf{e}_3, \ R_{232} = -\frac{3}{4}\mathbf{e}_3,$$

and

$$R_{1212} = -\frac{1}{4}, R_{1313} = \frac{1}{4}, R_{2323} = -\frac{3}{4}. \tag{9}$$

The set $\mathbb{D}$ of dual numbers is a commutative ring with the operations (+) and (.).
The set

$$\mathbb{D}^3_{\mathbf{Heis}^3} = \left\{\hat{x} : \hat{x} = x + \varepsilon x^*, x, x^* \in Heis^3\right\}$$

is a module over the ring $\mathbb{D}$.
Let us set

$$\hat{\varphi} = \varphi + \varepsilon\varphi^* = \begin{bmatrix} 1 & x + \varepsilon x^* & z + \varepsilon z^* \\ 0 & 1 & y + \varepsilon y^* \\ 0 & 0 & 1 \end{bmatrix},$$

where $\varphi = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \ \varphi^* = \begin{bmatrix} 1 & x^* & z^* \\ 0 & 1 & y^* \\ 0 & 0 & 1 \end{bmatrix}.$

The left-invariant Lorentz metric on $\mathbb{D}^3_{Heis^3}$ is

$$g(\hat{x}, \hat{y}) = g(x, y) + \varepsilon\left(g(x, y^*) + g(x^*, y)\right). \tag{10}$$

A dual vector $\hat{x}$ with norm 1 is called a dual unit vector.

$$\mathbb{S}^2_1 = \left\{\hat{x} = x + \varepsilon x^* : \|\hat{x}\| = (1,0), x, x^* \in Heis^3 \text{and the vector } x \text{ is spacelike}\right\}$$

is called the dual Lorentzian unit sphere in $\mathbb{D}^3_{Heis^3}$.

$$\mathbb{H}^2_1 = \left\{\hat{x} = x + \varepsilon x^* : \|\hat{x}\| = (1,0), x, x^* \in Heis^3 \text{and the vector } x \text{ is timelike}\right\}$$

is called the dual hyperbolic unit sphere in $\mathbb{D}^3_{Heis^3}$.

3. Spacelike dual biharmonic curves in the lorentzian dual heisenberg group $\mathbb{D}^3_{\mathbf{Heis}^3}$

An arbitrary curve $\hat{\gamma} = \gamma + \varepsilon\gamma^* : I \to \mathbb{D}^3_{\mathbf{Heis}^3}$ is spacelike, timelike or null, if all of its velocity vectors $\hat{\gamma}'(s)$ are, respectively, spacelike, timelike or null, for each $s \in I \subset \mathbb{R}$. Let $\hat{\gamma} : I \to \mathbb{D}^3_{\mathbf{Heis}^3}$ be a unit speed spacelike curve with timelike normal and $\{\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}\}$ being Frenet vector fields, then Frenet formulas are as follows:

$$\begin{aligned} \nabla_{\hat{\mathbf{t}}} \hat{\mathbf{t}} &= \hat{\kappa}\hat{\mathbf{n}}, \\ \nabla_{\mathbf{t}} \mathbf{n} &= \hat{\kappa}\hat{\mathbf{t}} + \hat{\tau}\hat{\mathbf{b}}, \\ \nabla_{\mathbf{t}} \mathbf{b} &= \hat{\tau}\hat{\mathbf{n}}, \end{aligned} \tag{11}$$

where $\hat{\kappa}$, $\hat{\tau}$ are dual curvature functions and

$$\begin{aligned} &g(\hat{\mathbf{t}}, \hat{\mathbf{t}}) = 1, g(\hat{\mathbf{n}}, \hat{\mathbf{n}}) = -1, g(\hat{\mathbf{b}}, \hat{\mathbf{b}}) = 1, \\ &g(\hat{\mathbf{t}}, \hat{\mathbf{n}}) = g(\hat{\mathbf{t}}, \hat{\mathbf{b}}) = g(\hat{\mathbf{n}}, \hat{\mathbf{b}}) = 0. \end{aligned} \tag{12}$$

We suppose that the dual torsion $\hat{\tau}$ is never pure-dual.

We write frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ of γ with respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as:

$$\begin{aligned} \mathbf{t} &= t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3, \\ \mathbf{n} &= n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3, \\ \mathbf{b} = \mathbf{t} \times \mathbf{n} &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3. \end{aligned} \tag{13}$$

Similarly, we write frenet frame $\{\mathbf{t}^*, \mathbf{n}^*, \mathbf{b}^*\}$ of γ^* with respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as follows:

$$\begin{aligned} \mathbf{t}^* &= t_1^*\mathbf{e}_1 + t_2^*\mathbf{e}_2 + t_3^*\mathbf{e}_3, \\ \mathbf{n}^* &= n_1^*\mathbf{e}_1 + n_2^*\mathbf{e}_2 + n_3^*\mathbf{e}_3, \end{aligned} \tag{14}$$

$$\mathbf{b}^* = \mathbf{t}^* \times \mathbf{n}^* = b_1^*\mathbf{e}_1 + b_2^*\mathbf{e}_2 + b_3^*\mathbf{e}_3.$$

