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# Bayesian and non-bayesian estimation of stress–strength model for Pareto type I distribution

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**Abstract**

This article examines statistical inference for $R = P(Y < X)$ where $X$ and $Y$ are independent but not identically distributed Pareto of the first kind (Pareto (I)) random variables with same scale parameter but different shape parameters. The Maximum likelihood, uniformly minimum variance unbiased and Bayes estimators with Gamma prior are used for this purpose. Simulation studies which compare the estimators are presented. Moreover, sensitivity of Bayes estimator to the prior parameters is considered.

***Keywords:*** Bayesian estimator; Maximum likelihood estimator (MLE); Pareto of first kind; uniformly minimum variance unbiased estimator (UMVUE); stress-strength model

## 1. Introduction

The problem of estimating $R = P(Y < X)$ arises in the context of mechanical reliability of a system with strength $X$ and stress $Y$, then reliability $R$ is chosen as a measure of system reliability. In a stress- strength model, the system fails if and only if $X \le Y$, at any time. This model, first considered by Birnbaum [1], is commonly used in many engineering applications such as civil, mechanical and aerospace. Recently, a number of papers have dealt with the stress-strength reliability problem. Several distributions have been used in the literature as failure models. For references see the book of Kotz et al. [2] and the articles [3-18].

The main aim of this paper is to discuss the inference of $R = P(Y < X)$, when $X$ and $Y$ are two independent, but not identically random variables belonging to Pareto (I) distribution with two parameters.

The Pareto distribution is a power law probability distribution that coincides with social, scientific, geophysical and many other types of observable phenomena. Recently it has been used to study ozone levels in the upper atmosphere and tensile strength of nylon carpet fibers. It has played a very important role in the investigation of city population occurrence of natural resources, insurance risk, and business failures.

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The probability density functions ( $pdf$ 's) of $X$ and $Y$ are given, respectively, by

$$f(x) = \theta_1 b^{\theta_1} x^{-(\theta_1+1)}, \quad x \ge b,\ \theta_1, b > 0 \tag{1}$$

$$g(y) = \theta_2 b^{\theta_2} y^{-(\theta_2+1)}, \quad y \ge b,\ \theta_2, b > 0\ . \tag{2}$$

Then, it can be shown that

$$R = P(Y < X) = \iint_{b<y<x} f(x,y)\,dx\,dy = \frac{\theta_2}{\theta_1+\theta_2}, \tag{3}$$

in which $R$ is free of $b$.

The paper is organized as follows: In Section 2, $MLE$, $UMVUE$ , and Bayesian estimators of $R$ are obtained. In Section 3, simulation studies are done to compare the different estimates of $R$ together with sensitivity of the Bayes estimators to the parameter of the prior distribution represented in Section 2.

## 2. Estimation of Reliability

### *2.1. Maximum Likelihood Estimation*

Let $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_m$ be the two independent random samples taken from the Pareto distribution with parameters $(\theta_1, b)$ and $(\theta_2, b)$, respectively, and let $b$ be known. Then, likelihood and log-likelihood functions based on the above samples are given by

$$L(x, y;\ \theta_1, \theta_2, b) = \theta_1^n \theta_2^m b^{n\theta_1 + m\theta_2} \prod_{i=1}^{n} x_i^{-(\theta_1+1)} \prod_{j=1}^{m} y_j^{-(\theta_1+1)}, \quad (4)$$

$$l(x, y;\ \theta_1, \theta_2, b) = n\log\theta_1 + m\log\theta_2 + (n\theta_1 + m\theta_2)\log b - (\theta_1+1)\sum_{i=1}^{n}\log x_i - (\theta_2+1)\sum_{j=1}^{m}\log y_j. \quad (5)$$

Using (5), the MLE's for $\theta_1$ and $\theta_2$ are, respectively,

$$\hat{\theta}_1 = n \Big/ \sum_{i=1}^{n} \log\left(x_i b^{-1}\right), \quad (6)$$

$$\hat{\theta}_2 = m \Big/ \sum_{j=1}^{m} \log\left(y_j b^{-1}\right). \quad (7)$$

Hence $\hat{R}_1$, the $MLE$ of $R$, is written as follows

$$\hat{R}_1 = \frac{\hat{\theta}_2}{\hat{\theta}_1 + \hat{\theta}_2}. \quad (8)$$

