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Korovkin type aproximation theorem through statistical lacunary summability

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Abstract

Korovkin type approximation theorems are useful tools to check whether a given sequence $(L_n)_{n\geq 1}$ of positive linear operators on $C[0,1]$ of all continuous functions on the real interval $[0,1]$ is an approximation process. That is, these theorems exhibit a variety of test functions which assure that the approximation property holds on the whole space if it holds for them. Such a property was discovered by Korovkin in 1953 for the functions $1, x$ and x^2 in the space $C[0,1]$ as well as for the functions *1, cos* and *sin* in the space of all continuous 2π-periodic functions on the real line. In this paper, we use the notion of statistical lacunary summability to improve the result of [Ann. Univ. Ferrara, 57(2) (2011) 373-381] by using the test functions $1, e^{-x}, e^{-2x}$ in place of $1, x$ and x^2. We apply the classical Baskakov operator to construct an example in support of our main result.

***Keywords**:* Statistical convergence; statistical lacunary summability; positive linear operator; Korovkin type approximation theorem

1. Introduction and Preliminaries

Let $K \subseteq \mathbb{N}$ and $K_n = \{k \leq n: k \in \mathbb{N}\}$. Then the *natural density* of K is defined by $\delta(K) = lim_n \frac{1}{n}|K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of the set K_n. A sequence $x = (x_k)^{\infty}{}_{k=0}$ of real numbers is said to be *statistically convergent* (c.f. [1]) to L provided that for every $\varepsilon > 0\_ > 0$ the set $K_\varepsilon := \{k \in \mathbb{N}: |x_k - L| \geq \varepsilon\}$ has natural density zero, i.e. for each $\varepsilon > 0$,

$$lim_n \frac{1}{n}|\{k \leq n: |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $L = st - limx$. Note that every convergent sequence is statistically convergent but not conversely.

By a *lacunary sequence* we mean an increasing sequence θ= (k_r) of positive integers such that $k_0 = 0, h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio k_r/k_{r-1} will be abbreviated by q_r.

Let $K \subseteq \mathbb{N}$. Then

$$\delta_\theta(K) = lim_r \frac{1}{h_r}|\{k_{r-1} < j \leq k_r: j \in K\}|$$

is said to be the *θ-density* of the set K.

A sequence $x = (x_k)$ is said to be *lacunary statistically convergent* (see [2]) to L if for every $\varepsilon > 0$, the set $K_\varepsilon := \{j \in \mathbb{N}: |x_j - L| \geq \varepsilon\}$ has θ-density zero, i.e. $\delta_\theta(K_\varepsilon) = 0$. In this case we write $L = S_\theta - limx$. That is,

$$lim_r \frac{1}{h_r}\left|\{k_{r-1} < j \leq k_r: |x_j - L| \geq \varepsilon\}\right| = 0.$$

Recently the concept of statistically lacunary summability has been introduced in [3].

A sequence $x = (x_k)$ is said to be *statistically lacunary summable* (or *statistically θ-summable*) to L if for every $\varepsilon > 0$, the set $K_\varepsilon(\theta) := \{r \in \mathbb{N}: |t_r(x) - L| \geq \varepsilon\}$ has natural density zero, i.e. $\delta(K_\varepsilon(\theta)) = 0$. In this case we write $L = S_\theta - limx$. That is,

$$lim_n \frac{1}{n}|\{r \leq n: |t_r(x) - L| \geq \varepsilon\}| = 0,$$

where $t_r(x) = \frac{1}{h_r}\sum_{j\in I_r} x_j$.

In other words, a sequence $x = (x_k)$ is statistically lacunary summable to L if and only if the sequence $(t_r(x))$ is statistically convergent to L. In this case we write $\theta_S - limx = L$. We denote the set of all statistically lacunary summable sequences by θ_S.

Note that if a sequence $x = (x_k)$ is bounded and lacunary statistically convergent to L then it is statistically lacunary summable to L [3].

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Let $C[a,b]$ be the space of all functions f continuous on $[a,b]$. We know that $C[a,b]$ is a Banach space with norm

$\| f \|_\infty = \sup_{x\in[a,b]} |f(x)|, f \in C[a,b].$

The classical Korovkin approximation theorem is stated as follows [4, 5]:

Let (T_n) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then $lim_n \| T_n(f,x) - f(x) \|_\infty = 0$ for all $f \in C[a,b]$ if and only if $lim_n \| T_n(f_i,x) - f_i(x) \|_\infty = 0$ for $i = 0,1, 2,$ where $f_0(x) = 0$, $f_1(x) = x$ and $f_2(x) = x^2$.

Quite recently, such type of approximation theorems are proved in [6] and [7] by using almost convergence of single and double sequences, respectively. Patterson and Savas [8] have proved the Korovkin type theorem for lacunary statistical convergence. In [9-12] authors have used different types of statistical summability methods. In [13] and [14] authors have used the notion of statistcal A-summability of double sequences to prove Korovkin type theorems for functions of two variables. Boyanov and Veselinov [15] have proved the Korovkin theorem on $C[0,\infty)$ by using the test functions $1, e^{-x}, e^{-2x}$. In this paper, we generalize the result of Boyanov and Veselinov by using the notion of statistical lacunary summability while using the same test functions $1, e^{-x}, e^{-2x}$. We also give an example to justify that our result is stronger than that of Boyanov and Veselinov [15].