If the formula (11) is separated into the real and dual part, we have

$$\begin{aligned}&\nabla_{\mathbf{t}}\hat{\mathbf{t}} = \kappa\mathbf{n} + \varepsilon(\kappa\mathbf{n}^* + \kappa^*\mathbf{n}),\\ &\nabla_{\mathbf{t}}\hat{\mathbf{n}} = \kappa\mathbf{t} + \tau\mathbf{b} + \varepsilon(\kappa\mathbf{t}^* + \kappa^*\mathbf{t} + \tau\mathbf{b}^* + \tau^*\mathbf{b}),\\ &\nabla_{\mathbf{t}}\hat{\mathbf{b}} = \tau\mathbf{n} + \varepsilon(\tau\mathbf{n}^* + \tau^*\mathbf{n}).\end{aligned} \tag{15}$$

Also, using $\varepsilon^2 = 0$ in (3.2), we obtain

$$\begin{aligned}&g(\mathbf{t},\mathbf{t}) = 1, g(\mathbf{n},\mathbf{n}) = -1, g(\mathbf{b},\mathbf{b}) = 1,\\ &g(\mathbf{t},\mathbf{t}^*) = 0, g(\mathbf{n},\mathbf{n}^*) = 0, g(\mathbf{b},\mathbf{b}^*) = 0,\\ &g(\mathbf{t},\mathbf{n}) = g(\mathbf{t},\mathbf{b}) = g(\mathbf{n},\mathbf{b}) = 0,\\ &g(\mathbf{t}^*,\mathbf{t}^*) = t_0^*, g(\mathbf{n}^*,\mathbf{n}^*) = n_0^*, g(\mathbf{b}^*,\mathbf{b}^*) = b_0^*,\end{aligned} \tag{16}$$

where t_0^*, n_0^*, b_0^* are fixed constants.

Lemma 3.1. Let $\hat{\gamma}: I \to \mathbb{D}^3_{\mathbf{Heis}^3}$ be a non-geodesic spacelike dual curve parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike dual biharmonic curve if and only if

$$\begin{aligned}&\kappa = \text{constant} \neq 0,\\ &\kappa^* = \text{constant} \neq 0,\\ &\kappa^2 + \tau^2 = b_1^2 - \frac{1}{4},\\ &\kappa\kappa^* + \tau\tau^* = 0,\\ &\tau' = n_1 b_1,\\ &\tau^{*\prime} = \mathrm{R},\end{aligned} \tag{17}$$

where $\mathrm{R} = R(\mathbf{t},\mathbf{n},\mathbf{t},\mathbf{b}^*) + R(\mathbf{t},\mathbf{n}^*,\mathbf{t},\mathbf{b})$.

Proof: From (4), we get the biharmonic equation of $\hat{\gamma}$

$$\tau_2(\hat{\gamma}) = \nabla^3_{\mathbf{t}}\hat{\mathbf{t}} - R(\hat{\mathbf{t}}, \nabla_{\mathbf{t}}\hat{\mathbf{t}})\hat{\mathbf{t}} = 0. \tag{18}$$

Next, using the Frenet equations (11) we obtain

$$\nabla^3_{\mathbf{t}}\hat{\mathbf{t}} = (3\hat{\kappa}\hat{\kappa}')\hat{\mathbf{t}} + (\hat{\kappa}'' + \hat{\kappa}^3 + \hat{\tau}^2\hat{\kappa})\hat{\mathbf{n}} + (2\hat{\kappa}'\hat{\tau} + \hat{\tau}'\hat{\kappa})\hat{\mathbf{b}}. \tag{19}$$

Thus, (18) and (19) imply

$$(3\hat{\kappa}\hat{\kappa}')\hat{\mathbf{t}} + (\hat{\kappa}'' + \hat{\kappa}^3 + \hat{\tau}^2\hat{\kappa})\hat{\mathbf{n}} + (2\hat{\kappa}'\hat{\tau} + \hat{\tau}'\hat{\kappa})\hat{\mathbf{b}} + \hat{\kappa}R(\hat{\mathbf{t}},\hat{\mathbf{n}})\hat{\mathbf{t}} = 0, \tag{20}$$

hence, we have

$$\hat{\kappa}\hat{\kappa}' = 0. \tag{21}$$

Also, from (21) we get

$$\hat{\kappa} = \text{constant}. \tag{22}$$

Using $\hat{\kappa} = \kappa + \varepsilon\kappa^*$, we get

$$\kappa = \text{constant and } \kappa^* = \text{constant}. \tag{23}$$

Then, (20) becomes

$$\begin{aligned}&\hat{\kappa}^2 + \hat{\tau}^2 = -R(\hat{\mathbf{t}},\hat{\mathbf{n}},\hat{\mathbf{t}},\hat{\mathbf{n}}),\\ &\hat{\tau}' = R(\hat{\mathbf{t}},\hat{\mathbf{n}},\hat{\mathbf{t}},\hat{\mathbf{b}}).\end{aligned} \tag{24}$$

By virtue of the above we have the following:

$$R(\hat{\mathbf{t}},\hat{\mathbf{n}},\hat{\mathbf{t}},\hat{\mathbf{n}}) = R(\mathbf{t}+\varepsilon\mathbf{t},\mathbf{n}+\varepsilon\mathbf{n},\mathbf{t}+\varepsilon\mathbf{t},\mathbf{n}+\varepsilon\mathbf{n}),$$

$$R(\hat{\mathbf{t}},\hat{\mathbf{n}},\hat{\mathbf{t}},\hat{\mathbf{b}}) = R(\mathbf{t}+\varepsilon\mathbf{t},\mathbf{n}+\varepsilon\mathbf{n},\mathbf{t}+\varepsilon\mathbf{t},\mathbf{b}+\varepsilon\mathbf{b}).$$