*2.2. Some properties of* $\hat{R}_1$

Since $b$ is known, $\hat{R}_1$ can be expressed as

$$\hat{R}_1 = 1 \Bigg/ \left(1 + \frac{n\sum_{j=1}^{m}\log(y_j b)}{m\sum_{i=1}^{n}\log(x_i b)}\right). \quad (9)$$

Since

$$2\theta_1 \sum_{i=1}^{n} \log\left(x_i b^{-1}\right) \sim x_{2n}^2,$$

and

$$2\theta_2 \sum_{j=1}^{m} \log\left(y_j b^{-1}\right) \sim x_{2m}^2.$$

So

$$\frac{\theta_2}{\theta_1}\left(\frac{1}{\hat{R}_1} - 1\right) = \frac{n\theta_2\sum_{j=1}^{m}\log(y_j b^{-1})}{m\theta_1\sum_{i=1}^{n}\log(x_i b^{-1})} \sim F_{(2m,2n)}.$$

From this fact we shall study some properties of $\hat{R}_1$ (Lindley [19]) and it could be shown that

$$E\left(\hat{R}_1\right) = \frac{\theta_2(n-1)}{\theta_2(n-1) + \theta_1 n}. \left[1 + \frac{n+m-1}{m(n-2)}\left(1 - \frac{\theta_2(n-1)}{\theta_2(n-1)+\theta_1 n}\right)^2\right]. \quad (10)$$

Note that if $m$ is fixed,

$$\lim_{n\to\infty} E\left(\hat{R}_1\right) = R\left[1 + \frac{1}{m}(1-R)^2\right], \quad (11)$$

and

$$\lim_{m,n\to\infty} E\left(\hat{R}_1\right) = R, \quad (12)$$

$\hat{R}_1$ is asymptotically unbiased,

$$V\left(\hat{R}_1\right) = \frac{n+m-1}{m(n-2)}\left(1 - \frac{\theta_2(n-1)}{\theta_2(n-1)+\theta_1 n}\right)^2\left(\frac{\theta_2(n-1)}{\theta_2(n-1)+\theta_1 n}\right)^2, \quad (13)$$

then

$$\lim_{n\to\infty} V\left(\hat{R}_1\right) = \frac{1}{m}\left(1 - \frac{\theta_2}{\theta_2+\theta_1}\right)^2\left(\frac{\theta_2}{\theta_2+\theta_1}\right)^2, \quad (14)$$

and

$$\lim_{m,n\to\infty} V\left(\hat{R}_1\right) = 0, \quad (15)$$

i.e. $\hat{R}_1$ is consistent estimator for $R$.

The mean square error ($MSE$) of $\hat{R}_1$ is given as

$$MSE\left(\hat{R}_1\right) = V\left(\hat{R}_1\right) + \left[R - E\left(\hat{R}_1\right)\right]^2$$
$$= \frac{n+m-1}{m(n-2)}\left(1 - \frac{\theta_2(n-1)}{\theta_2(n-1)+\theta_1 n}\right)^2\left(\frac{\theta_2(n-1)}{\theta_2(n-1)+\theta_1 n}\right)^2 + \left\{\frac{\theta_2}{\theta_2+\theta_1} - \frac{\theta_2(n-1)}{\theta_2(n-1)+\theta_1 n}.\right.$$

$$\left[1+\frac{n+m-1}{m(n-2)}\left(1-\frac{\theta_2(n-1)}{\theta_2(n-1)+\theta_1 n}\right)^2\right]\Bigg\}^2, \quad (16)$$

this tends to zero as $m,n \to \infty$.

*2.3. Uniformly Minimum Variance Unbiased Estimator $(UMVUE)$ of $R$*

Let $X_1, X_2, \ldots, X_n$ be a random sample from Pareto $(\theta_1, b)$ where $b$ is known. Set $W_i = \log(X_i b^{-1}),\ i = 1,2,\ldots,n,$ then $W_i$ is Expo$(\theta_1)$, and hence $\sum_{i=1}^{n} W_i$ is sufficient statistic for $\theta_1$. Similarly, if $Y_1, Y_2, \ldots, Y_m$ is a random sample from Pareto $(\theta_2, b)$ where $b$ is known and $V_j = \log(Y_j b^{-1}),\ j = 1,2,\ldots,m,$ then $\sum_{j=1}^{m} V_j$ is sufficient statistic for $\theta_2$. Moreover, $\left(\sum_{i=1}^{n} W_i, \sum_{j=1}^{m} V_j\right)$ is jointly sufficient statistic for $(\theta_1, \theta_2)$.