2. Main result

Let $C(I)$ be the Banach space with the uniform norm $\|.\|_\infty$of all real-valued continuous functions on $I = [0;1)$, provided that $\lim_{x\to\infty} f(x)$ is finite. Suppose that $L_n: C(I) \to C(I)$. We write $L_n(f;x)$ for $L_n(f(s);x)$; and we say that L is a positive operator if $L(f;x) \geq 0$ for all $(f(x) \geq 0$.

We prove the following generalization of Boyanov and Veselinov [15] for statistical lacunary summability.

Theorem 2.1. Let (T_k) be a sequence of positive linear operators from $C(I)$ into $C(I)$. Then for all $f \in C(I)$

$$\theta_S - \lim_{k\to\infty} \| T_k(f;x) - f(x) \|_\infty = 0. \quad (1)$$

If and only if

$$\theta_S - \lim_{k\to\infty} \| T_k(1;x) - 1 \|_\infty = 0, \quad (2)$$

$$\theta_S - \lim_{k\to\infty} \| T_k(e^{-s};x) - e^{-x} \|_\infty = 0, \quad (3)$$

$$\theta_S - \lim_{k\to\infty} \| T_k(e^{-2s};x) - e^{-2x} \|_\infty = 0. \quad (4)$$

Proof: Since each $1, e^{-x}, e^{-2x}$ belongs to $C(I)$, conditions (2)--(4) follow immediately from (1). Let $f \in C(I)$. Then there exists a constant $M > 0$ such that $|f(x)| \leq M$ for $x \in I$. Therefore,

$$|f(s) - f(x)| \leq 2M, -\infty < s,x < \infty. \quad (5)$$

It is easy to prove that for a given $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(s) - f(x)| \leq \varepsilon, \quad (6)$$

whenever $|e^{-s} - e^{-x}| < \delta$ for all $x \in I$.

Using (5), (6), putting $\Psi_1 = \Psi_1(s,x) = (e^{-s} - e^{-x})^2$, we get

$$|f(s) - f(x)| < \varepsilon + \frac{2M}{\delta^2}\Psi_1, \text{ for every } |s - x| < \delta.$$

That is,

$$-\varepsilon - \frac{2M}{\delta^2}\Psi_1 < f(s) - f(x) < \varepsilon + \frac{2M}{\delta^2}\Psi_1.$$

Now, operating $T_k(1;x)$ to this inequality and using the monotonicity and linearity of (T_k), we obtain

$$T_k(1;x)\left(-\varepsilon - \frac{2M}{\delta^2}\Psi_1\right) < T_k(1;x)\big(f(s) - f(x)\big) < T_k(1;x)\left(\varepsilon + \frac{2M}{\delta^2}\Psi_1\right).$$

Note that x is fixed and so *f(x)* is a constant number. Therefore

$$-\varepsilon T_k(1;x) - \frac{2M}{\delta^2}T_k(\Psi_1;x) < T_k(f;x) - f(x)T_k(1;x) < \varepsilon T_k(1;x) + \frac{2M}{\delta^2}T_k(\Psi_1;x). \quad (7)$$

But

$$T_k(f;x) - f(x) = T_k(f;x) - f(x)T_k(1;x) + f(x)T_k(1;x) - f(x)$$
$$=[T_k(f;x) - f(x)T_k(1;x)] + f(x)[T_k(1;x) - 1]. \quad (8)$$

Using (7) and (8), we have

$$T_k(f;x) - f(x) < \varepsilon T_k(f;x) + \frac{2M}{\delta^2}T_k(\Psi_1;x) + f(x)[T_k(1;x) - 1]. \quad (9)$$

Now

$$T_k(\Psi_1;x) = T_k((e^{-s} - e^{-x})^2;x) = T_k(e^{-2s} - 2e^{-s}e^{-x} + e^{-2x};x)$$
$$=T_k(e^{-2s};x) - 2e^{-x}T_k(e^{-s};x) + e^{-2x}T_k(1;x)$$
$$=[T_k(e^{-2s};x) - e^{-2x}] - 2e^{-x}[T_k(e^{-s};x) - e^{-x}] + e^{-2x}[T_k(1;x) - 1].$$

Using (9), we obtain

$$T_k(f;x) - f(x) < \varepsilon T_k(f;x)$$

$$+\frac{2M}{\delta^2}\{T_k(e^{-2s};x)-e^{-2x}]-2e^{-x}[T_k(e^{-s};x)-e^{-x}]+e^{-2x}[T_k(1;x)-1]\}+f(x)[T_k(1;x)-1]$$

$$=\varepsilon[T_k(1;x)-1]+\varepsilon+\frac{2M}{\delta^2}\{T_k(e^{-2s};x)-e^{-2x}]-2e^{-x}[T_k(e^{-s};x)-e^{-x}]+e^{-2x}[T_k(1;x)-1]\}+f(x)[T_k(1;x)-1].$$