Also, using $\hat{\kappa} = \kappa + \varepsilon\kappa^*$ and $\hat{\tau} = \tau + \varepsilon\tau^*$ we obtain

$$\kappa^2 + \tau^2 + \varepsilon(2\kappa\kappa^* + 2\tau\tau^*) = -R(\mathbf{t}+\varepsilon\mathbf{t}^*,\mathbf{n}+\varepsilon\mathbf{n}^*,\mathbf{t}+\varepsilon\mathbf{t}^*,\mathbf{n}+\varepsilon\mathbf{n}^*), \tag{25}$$

$$\tau' + \varepsilon\tau^{*\prime} = R(\mathbf{t}+\varepsilon\mathbf{t}^*,\mathbf{n}+\varepsilon\mathbf{n}^*,\mathbf{t}+\varepsilon\mathbf{t}^*,\mathbf{b}+\varepsilon\mathbf{b}^*).$$

Besides, using the formulae of the curvature, we express

$$\begin{aligned}R(\mathbf{t}+\varepsilon\mathbf{t}^*,\mathbf{n}+\varepsilon\mathbf{n}^*,\mathbf{t}+\varepsilon\mathbf{t}^*,\mathbf{n}+\varepsilon\mathbf{n}^*) &= R(\mathbf{t},\mathbf{n},\mathbf{t},\mathbf{n}) + \varepsilon R(\mathbf{t},\mathbf{n},\mathbf{t},\mathbf{n}^*)\\ &+ \varepsilon R(\mathbf{t},\mathbf{n},\mathbf{t}^*,\mathbf{n}) + \varepsilon R(\mathbf{t},\mathbf{n}^*,\mathbf{t},\mathbf{n}) + \varepsilon R(\mathbf{t}^*,\mathbf{n},\mathbf{t},\mathbf{n}).\end{aligned} \tag{26}$$

Also, we can obtain

$$\begin{aligned}&R(\mathbf{t},\mathbf{n},\mathbf{t}^*,\mathbf{n}) = -R(\mathbf{t}^*,\mathbf{n},\mathbf{t},\mathbf{n}),\\ &R(\mathbf{t},\mathbf{n}^*,\mathbf{t},\mathbf{n}) = -R(\mathbf{t},\mathbf{n},\mathbf{t},\mathbf{n}^*).\end{aligned} \tag{27}$$

Substituting the system of (27) into (26) we obtain

$$R(\mathbf{t}+\varepsilon\mathbf{t}^*,\mathbf{n}+\varepsilon\mathbf{n}^*,\mathbf{t}+\varepsilon\mathbf{t}^*,\mathbf{n}+\varepsilon\mathbf{n}^*) = R(\mathbf{t},\mathbf{n},\mathbf{t},\mathbf{n}). \tag{28}$$

A direct computation using (9) yields

$$R(\mathbf{t},\mathbf{n},\mathbf{t},\mathbf{n}) = \frac{1}{4} - b_1^2, \tag{29}$$

Substituting (29) in (28), we obtain

$$R(\mathbf{t}+\varepsilon\mathbf{t}^*,\mathbf{n}+\varepsilon\mathbf{n}^*,\mathbf{t}+\varepsilon\mathbf{t}^*,\mathbf{n}+\varepsilon\mathbf{n}^*) = \frac{1}{4} - b_1^2. \tag{30}$$

Similarly, using the formulae of the curvature, we express

$$\begin{aligned}R(\mathbf{t}+\varepsilon\mathbf{t}^*,\mathbf{n}+\varepsilon\mathbf{n}^*,\mathbf{t}+\varepsilon\mathbf{t}^*,\mathbf{b}+\varepsilon\mathbf{b}^*) &= R(\mathbf{t},\mathbf{n},\mathbf{t},\mathbf{b}) + \varepsilon R(\mathbf{t},\mathbf{n},\mathbf{t},\mathbf{b}^*)\\ &+ \varepsilon R(\mathbf{t},\mathbf{n},\mathbf{t}^*,\mathbf{b}) + \varepsilon R(\mathbf{t},\mathbf{n}^*,\mathbf{t},\mathbf{b}) + \varepsilon R(\mathbf{t}^*,\mathbf{n},\mathbf{t},\mathbf{b}).\end{aligned} \tag{31}$$

Also, we can obtain

$$R(\mathbf{t}^*,\mathbf{n},\mathbf{t},\mathbf{b}) = -R(\mathbf{t},\mathbf{n},\mathbf{t}^*,\mathbf{b}). \tag{32}$$

Substituting (32) into (31), we obtain

$$\begin{aligned}R(\mathbf{t}+\varepsilon\mathbf{t}^*,\mathbf{n}+\varepsilon\mathbf{n}^*,\mathbf{t}+\varepsilon\mathbf{t}^*,\mathbf{b}+\varepsilon\mathbf{b}^*) &= R(\mathbf{t},\mathbf{n},\mathbf{t},\mathbf{b})\\ &+ \varepsilon R(\mathbf{t},\mathbf{n},\mathbf{t},\mathbf{b}^*) + \varepsilon R(\mathbf{t},\mathbf{n}^*,\mathbf{t},\mathbf{b}).\end{aligned} \tag{33}$$