Let

$$T = \begin{cases} 1 & if \quad v_1 < w_1 \\ 0 & if \quad v_1 \ge w_1. \end{cases}$$

Then

$$E(T) = P(V_1 < W_1) = P\left(\log(y_1 b^{-1}) < \log(x_1 b^{-1})\right)$$
$$= P\left(y_1 < x_1\right) = P\left(y < x\right) = R. \quad (17)$$

Therefore, $T$ is an unbiased estimator for $R$. By using either Rao-Blackwell and Lehmann-Scheffe's Theorems, or Tong's results [20, 21], we obtain $UMVUE$ for $R$ as

$$\hat{R}_2 = E\left(T \,\middle|\, \sum_{i=1}^{n} W_i, \sum_{j=1}^{m} V_j\right)$$

$$= \begin{cases} \sum_{i=0}^{n-1} a_i Q^{-i} & if \quad Q \ge 1 \\ 1 - \sum_{j=0}^{m-1} c_j Q^j & if \quad Q < 1, \end{cases} \quad (18)$$

where

$$Q = \frac{\sum_{i=1}^{n} W_i}{\sum_{j=1}^{m} V_j}$$

and

$$a_i = (-1)^i \frac{(n-1)!\,(m-1)!}{(n-i-1)!\,(m+i-1)!},$$

$$c_j = (-1)^j \frac{(n-1)!\,(m-1)!}{(n+j-1)!\,(m-j-1)!}.$$

To find the $MSE$ of $\hat{R}_2$ which is given by

$$MSE(\hat{R}_2) = E(\hat{R}_2 - R)^2 = E(\hat{R}_2)^2 - R^2, \quad (19)$$

we proceed to get

$$E\left[(\hat{R}_2)^2\right] = \sum_{i=0}^{n-1}\sum_{k=0}^{n-1} a_i a_k E(Q^{-i-k} \mid Q \ge 1) P(Q \ge 1)$$
$$+ P(Q<1) - 2\sum_{j=0}^{m-1} c_j E(Q^j \mid Q<1) P(Q<1)$$
$$+ \sum_{j=0}^{m-1}\sum_{s=0}^{m-1} c_j c_s E(Q^{j+s} \mid Q<1) P(Q<1), \quad (20)$$

Equation (20) depends on the evaluation of $E(Q^l \mid Q<1) P(Q<1)$ and $E(Q^{-l} \mid Q \ge 1) P(Q \ge 1)$. For this purpose, we first obtain the $pdf$ of $Q$. Since $\sum_{i=1}^{n} W_i, \sum_{j=1}^{m} V_j$ have respectively the independent distributions $\Gamma(n, \theta_1)$ and $\Gamma(m, \theta_2)$, then $\frac{n\rho}{mQ} \sim F_{(2m,2n)}$ where,

$$\rho = \frac{\theta_2}{\theta_1},\ Q = \frac{\sum_{i=1}^{n} W_i}{\sum_{j=1}^{m} V_j},$$

thus the $pdf$ of $Q$ is

$$f(q) = \frac{\rho^{-n}}{\beta(n,m)} q^{n-1} \left(1 + \frac{q}{\rho}\right)^{-(n+m)},\quad q > 0. \quad (21)$$

Then, for $l > 0$

$$E(Q^l \mid Q<1)\,P(Q<1)$$
$$= \frac{\rho^{-n}}{\beta(n,m)} \int_0^1 q^{l+n-1}(1+q/\rho)^{-(n+m)}\,dq$$
$$= \frac{\rho^{-n}}{(l+n)\beta(n,m)} F_{2:1}\left(l+n, n+m, l+n+1, -\frac{1}{\rho}\right). \quad (22)$$

Interchanging $n$ with $m$ and replacing $\rho$ by $\frac{1}{\rho}$ in (22), we have

$$E(Q^{-l} \mid Q \geq 1)\,P(Q \geq 1) = \frac{\rho^m}{(l+m)\beta(n,m)} F_{2:1}(l+m, n+m, l+m+1, -\rho). \quad (23)$$

We let $l = 0$ in (22), $P(Q<1)$ has the form

$$P(Q<1) = \frac{\rho^{-n}}{n\,\beta(n,m)} F_{2:1}\left(n, n+m, n+1, -\frac{1}{\rho}\right), \quad (24)$$

where $F_{2:1}(.,.,.,.)$ is the generalized hypergeometric function, [22]. Substituting (22)-(24) in (20), we then show that, using (19), the $MSE$ of $\hat{R}_2$ is represented by hypergeometric function.