Since ε is arbitrary, we can write

$$T_k(f;x)-f(x)\le\ \varepsilon[T_k(1;x)-1]+\frac{2M}{\delta^2}\{T_k(e^{-2s};x)-e^{-2x}]-2e^{-x}[T_k(e^{-s};x)-e^{-x}]+e^{-2x}[T_k(1;x)-1]\}+f(x)[T_k(1;x)-1].$$

Therefore

$$|T_k(f;x)-f(x)|<\varepsilon+(\varepsilon+M)|T_k(1;x)-1|+\frac{2M}{\delta^2}|e^{-2x}||T_k(1;x)-1|+\frac{2M}{\delta^2}|T_k(e^{-2s};x)-e^{-2x}|+\frac{4M}{\delta^2}|e^{-x}||T_k(e^{-s};x)-e^{-x}|$$
$$\le\ \varepsilon+\left(\varepsilon+M+\frac{4M}{\delta^2}\right)|T_k(1;x)-1|+\frac{2M}{\delta^2}|T_k(1;x)-1|+\frac{2M}{\delta^2}|T_k(e^{-2s};x)-e^{-2x}|+\frac{4M}{\delta^2}|T_k(e^{-s};x)-e^{-x}|, \quad (10)$$

since $|e^{-x}|\le 1$ for all $x\in I$. Now, taking $sup_{x\in I}$, we get

$$\|T_k(f;x)-f(x)\|_\infty\le\varepsilon+K(\|T_k(1;x)-1\|_\infty+\|T_k(e^{-s};x)-e^{-x}\|_\infty+\|T_k(e^{-2s};x)-e^{-2x}\|_\infty), \quad (11)$$

where $K=\max\left\{\varepsilon+M+\frac{4M}{\delta^2},\frac{2M}{\delta^2}\right\}$. Now replacing $T_k(.;x)$ by $B_r(.;x)=\frac{1}{h_r}\sum_{k\in I_r}T_k(.;x)$ in (11) on both sides. For a given $r>0$ choose $\varepsilon'>0$ such that $\varepsilon'<r$. Define the following sets

$$D=\{r\le n:\|B_r(f,x)-f(x)\|_\infty\ge r\},$$
$$D_1=\left\{r\le n:\|B_r(1,x)-1\|_\infty\ge\frac{r-\varepsilon'}{4K}\right\},$$
$$D_2=\left\{r\le n:\|B_r(e^{-s};x)-e^{-x}\|_\infty\ge\frac{r-\varepsilon'}{4K}\right\},$$
$$D_3=\left\{r\le n:\|B_r(e^{-2s};x)-e^{-2x}\|_\infty\ge\frac{r-\varepsilon'}{4K}\right\}.$$

Then $D\subset D_1\cup D_2\cup D_3$, and so $\delta(D)\le\delta(D_1)+\delta(D_2)+\delta(D_3)$. Therefore, using conditions (2), (3) and (4), we get

$$\theta_S-\lim_{k\to\infty}\|T_k(f;x)-f(x)\|_\infty=\ 0.$$

This completes the proof of the paper.

3. Example

In the following we construct an example of a sequence of positive linear operators satisfying the conditions of Theorem 2.1, but not satisfying the conditions of the Korovkin approximation theorem due to that of Boyanov and Veselinov [15].

Consider the sequence of classical Baskakov operators [16]

$$V_n(f;x)=\sum_k f\left(\frac{k}{n}\right)\binom{n-1+k}{k}x^k(1+x)^{-n-k};$$

where $0\le x,y<\infty$.

Let θ= (k_r) be any lacunary sequence and $x=(x_n)$ be defined by

$$x_k=\begin{cases}\sqrt{n}, \text{if } n \text{ is square,}\\ 0, \text{otherwise.}\end{cases}$$

Let $L_n: C(I)\to C(I)$ be defined by

$$L_n(f;x)=(1+x_n)V_n(f;x).$$

Note that this sequence is statistically lacunary summable to 0 but not convergent. Now

$$L_n(1;x)=1,$$
$$L_n(e^{-s};x)=(1+x-xe^{\frac{-1}{n}})^{-n},$$
$$L_n(e^{-2s};x)=(1+x-xe^{\frac{-2}{n}})^{-2n},$$

we have that the sequence (L_n) satisfies the conditions (2), (3) and (4). Hence by Theorem 2.1, we have

$$\theta_S-\lim_{n\to\infty}\|L_n(f)-f\|_\infty=\ 0.$$

On the other hand, we get $L_n(f;0)=(1+x_n)V_n(f;0)$, since $V_n(f;0)=f(0)$, and hence

$$\|L_n(f;x)-f(x)\|_\infty\ge|L_n(f;0)-f(0)|=x_n|f(0)|.$$

We see that (L_n) does not satisfy the conditions of the theorem of Boyanov and Veselinov, since $\lim_{n\to\infty}x_n$ does not exist. Hence our Theorem 2.1 is stronger than that of Boyanov and Veselinov [15].

Remark: Most recently in [17-21], one can find such type of theorems for different summability methods through different set of test functions.

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