Additionally, direct computations using (9) yields

$$R(\mathbf{t},\mathbf{n},\mathbf{t},\mathbf{b}) = n_1 b_1. \tag{34}$$

and we put

$$\mathsf{R} = R(\mathbf{t},\mathbf{n},\mathbf{t},\mathbf{b}^*) + R(\mathbf{t},\mathbf{n}^*,\mathbf{t},\mathbf{b}), \tag{35}$$

then, substituting (34) and (35) in (33), we have

$$R(\mathbf{t}+\varepsilon\mathbf{t}^*,\mathbf{n}+\varepsilon\mathbf{n}^*,\mathbf{t}+\varepsilon\mathbf{t}^*,\mathbf{b}+\varepsilon\mathbf{b}^*) = n_1 b_1 + \mathsf{R}. \tag{36}$$

This concludes the proof of lemma.

Theorem 3.2. Let $\hat{\gamma}: I \to \mathsf{D}^3_{\mathbf{Heis}^3}$ be a non-geodesic spacelike dual curve parametrized by arc length. If $\hat{\gamma}$ is non-geodesic spacelike dual biharmonic curve, then $\hat{\gamma}$ is a spacelike dual helix.

Proof: Suppose that $\hat{\gamma}$ is not a dual helix parametrized by arc length. We shall derive a contradiction by showing that $\hat{\gamma}$ must be a dual helix.

Using Frenet formulas (11), (12) and (13), we get the following:

$$\begin{aligned} t_1' &= n_1, \\ n_1' + t_2 n_3 - t_3 n_2 &= \kappa t_1 + \tau b_1, \\ b_1' + t_2 b_3 - t_3 b_2 &= \tau n_1 \end{aligned} \tag{37}$$

Differentiating the third equation of (17), we have

$$\tau\tau' = b_1 b_1'.$$

Using the fifth equation of (17), we obtain

$$\tau n_1 b_1 = b_1 b_1'. \tag{38}$$

We substitute b_1' in (38), we find

$$\tau = \frac{n_1(1-\tau)}{n_1}. \tag{39}$$

Thus (39) becomes

$$\tau = \frac{1}{2} = \text{constant}.$$

Thus, we find that $\tau =$ constant. Therefore, we have a contradiction. Now we show that τ^* is constant.

Also, using (17) we have

$$\tau^* = -\frac{\kappa\kappa^*}{\tau}. \tag{40}$$

From the above proof and (17), we have

$$\begin{aligned} \kappa &= \text{constant}, \\ \kappa^* &= \text{constant}, \\ \tau &= \text{constant}. \end{aligned} \tag{41}$$

If we substitute the equation (41) in the (40), we have

$$\tau^* = \text{constant}.$$

Therefore, $\hat{\gamma}$ is a spacelike dual helix. This completes the proof of the theorem.

As an immediate consequence we have

Corollary 3.3. Let $\hat{\gamma}: I \to \mathsf{D}^3_{\mathbf{Heis}^3}$ be a non-geodesic spacelike dual curve parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike dual biharmonic curve if and only if

$$\begin{aligned} &\kappa = \text{constant} \neq 0, \\ &\kappa^* = \text{constant} \neq 0, \\ &\tau = \text{constant} \neq 0, \\ &\tau^* = \text{constant} \neq 0, \\ &\kappa^2 + \tau^2 = b_1^2 - \frac{1}{4}, \\ &\kappa\kappa^* + \tau\tau^* = 0. \end{aligned} \tag{42}$$

Theorem 3.4. The parametric equations of spacelike dual biharmonic curve in the dual Lorentzian Heisenberg group $\mathsf{D}^3_{\mathbf{Heis}^3}$ are

$$\begin{aligned} \hat{\gamma}_1(s) &= \frac{1}{\beta}\sin(\varphi)\cosh(\beta s+\lambda) + \varepsilon\frac{1}{\beta}\varphi^*\cos(\varphi)\cosh(\beta s+\lambda) \\ &- \frac{\beta^*}{\beta^2}\sin(\varphi)\cosh(\beta s+\lambda) + \frac{1}{\beta}(\beta^* s+\lambda^*)\sin(\varphi)\sinh(\beta s+\lambda)] + \hat{p}_1, \\ \hat{\gamma}_2(s) &= \frac{1}{\beta}\sin(\varphi)\sinh(\beta s+\lambda) + \varepsilon\frac{1}{\beta}\varphi^*\cos(\varphi)\sinh(\beta s+\lambda) \\ &- \frac{\beta^*}{\beta^2}\sin(\varphi)\sinh(\beta s+\lambda) + \frac{1}{\beta}(\beta^* s+\lambda^*)\sin(\varphi)\cosh(\beta s+\lambda)] + \hat{p}_2, \\ \hat{\gamma}_3(s) &= \cos(\varphi)s - \frac{p_1}{\beta}\sin(\varphi)\sinh(\beta s+\lambda) \\ &- \frac{1}{4\beta^2}\sin^2(\varphi)[2(\beta s+\lambda)+\sinh 2(\beta s+\lambda)] \\ &\varepsilon - \varphi^*\sin(\varphi)s - \frac{1}{24\beta^4}[24\beta(\beta\varphi^*\cos(\varphi) \\ &- \beta^*\sin(\varphi))(\varphi^*\cos(\varphi)+\beta p_1^*)\cosh(\beta s+\lambda) \\ &+ \sin(\varphi)[12\beta^2\beta^*\varphi^*\sin(\varphi)s + 12\beta^3(\lambda^*)^2\sin(\varphi)s \\ &+ 12\beta^3\lambda^*\beta^*\sin(\varphi)s^2 + 4\beta^3\sin(\varphi)(\beta^*)^2 s \\ &+ 6\beta(\beta\varphi^*\cos(\varphi) - 2\beta^*\sin(\varphi))(\beta^* s+\lambda^*)\cosh 2(\beta s+\lambda) \\ &+ 6\beta^2(\lambda^*)^2\sin(\varphi)\sinh 2(\beta s+\lambda) + 12\beta^2\beta^*\lambda^*\sin(\varphi)\sin(\varphi)\sinh 2(\beta s+\lambda) \\ &+ 6\beta^2(\beta^*)^2\sin(\varphi)s^2\sinh 2(\beta s+\lambda)]] + \hat{p}_3, \end{aligned} \tag{43}$$