### 2.4. Bayes Estimation for $R$

It is remarkable that most of the Bayesian inference procedures have been developed under the usual squared error (SE) loss function (quadratic loss), which is symmetrical and associates equal importance to the losses due to overestimation and underestimation of equal magnitude. However, such a restriction may be impractical. For example, in the estimation of reliability and failure rate functions, an overestimate is usually much more serious than an underestimate; in this case the use of a symmetrical loss function might be inappropriate, as has been recognized by Basu and Ebrahimi [23] and Canfield [24]. Feynman [25] states that in the disaster of a space shuttle, the management overestimated the average life or reliability of the solid fuel rocket booster. This is an example of an asymmetrical loss function.

A useful asymmetric loss known as the LINEX (linear-exponential) loss function was introduced in [26], and was widely used in several papers [23, 27-29]. This function rises approximately exponentially on one side of zero, and approximately linearly on the other side. Under the assumption that the minimal loss occurs at $\tilde{u} = u$, the LINEX loss function can be expressed as

$$L(\Delta) \propto e^{c\Delta} - c\Delta - 1; \quad c \neq 0, \quad (25)$$

where $\Delta = (\tilde{u} - u)$, $\tilde{u}$ is an estimate of $u$.

The sign and magnitude of '$c$' represent the direction and degree of symmetry, respectively. ($c > 0$ means overestimation is more serious than underestimation, and $c < 0$ means the opposite). For '$c$' closed to zero, the LINEX loss function is approximately the Squared Error (SE) loss, and therefore almost symmetric. The posterior $s$-expectation of the LINEX loss function of (25) is

$$E_u((L(\hat{u}-u)) \propto \exp(cu).E_u(\exp(-cu)) - c.(\hat{u} - E_u(u)) - 1. \quad (26)$$

$E_u$ is equivalent to the posterior s-expectation with respect to the posterior pdf. The Bayes estimator $\hat{u}_{BL}$ of $u$ under the LINEX loss function is the value $\hat{u}$, which minimizes (26).

$$\hat{u}_{BL} = -\frac{1}{c}\log(E_u[\exp(-cu)]), \quad (27)$$

provided that $E_u[\exp(-cu)]$ exists, and is finite. Another useful asymmetric loss function is the General Entropy (GE) loss

$$L_2(\hat{u}, u) \propto \left(\frac{\hat{u}}{u}\right)^q - q\,log\left(\frac{\hat{u}}{u}\right) - 1, \quad (28)$$

whose minimum occurs at $\hat{u} = u$. This loss function is a generalization of the Entropy-loss used in several papers where $q = 1$, see [30] and [31]. When $q > 0$, a positive error ($\hat{u} > u$) causes more serious consequences than a negative error. The Bayes estimate $\hat{u}_{BG}$ of $u$ under GE loss (28) is

$$\hat{u}_{BG} = (E_u[u^{-q}])^{\frac{-1}{q}}, \quad (29)$$

provided that $E_u[u^{-q}]$ exists, and is finite.

Bayes estimation of $R$ can be obtained if $\theta_1$ and $\theta_2$ are assumed to be random variables which have independent gamma prior distributions with probability density functions

$$h_i(\theta_i) = \frac{\beta_i^{\varphi_i}}{\Gamma\varphi_i}\theta_i^{\varphi_i - 1}\exp(-\beta_i\theta_i), \theta_i, \varphi_i, \beta_i > 0, i = 1, 2. \quad (30)$$

It can be shown that the posterior density of $(\theta_1, b)$ given $\underline{X}$ is, from (4) and (30),

$$h(\theta_1, b \mid \underline{x}) \propto \theta_1^{\delta_1 - 1}\exp(-\theta_1\lambda_1), \quad (31)$$

where $\delta_1 = n + \varphi_1$ and

$$\lambda_1 = \beta_1 - n\log b + \sum_{i=1}^{n} \log x_i \,.$$

Similarly, the posterior density of $(\theta_2, b)$ can be written as

$$h\left(\theta_2, b \middle| \underline{y}\right) \propto \theta_2^{\delta_2 - 1} \exp\left(-\theta_2 \lambda_2\right), \tag{32}$$

where $\delta_2 = n + \varphi_2$ and

$$\lambda_2 = \beta_2 - m\log b + \sum_{j=1}^{m} \log y_j \,.$$

The joint posterior density function of $(\theta_1, \theta_2)$ is then

$$h(\theta_1, \theta_2 \big| \underline{x}, \underline{y}) = \frac{\lambda_1^{\delta_1} \lambda_2^{\delta_2}}{\Gamma(\delta_1)\Gamma(\delta_2)} \theta_1^{\delta_1 - 1} \theta_2^{\delta_2 - 1}.$$
$$\exp(-\theta_1\lambda_1 - \theta_2\lambda_2). \tag{33}$$

Put $R = \dfrac{\theta_2}{\theta_1 + \theta_2}$, and using standard procedure of transformations of random variables we obtain the density of $R$ as

$$g_R(r) = \frac{\Gamma(\delta_1 + \delta_2)}{\Gamma(\delta_1)\Gamma(\delta_2)} \lambda_1^{\delta_1} \lambda_2^{\delta_2} (1-r)^{\delta_1 - 1} r^{\delta_2 - 1}.$$
$$\left((1-r)\lambda_1 + r\lambda_2\right)^{-(\delta_1 + \delta_2)}, \quad 0 < r < 1 .$$

**A. Symmetric Bayes Estimation**

Under the squared error (SE) loss function, the Bayes estimator of R is

$$\hat{R}_3 = E(R \big| x, y, b) = \frac{\Gamma(\delta_1 + \delta_2)}{\Gamma(\delta_1)\Gamma(\delta_2)} \lambda_1^{-\delta_2} \lambda_2^{\delta_2}.$$
$$\int_0^1 (1-r)^{\delta_1 - 1} r^{\delta_2} \left(1 - \frac{\lambda_1 - \lambda_2}{\lambda_1} r\right)^{-(\delta_1 + \delta_2)} dr. \tag{34}$$

**B. Asymmetric Bayes Estimation**

**LINEX Loss:**

If in (27), $u = r$, then the Bayes estimator of R is

$$\hat{R}_4 = \frac{-1}{c} \log\{ \frac{\Gamma(\delta_1 + \delta_2)}{\Gamma(\delta_1)\Gamma(\delta_2)} \lambda_1^{\delta_1} \lambda_2^{\delta_2}.$$
$$\int_0^1 e^{-cr} (1-r)^{\delta_1 - 1} r^{\delta_2 - 1} \left((1-r)\lambda_1 + r\lambda_2\right)^{-(\delta_1 + \delta_2)} dr\}. \tag{35}$$

**General Entropy Loss**: Set $u = r$ in (29), then the Bayes estimate $\hat{R}_5$ of $R$ relative to the general Entropy loss function (28) is

$$\hat{R}_5 = \{ \frac{\Gamma(\delta_1 + \delta_2)}{\Gamma(\delta_1)\Gamma(\delta_2)} \lambda_1^{\delta_1} \lambda_2^{\delta_2}.$$
$$\{\int_0^1 (1-r)^{\delta_1 - 1} r^{\delta_2 - q - 1} \left((1-r)\lambda_1 + r\lambda_2\right)^{-(\delta_1 + \delta_2)} dr\}^{\frac{-1}{q}}. \tag{36}$$

*2.5. Two-Sided Confidence Intervals*

To determine the confidence interval for the reliability $R$, we first note that $\dfrac{\rho}{\hat{\rho}}$ has an $F$-distribution with $2m, 2n$ degrees of freedom, where $R$ may be written as

$$R = P(V < W) = P(\frac{\rho}{Q} < \rho),$$

where $W = \sum_{i=1}^{n} W_i$, $V = \sum_{j=1}^{m} V_j$ and $\dfrac{\rho}{Q} \sim F_{(2,2)}$.