where $\hat{p}_1 = p_1 + \varepsilon p_1^*$, $\hat{p}_2 = p_2 + \varepsilon p_2^*$, $\hat{p}_3 = p_3 + \varepsilon p_3^*$, $\hat{\lambda} = \lambda + \varepsilon\lambda^*$ are dual constants of integration and

$$\left(\frac{\hat{\kappa}^2 - \sin 2\hat{\varphi}}{\sin\hat{\varphi}}\right)^{\frac{1}{2}} = \beta + \varepsilon\beta^*.$$

Proof: Since $\hat{\gamma}$ is spacelike biharmonic curve, γ is a spacelike helix. So, without loss of generality, we take the axis of γ is parallel to the spacelike vector $\mathbf{e}_1$. Then

$$g(\hat{\mathbf{t}}, \mathbf{e}_1) = \cos\hat{\varphi}, \tag{44}$$

where $\hat{\varphi} = \varphi + \varepsilon\varphi^*$ is dual constant angle.

So, substituting the components $\hat{t}_1$, $\hat{t}_2$ and $\hat{t}_3$ in $\hat{\mathbf{t}}$, we have the following equation

$$\hat{\mathbf{t}} = \cos\hat{\varphi}\mathbf{e}_1 + \sin\hat{\varphi}\cosh\hat{\mu}\mathbf{e}_2 + \sin\hat{\varphi}\sinh\hat{\mu}\mathbf{e}_3. \tag{45}$$

Using (8) and (45), we have

$$\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}} = (\hat{\mu}'\sin\hat{\varphi}\sinh\hat{\mu} + 2\sin\hat{\varphi}\cos\hat{\varphi}\sinh\hat{\mu})\mathbf{e}_2 + (\hat{\mu}'\sin\hat{\varphi}\cosh\hat{\mu} + 2\sin\hat{\varphi}\cos\hat{\varphi}\cosh\hat{\mu})\mathbf{e}_3.$$

Since $|\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}}| = \hat{\kappa}$, we obtain

$$\hat{\mu} = \left(\frac{\hat{\kappa}^2 - \sin 2\hat{\varphi}}{\sin\hat{\varphi}}\right)s + \hat{\lambda}, \tag{46}$$

where $\hat{\lambda}$ is dual constant of integration. Additionally, we put

$$\left(\frac{\hat{\kappa}^2 - \sin 2\hat{\varphi}}{\sin\hat{\varphi}}\right) = \hat{\beta} = \beta + \varepsilon\beta^*. \tag{47}$$

Thus (45) and (46) imply

$$\hat{\mathbf{t}} = \cos(\varphi + \varepsilon\varphi^*)\mathbf{e}_1 + \sin(\varphi + \varepsilon\varphi^*)\cosh(\hat{\beta}s + \hat{\lambda})\mathbf{e}_2 + \sin(\varphi + \varepsilon\varphi^*)\sinh(\hat{\beta}s + \hat{\lambda})\mathbf{e}_3. \tag{48}$$

and

$$\hat{\mathbf{t}} = \cos(\varphi + \varepsilon\varphi^*)\mathbf{e}_1 + \sin(\varphi + \varepsilon\varphi^*)\cosh(\beta s + \lambda + \varepsilon(\beta^* s + \lambda^*))\mathbf{e}_2 + \sin(\varphi + \varepsilon\varphi^*)\sinh(\beta s + \lambda + \varepsilon(\beta^* s + \lambda^*))\mathbf{e}_3. \tag{49}$$

Using Maclaurine series expansion of dual functions, we have

$$\begin{aligned} \cos(\varphi + \varepsilon\varphi^*) &= \cos(\varphi) - \varepsilon\varphi^*\sin(\varphi), \\ \sin(\varphi + \varepsilon\varphi^*) &= \sin(\varphi) + \varepsilon\varphi^*\cos(\varphi), \\ \sinh(\beta s + \lambda + \varepsilon(\beta^* s + \lambda^*)) &= \sinh(\beta s + \lambda) + \varepsilon(\beta^* s + \lambda^*)\cosh(\beta s + \lambda), \\ \cosh(\beta s + \lambda + \varepsilon(\beta^* s + \lambda^*)) &= \cosh(\beta s + \lambda) + \varepsilon(\beta^* s + \lambda^*)\sinh(\beta s + \lambda). \end{aligned} \tag{50}$$