Then $R = F_F(\rho, 2, 2)$, where $F_F(., 2, 2)$ is the cumulative distribution function of an $F$ random variable with $2, 2$ degrees of freedom.

The $MLE's$ of $\theta_1$ and $\theta_2$, by (6), (7), are $\hat{\theta}_1 = \dfrac{n}{W}$ and $\hat{\theta}_2 = \dfrac{m}{V}$, respectively.

If $\hat{R}_1$ is the $MLE$ of $R$, then

$\hat{R}_1 = F_F(\hat{\rho}, 2, 2)$, where $\hat{\rho} = \dfrac{\hat{\theta}_2}{\hat{\theta}_1}$,

so

$$\frac{\rho}{\hat{\rho}} = \frac{\theta_2 \,\hat{\theta}_1}{\theta_1 \,\hat{\theta}_2} = \frac{n\theta_2 V}{m\theta_1 W} \sim F_{(2m, 2n)}$$

with $1 - \alpha = 1 - (\alpha_1 + \alpha_2)$ confidence coefficient, where $F_{\alpha_1}$ has $1 - \alpha_1$ confidence coefficient and $F_{\alpha_2}$ has $1 - \alpha_2$ confidence coefficient with $2m, 2n$ degrees of freedom. We can obtain the desired interval by

$$1 - \alpha = P\left( F_{\alpha_1}^{-1} \le \frac{\rho}{\hat{\rho}} \le F_{\alpha_2} \right)$$
$$= P\left(\hat{\rho}\, F_{\alpha_1}^{-1} \le \rho \le \hat{\rho}\, F_{\alpha_2}\right)$$
$$= P\left(F(\hat{\rho}F_{\alpha_1}^{-1}, 2, 2) \le R \le F(\hat{\rho}F_{\alpha_2}, 2, 2)\right),$$

so the two- sided $(1-\alpha)100\%$ confidence interval for $R$ is given by

$$\left[ F(\hat{\rho} F_{\alpha_1}^{-1}, 2, 2), F(\hat{\rho} F_{\alpha_2}, 2, 2) \right],$$

i.e.,

$$\left[ \frac{1}{\frac{1}{\hat{\rho}} F_{\alpha_1} + 1}, \ \frac{1}{\frac{1}{\hat{\rho}} F_{\alpha_2}^{-1} + 1} \right].$$

If $\alpha_1 = \alpha_2 = \dfrac{\alpha}{2}$, then the confidence interval takes the form $\left[ \frac{1}{\frac{1}{\hat{\rho}} F_{\alpha/2}+1}, \ \frac{1}{\frac{1}{\hat{\rho}} F_{\alpha/2}^{-1}+1} \right]$, where $F_{\alpha}^{-1}$ denotes the inverse function of $F$ with $1-\alpha$ confidence coefficient and $2n, 2m$ degrees of freedom.

## 3. Simulation Study for the Different Estimators

It is clear that, the computations of $\hat{R}_i, i = 1, \ldots, 5$ and their mean square errors are very complicated. Therefore, Mathematical 7 is used to evaluate $\hat{R}_i$ and the $MSE$'s of $\hat{R}_1, \hat{R}_2$ and $\hat{R}_3$, as shown in the following tables. Table (1) shows the simulation results for the $MSE$'s of the *MLE, UMVUE* and Bayes estimators for *R,* for different values of the sample sizes. It is observed that the MSE of Bayes is smaller than the MSE of maximum likelihood. Table (2) illustrates the sensitivity of the Bayes estimators relative to asymmetric loss functions (LINEX and General Entropy) $\hat{R}_4$ and $\hat{R}_5$ for the values of the shape parameters c and q. Table (3) simplifies the sensitivity of $\hat{R}_3$ to the prior parameters $\varphi_1, \varphi_2, \beta_1$ and $\beta_2$. It is clear first that, the Bayes estimator is sensitive to the prior parameters, and secondly, on keeping $\beta_2$ constant, *MSE* is increasing as $\beta_1$.

**Table 1.** Maximum Likelihood Estimator, Uniformly Minimum Variance Unbiased and Bayes Estimator with Empirical Estimators for the Prior Parameters (The effect of changing two sample sizes over the $MSE$ for the $MLE$ )

| $\theta_1$ =1, $\theta_2$= 2, b= .2, R = .66667 | | | | | | | |
|---|---|---|---|---|---|---|---|
| n | m | $\hat{R}_1$ | $MSE\hat{R}_1$ | $\hat{R}_2$ | $MSE\hat{R}_2$ | $\hat{R}_3$ | $MSE\hat{R}_3$ |
| 10 | 10 | 0.528 | 0.012 | 0.689 | 0.010 | 0.696 | 0.0044 |
| | 40 | 0.610 | 0.008 | 0.654 | 0.007 | 0.671 | 0.0003 |
| | 60 | 0.727 | 0.008 | 0.592 | 0.007 | 0.597 | 0.0004 |
| 20 | 10 | 0.834 | 0.008 | 0.714 | 0.006 | 0.528 | 0.0064 |
| | 20 | 0.673 | 0.005 | 0.708 | 0.004 | 0.637 | 0.0015 |
| | 60 | 0.560 | 0.004 | 0.658 | 0.003 | 0.706 | 0.0001 |
| 30 | 10 | 0.655 | 0.007 | 0.772 | 0.004 | 0.522 | 0.0070 |
| | 30 | 0.672 | 0.003 | 0.629 | 0.004 | 0.573 | 0.0008 |
| | 60 | 0.667 | 0.003 | 0.705 | 0.002 | 0.655 | 0.0001 |
| 40 | 10 | 0.650 | 0.006 | 0.746 | 0.005 | 0.588 | 0.0075 |
| | 40 | 0.597 | 0.003 | 0.610 | 0.003 | 0.592 | 0.0005 |
| | 60 | 0.710 | 0.002 | 0.635 | 0.002 | 0.653 | 0.0002 |
| 50 | 10 | 0.759 | 0.006 | 0.649 | 0.006 | 0.579 | 0.0078 |
| | 50 | 0.732 | 0.002 | 0.635 | 0.002 | 0.658 | 0.0003 |
| | 60 | 0.660 | 0.002 | 0.676 | 0.002 | 0.644 | 0.0002 |
| 60 | 10 | 0.634 | 0.006 | 0.650 | 0.007 | 0.685 | 0.0081 |
| | 60 | 0.612 | 0.002 | 0.666 | 0.002 | 0.678 | 0.0002 |

**Table 2.** Bayes Estimates of the Reliability Function

| n | m | Prior Parameters $\varphi_1$= 1, $\varphi_2$= 2, $\beta_1$= 3, $\beta_2$= 4<br>$\theta_1$ =1, $\theta_2$= 2, b= .2, R = .66667 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{R}_1$ | $\hat{R}_3$ | $\hat{R}_4$ | | | | $\hat{R}_5$ | | | |
| | | | | C=100 | C=150 | C=-50 | C=-100 | q =1 | q = 3 | q = - .5 | q = - 2 |
| 20 | 10 | 0.638 | 0.329 | 0.438 | 0.466 | 0.699 | 0.563 | 0.580 | 0.638 | 0.539 | 0.553 |
| 60 | 10 | 0.666 | 0.307 | 0.456 | 0.482 | 0.661 | 0.544 | 0.546 | 0.557 | 0.596 | 0.508 |
| 60 | 30 | 0.666 | 0.2083 | 0.325 | 0.375 | 0.928 | 0.674 | 0.561 | 0.595 | 0.608 | 0.578 |
| 100 | 10 | 0.666 | 0.158 | 0.458 | 0.483 | 0.654 | 0.539 | 0.529 | 0.608 | 0.557 | 0.596 |
| 100 | 30 | 0.648 | 0.158 | 0.351 | 0.398 | 0.876 | 0.645 | 0.569 | 0.629 | 0.594 | 0.595 |