If we substitute the equations (50) in (49), we have

$$\begin{aligned} \hat{\mathbf{t}} &= [\cos(\varphi) - \varepsilon\varphi^*\sin(\varphi)]\mathbf{e}_1 \\ &+ [\sin(\varphi) + \varepsilon\varphi^*\cos(\varphi)][\cosh(\beta s + \lambda) + \varepsilon(\beta^* s + \lambda^*)\sinh(\beta s + \lambda)]\mathbf{e}_2 \\ &+ [\sin(\varphi) + \varepsilon\varphi^*\cos(\varphi)][\sinh(\beta s + \lambda) + \varepsilon(\beta^* s + \lambda^*)\cosh(\beta s + \lambda)]\mathbf{e}_3. \end{aligned} \tag{51}$$

Thus, through (3.41), we have

$$\begin{aligned} \hat{\mathbf{t}} &= [\cos(\varphi) - \varepsilon\varphi^*\sin(\varphi)]\mathbf{e}_1 \\ &+ [\sin(\varphi)\sinh(\beta s + \lambda) + \varepsilon\varphi^*\cos(\varphi)\cosh(\beta s + \lambda) \\ &+ \sin(\varphi)(\beta^* s + \lambda^*)\sinh(\beta s + \lambda)]]\mathbf{e}_2 \\ &+ [\sin(\varphi)\cosh(\beta s + \lambda) + \varepsilon\varphi^*\cos(\varphi)\sinh(\beta s + \lambda) \\ &+ \sin(\varphi)(\beta^* s + \lambda^*)\cosh(\beta s + \lambda)]]\mathbf{e}_3. \end{aligned} \tag{52}$$

The formula (3.42) is separated into the real and dual part, we have

$$\mathbf{t} = \cos(\varphi)\mathbf{e}_1 + \sin(\varphi)\cosh(\beta s + \lambda)\mathbf{e}_2 + \sin(\varphi)\sinh(\beta s + \lambda)\mathbf{e}_3, \tag{53}$$

and

$$\begin{aligned} \mathbf{t}^* &= -\varphi^*\sin(\varphi)\mathbf{e}_1 + [\varphi^*\cos(\varphi)\cosh(\beta s + \lambda) \\ &+ \sin(\varphi)(\beta^* s + \lambda^*)\sinh(\beta s + \lambda)]\mathbf{e}_2 \\ &+ [\varphi^*\cos(\varphi)\sinh(\beta s + \lambda) \\ &+ \sin(\varphi)(\beta^* s + \lambda^*)\cosh(\beta s + \lambda)]\mathbf{e}_3. \end{aligned} \tag{54}$$

On the other hand, using our left-invariant vector fields, we obtain

$$\begin{aligned} \mathbf{t} = (&\sin(\varphi)\sinh(\beta s + \lambda), \sin(\varphi)\cosh(\beta s + \lambda), \cos(\varphi) \\ &- (\frac{1}{\beta}\sin(\varphi)\cosh(\beta s + \lambda) + p_1)\sin(\varphi)\cosh(\beta s + \lambda)), \end{aligned} \tag{55}$$

and

$$\begin{aligned} \mathbf{t}^* = (&[\varphi^*\cos(\varphi)\sinh(\beta s + \lambda) + \sin(\varphi)(\beta^* s + \lambda^*)\cosh(\beta s + \lambda)], \\ &[\varphi^*\cos(\varphi)\cosh(\beta s + \lambda) + \sin(\varphi)(\beta^* s + \lambda^*)\sinh(\beta s + \lambda)], \\ &-\varphi^*\sin(\varphi) - [\varphi^*\cos(\varphi)\cosh(\beta s + \lambda) + \sin(\varphi)(\beta^* s + \lambda^*)\sinh(\beta s + \lambda)] \\ &.[\frac{1}{\beta}\varphi^*\cos(\varphi)\cosh(\beta s + \lambda) - \frac{\beta^*}{\beta^2}\sin(\varphi)\cosh(\beta s + \lambda) + \frac{1}{\beta}(\beta^* s + \lambda^*)\sin(\varphi)\sinh(\beta s + \lambda) + p_1^*]). \end{aligned} \tag{56}$$

Substituting (55) and (56) to $\hat{\mathbf{t}}$ and integrating both sides, we have (43) as desired.