**Table 3.** Sensitivity of Bayes Estimator to the Prior Parameters

| $\theta_1 = 1, \theta_2 = 2, b = 2, R = .66667, n = 10, m = 20$ | | | | | | |
|---|---|---|---|---|---|---|
| $\varphi_1$ | $\varphi_2$ | $\beta_1$ | $\beta_2$ | $\hat{R}_3$ | $Bias$ | $MSE\ \hat{R}_3$ |
| 1 | 2 | 3 | 2 | 0.675807 | 0.009140 | 0.000540 |
| | | | 4 | 0.705948 | 0.039282 | 0.000792 |
| | | | 6 | 0.647106 | - 0.019560 | 0.003069 |
| | | 5 | 2 | 0.799059 | 0.132392 | 0.001352 |
| | | | 4 | 0.678427 | 0.011760 | 0.000428 |
| | | | 6 | 0.609685 | - 0.056980 | 0.001507 |
| | 4 | 3 | 2 | 0.707073 | 0.040406 | 0.003145 |
| | | | 4 | 0.597962 | - 0.068700 | 0.001105 |
| | | | 6 | 0.637956 | - 0.028710 | 0.001108 |
| | | 5 | 2 | 0.676283 | 0.009617 | 0.005345 |
| | | | 4 | 0.726487 | 0.059820 | 0.001917 |
| | | | 6 | 0.714830 | 0.048163 | 0.001052 |
| | 6 | 3 | 2 | 0.710516 | 0.043849 | 0.007151 |
| | | | 4 | 0.738799 | 0.072132 | 0.003222 |
| | | | 6 | 0.705436 | 0.038770 | 0.001764 |
| | | 5 | 2 | 0.708526 | 0.041859 | 0.009894 |
| | | | 4 | 0.683676 | 0.017010 | 0.005639 |
| | | | 6 | 0.755847 | 0.089180 | 0.002477 |
| 3 | 2 | 3 | 2 | 0.749162 | 0.082495 | 0.000802 |
| | | | 4 | 0.599080 | - 0.067590 | 0.003771 |
| | | | 6 | 0.388317 | - 0.278350 | 0.008626 |
| | | 5 | 2 | 0.564465 | - 0.102200 | 0.000318 |
| | | | 4 | 0.684025 | 0.017359 | 0.001810 |
| | | | 6 | 0.516416 | - 0.150250 | 0.005592 |
| | 4 | 3 | 2 | 0.710881 | 0.044214 | 0.000932 |
| | | | 4 | 0.618519 | - 0.048150 | 0.001211 |
| | | | 6 | 0.554267 | - 0.112400 | 0.003576 |
| | | 5 | 2 | 0.673980 | 0.007314 | 0.001707 |
| | | | 4 | 0.643080 | - 0.023590 | 0.000811 |
| | | | 6 | 0.618309 | - 0.048360 | 0.001815 |
| | 6 | 3 | 2 | 0.681611 | 0.014944 | 0.002772 |
| | | | 4 | 0.618714 | - 0.047950 | 0.001313 |
| | | | 6 | 0.550589 | - 0.116080 | 0.001693 |
| | | 5 | 2 | 0.780480 | 0.113814 | 0.004810 |
| | | | 4 | 0.709575 | 0.042909 | 0.002034 |
| | | | 6 | 0.609194 | - 0.057470 | 0.001402 |
| 5 | 2 | 3 | 2 | 0.662994 | - 0.003670 | 0.003476 |
| | | | 4 | 0.635475 | - 0.031190 | 0.008958 |
| | | | 6 | 0.490599 | - 0.176070 | 0.016216 |
| | | 5 | 2 | 0.618631 | - 0.048040 | 0.001596 |
| | | | 4 | 0.516726 | - 0.149940 | 0.005709 |
| | | | 6 | 0.435846 | - 0.230820 | 0.011371 |
| | 4 | 3 | 2 | 0.599962 | - 0.066700 | 0.000940 |
| | | | 4 | 0.707833 | 0.041166 | 0.003461 |
| | | | 6 | 0.595427 | - 0.071240 | 0.008232 |
| | | 5 | 2 | 0.695973 | 0.029306 | 0.000726 |
| | | | 4 | 0.579451 | - 0.087220 | 0.001940 |
| | | | 6 | 0.533457 | - 0.133210 | 0.005283 |
| | 6 | 3 | 2 | 0.567377 | - 0.099290 | 0.001239 |
| | | | 4 | 0.633209 | - 0.033460 | 0.001638 |
| | | | 6 | 0.624459 | - 0.042210 | 0.003994 |
| | | 5 | 2 | 0.635440 | - 0.031230 | 0.001992 |
| | | | 4 | 0.673176 | 0.006510 | 0.001081 |
| | | | 6 | 0.626982 | - 0.039680 | 0.002362 |

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