Now, we illustrate theorem 3.4 in Figs. 1 & 2. Spacelike dual biharmonic curve may be written by the aid Mathematica program in Figs. 1 & 2:

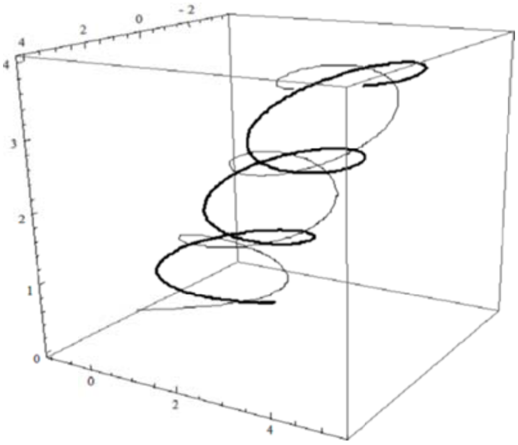

Fig. 1. Spacelike dual biharmonic curve in the dual Lorentzian Heisenberg group

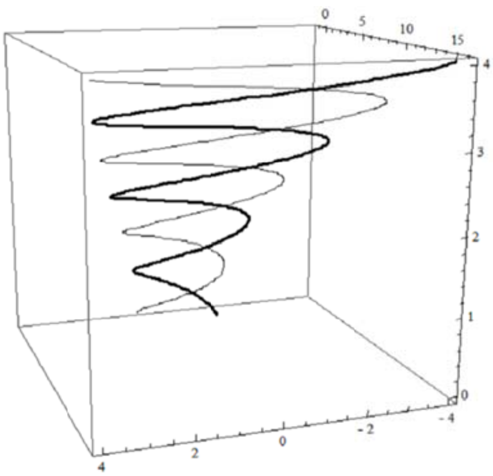

Fig. 2. Spacelike dual biharmonic curve in the dual Lorentzian Heisenberg group

Therefore, Figs. 1 & 2 show that these curves are parallel to each other.

Corollary 3.5. The parametric equations real part of spacelike dual biharmonic curve in the dual Lorentzian Heisenberg group $\mathbb{D}^3_{\mathbf{Heis}^3}$ are

$$\begin{aligned}
\gamma_1(s) &= \frac{1}{\beta}\sin(\varphi)\cosh(\beta s+\lambda)+p_1,\\
\gamma_2(s) &= \frac{1}{\beta}\sin(\varphi)\sinh(\beta s+\lambda)+p_2,\\
\gamma_3(s) &= \cos(\varphi)s-\frac{p_1}{\beta}\sin(\varphi)\sinh(\beta s+\lambda)\\
&-\frac{1}{\beta^2}\sin^2(\varphi)[2(\beta s+\lambda)+\sinh 2(\beta s+\lambda)]]+p_3,
\end{aligned} \tag{57}$$

where p_1, p_2, p_3, λ are constants of integration and

$$(\frac{\hat{\kappa}^2-\sin 2\hat{\varphi}}{\sin\hat{\varphi}})=\beta+\varepsilon\beta^*.$$

Proof: Using above Theorem we get (57). This completes the proof.

Corollary 3.6. The parametric equations dual part of spacelike dual biharmonic curve in the dual Lorentzian Heisenberg group $\mathbb{D}^3_{Heis^3}$ are

$$\begin{aligned}
\gamma_1^*(s) &= \frac{1}{\beta}\varphi^*\cos(\varphi)\cosh(\beta s+\lambda)-\frac{\beta^*}{\beta^2}\sin(\varphi)\cosh(\beta s+\lambda)\\
&+\frac{1}{\beta}(\beta^* s+\lambda^*)\sin(\varphi)\sinh(\beta s+\lambda)+p_1^*,\\
\gamma_2^*(s) &= \frac{1}{\beta}\varphi^*\cos(\varphi)\sinh(\beta s+\lambda)-\frac{\beta^*}{\beta^2}\sin(\varphi)\sinh(\beta s+\lambda)\\
&+\frac{1}{\beta}(\beta^* s+\lambda^*)\sin(\varphi)\cosh(\beta s+\lambda)+p_2^*,\\
\gamma_3^*(s) &= -\varphi^*\sin(\varphi)s-\frac{1}{24\beta^4}[24\beta(\beta\varphi^*\cos(\varphi)\\
&-\beta^*\sin(\varphi))(\varphi^*\cos(\varphi)+\beta p_1^*)\cosh(\beta s+\lambda)\\
&+\sin(\varphi)[12\beta^2\beta^*\varphi^*\sin(\varphi)s+12\beta^3(\lambda^*)^2\sin(\varphi)s\\
&+12\beta^3\lambda^*\beta^*\sin(\varphi)s^2+4\beta^3\sin(\varphi)(\beta^*)^2 s\\
&+6\beta(\beta\varphi^*\cos(\varphi)-2\beta^*\sin(\varphi))(\beta^* s+\lambda^*)\cosh 2(\beta s+\lambda)\\
&+24\beta^2(\varphi^*\cos(\varphi)+\beta p_1^*)(\beta^* s+\lambda^*)\sinh(\beta s+\lambda)\\
&-9\beta\beta^*\varphi^*\cos(\varphi)\sinh 2(\beta s+\lambda)+6(\beta^*)^2\sin(\varphi)\sinh 2(\beta s+\lambda)\\
&+6\beta^2(\lambda^*)^2\sin(\varphi)\sinh 2(\beta s+\lambda)+12\beta^2\beta^*\lambda^*\sin(\varphi)\sin(\varphi)\sinh 2(\beta s+\lambda)\\
&+6\beta^2(\beta^*)^2\sin(\varphi)s^2\sinh 2(\beta s+\lambda)]]+p_3^*,
\end{aligned}$$

where p_1^*, p_2^*, p_3^*, λ, λ^* are constants of integration and

$$(\frac{\hat{\kappa}^2-\sin 2\hat{\varphi}}{\sin\hat{\varphi}})=\beta+\varepsilon\beta^*.$$

The proof of the Corollary 3.6. is similar to the proof of the Corollary 3.5.

References

[1] Asil, V. (2007). Velocities of dual homothetic exponential motions in $\mathbb{D}^3$. *IJST-Trans. A, 31*(4), 265-271.

[2] Baky, R. A. A. (2002). An explicit characterization of dual spherical curves. *Commun. Fac. Sci. Univ. Ank. Series A.*, *51*(2), 1-9.

[3] Clifford, W. K. (1873). Preliminary sketch of biquaternions. *Proceedings of London Math. Soc.*, *4*, 361-395.

[4] Guggenheimer, W. (1963). *Differential Geometry*, New York, McGraw-Hill.

[5] Köse, Ö., Nizamoğlu S. & Sezer, M. (1988). An explicit characterization of dual spherical curve. *Doğa Mat.*, *12*(3), 105-113.

[6] Uğurlu H. & Çalışkan, A. (1996). The study mapping for directed space-like and time-like lines in Minkowski 3-Space. *Math. Comput. Appl. 1*(2), 142-148.

[7] Veldkamp, G. R. (1976). On the use of dual numbers, vectors, and matrices in instantaneous spatial kinematics. *Mech. Mach. Theory*, *11*, 141-156.

[8] A.T. Yang, Application of Quaternion Algebra and Dual Numbers to the Analysis of Spatial Mechanisms, (Doctoral Dissertation, Columbia University, 1963).

[9] Yucesan, A., Coken A. C. & Ayyildiz, N. (2002). On the dual Darboux rotation axis of the timelike dual space curve. *Balkan J. Geom. Appl. 7*(2), 137-142.

[10] Sezer, M., Köse Ö. & Nizamoğlu, S. (1990). A criterion for a ruled surface to be closed. *Doğa Mat. 14*(1), 39-47.

[11] Eells J. & Lemaire, L. (1978). A report on harmonic maps. *Bull. London Math. Soc.*, *16*, 1-68.

[12] Jiang, G. Y. (1986). 2-harmonic isometric immersions between Riemannian manifolds, *Chinese Ann. Math. Ser. A*7, 130-144.

[13] Jiang, G. Y. (1986). 2-harmonic maps and their first and second variational formulas. *Chinese Ann. Math. Ser. A*7, 389-402.

[14] Arslan, K., Ezentas, R., Murathan, C. & Sasahara, T. (2005). Biharmonic submanifolds 3-dimensional (κ,μ)-manifolds. *Internat. J. Math. Math. Sci. 22*, 3575-3586.

[15] Caddeo, R., Montaldo, S. & Piu, P. (2001). Biharmonic curves on a surface. *Rend. Mat. Appl. 21*, 143-157.

[16] Caddeo, R., Montaldo S. & Oniciuc, C. (2002). Biharmonic submanifolds in spheres. *Israel J. Math, 130*, 109-123.

[17] Chen, B. Y. & Ishikawa, S. (1991). Biharmonic surfaces in pseudo-Euclidean Spaces. *Mem. Fac. Sci. Kyushu Univ. Ser. A, 45*(2), 323-347.

[18] Turhan, E. & Körpınar, T. (2010). Position vector of spacelike biharmonic slant helices with timelike principal normal according to Bishop frame in Minkowski 3-space. *International Journal of Physical Sciences*, *5*(12), 1824-1829.

[19] Körpınar, T. & Turhan, E. (2010). On Horizontal Biharmonic Curves In The Heisenberg Group Heis³. *Arab. J. Sci. Eng. Sect. A Sci. 35*(1), 79-85.

[20] Körpınar, T., Turhan, E. & Asil, V. (2012). Tangent Bishop spherical images of a biharmonic B-slant helix in the Heisenberg group Heis³. *IJST-Trans. A, 35*(A4), 265-271.

[21] Körpınar, T. & Turhan, E. (2012). Tubular Surfaces Around Timelike Biharmonic Curves in Lorentzian Heisenberg Group Heis³. *Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica, 20*(1), 431-445.

[22] Körpınar, T. & Turhan, E. (2011). On characterization of B-canal surfaces in terms of biharmonic B-slant helices according to Bishop frame in Heisenberg group Heis³. *Journal of Mathematical Analysis and Applications 382*(1), 57-65.

[23] Happel, J. & Brenner, H. (1965). *Low Reynolds Number Hydrodynamics with Special Applications to Particulate Media*. New Jersey. Prentice-Hall.

[24] Rahmani, S. (1992). Metriqus de Lorentz sur les groupes de Lie unimodulaires de dimension trois. *Journal of Geometry and Physics*, *9*, 295-302.

[25] Turhan, E. & Körpınar, T. (2011). On Characterization Canal Surfaces around Timelike Horizontal Biharmonic Curves in Lorentzian Heisenberg Group Heis³: *Zeitschrift für Naturforschung A- A Journal of Physical Sciences 66*a, 441-449.

[26] Turhan, E. & Körpınar, T. (2011). Position vector of spacelike biharmonic curves in the Lorentzian Heisenberg group Heis³. *Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica, 19*(1), 285-296.

[27] Turhan, E. & Körpınar, T. (2010). On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group Heis³. *Zeitschrift für Naturforschung A-A Journal of Physical Sciences 65*a, 641-648.