Direct and fixed point methods approach to the generalized Hyers–Ulam stability for a functional equation having monomials as solutions

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Abstract

The main goal of this paper is the study of the generalized Hyers-Ulam stability of the following functional equation

\[ f(2x + y) + f(2x - y) + (n-1)(n-2)(n-3)f(y) = 2^{n-2}[f(x + y) + f(x - y) + 6f(x)] \]

where \( n = 1, 2, 3, 4 \), in non-Archimedean spaces, by using direct and fixed point methods.

Keywords: Hyers- Ulam stability; non-Archimedean normed space; \( p \)-adic field

1. Introduction

A classical question in the theory of functional equations is the following: when is it true that a function which approximately satisfies a functional equation \( D \) must be close to an exact solution of \( D \)? If the problem accepts a solution, we say that the equation \( D \) is stable. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940.

In the next year, D. H. Hyers [2] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces.

In 1978, Th. M. Rassias proved a generalization of Hyers’ theorem for additive mappings. The result of Th. M. Rassias has influenced the development of what is now called the Hyers-Ulam-Rassias stability theory for functional equations.

Theorem 1. ([3]): Let \( f : E \to E' \) be a mapping from a normed vector space \( E \) into a Banach space \( E' \) subject to the inequality

\[ \|f(x + y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^p + \|y\|^p) \]

for all \( x, y \in E \) where \( \varepsilon \) and \( p \) are constants with \( \varepsilon > 0 \) and \( 0 \leq p < 1 \). Then the limit

\[ L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \]

exists for all \( x \in E \) and \( L : E \to E' \) is the unique additive mapping which satisfies

\[ \|f(x) - L(x)\| \leq \frac{2\varepsilon}{2-2p} \|x\|^p \]

for all \( x \in E \). Also, if for each \( x \in E \) the function \( f(tx) \) is continuous in \( t \in R \), then \( L \) is linear.

In 1994, a generalization of Rassias’ theorem was obtained by Gavruta [4] by replacing the bound \( \varepsilon (\|x\|^p + \|y\|^p) \) with a general control function \( \varphi(x, y) \).

Let \( X \) and \( Y \) be vector spaces and let \( f : X \to Y \) be a mapping for each \( n = 1, 2, 3 \), consider the functional equation

\[ f(2x + y) + f(2x - y) = 2^{n-2}[f(x + y) + f(x - y) + 6f(x)] \]  \hspace{1cm} (1)

Also, consider the functional equation

\[ f(2x + y) + f(2x - y) + 6f(y) = 4[f(x + y) + f(x - y) + 6f(x)] \]  \hspace{1cm} (2)

For \( X = Y = R \), the monomial \( f(x) = cx^n \) is a solution of (1) for each \( n = 1, 2, 3 \) and the monomial

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Received: 18 January 2011 / Accepted: 28 June 2011
\[ f(x) = cx^4 \] is a solution of (2). It is easy to show that, a mapping \( f : X \to Y \) satisfies (1) for \( n = 1 \) if and only if it also satisfies the Cauchy functional equation \[ f(x + y) = f(x) + f(y). \]

For \( n = 2 \), in [5] it was shown that the equation (1) is equivalent to the quadratic functional equation
\[ f(x^2) + f(y^2) = 2f(x^2) + 2f(y^2). \]

\[ f(x^2) + f(y^2) = 2f(x^2) + 2f(y^2) - 2f(x) - 2f(y). \]

In [8], the equation (2) was shown to be equivalent to the above equation.

\[ f(x^2) + f(y^2) = 2f(x^2) + 2f(y^2) - 2f(x) - 2f(y). \]

In [8], the equation (2) was shown to be equivalent to the above equation.

In 1897, Hensel [9] introduced a normed space which does not have the Archimedean property.

In this paper, the generalized Hyers-Ulam stability of functional equation
\[ f(x^2) + f(y^2) = 2f(x^2) + 2f(y^2) - 2f(x) - 2f(y). \]

will be investigated in non-Archimedean normed space.

In [8], Bae and Park obtained the general solution of the functional equation (4) and proved the generalized Hyers-Ulam stability of this functional equation in Banach * -algebra.

**Remark 1.** For convenience, for all \( x, y \), let
\[ \Omega_2(x, y) = f(x^2) + f(y^2) - 2f(x^2) + 2f(y^2) - 2f(x) - 2f(y). \]

**2. Preliminaries**

**Definition 1.** By a non-Archimedean field, we mean a field \( K \) equipped with a function (valuation): \( K \to [0, \infty) \) such that for all \( r, s \in K \), the following conditions hold:
(i) \( |r| = 0 \) if and only if \( r = 0 \)
(ii) \( |rs| = |r||s| \)
(iii) \( |r + s| \leq \max\{|r|, |s|\} \).

**Definition 2.** Let \( X \) be a vector space over a scalar field \( K \) with a non-Archimedean non-trivial valuation. A function \( |\cdot| : X \to R \) is a non-Archimedean norm (valuation) if it satisfies the following conditions:
(i) \( |r| = 0 \) if and only if \( r = 0 \)
(ii) \( |r| = |r||s| \)
(iii) the strong triangle inequality (ultra-metric), namely
\[ |r + s| \leq \max\{|r||s|, |r||s|\}, \quad r, s \in X \]

Then \( (X, |\cdot|) \) is called a non-Archimedean space.

Due to the fact that
\[ |r_n - x_n| \leq \max\{|r_{j+1} - x_{j+1}| : m \leq j < n\} \quad (n > m) \]

**Definition 3.** A sequence \( \{x_n\} \) is Cauchy if and only if \( \{x_{n+1} - x_n\} \) converges to zero in a non-Archimedean space. By a complete non-Archimedean space, that is, one in which every Cauchy sequence is convergent.

The most important examples of non-Archimedean spaces are \( p \)-adic numbers. A key property of \( p \)-adic numbers is that they do not satisfy the Archimedean axiom: for all \( x, y > 0 \), there exists an integer \( n \) such that \( x < ny \).

**Example 1.** Fix a prime number \( p \). For any nonzero rational number \( x \), there exists a unique integer \( n \in \mathbb{Z} \) such that \( x = \frac{a}{b} \cdot p^n \), where \( a \) and \( b \) are integers not divisible by \( p \). Then \( \lfloor x \rfloor_p \) defines a non-Archimedean norm on \( Q \). The completion of \( Q \) with respect to the metric \( d(x, y) = |x - y|_p \) is denoted by \( Q_p \), which is called the \( p \)-adic number field. In fact, \( Q_p \) is the set of all formal series \( x = \sum_{k \geq n} a_k p^k \) where \( |a_k| \leq p - 1 \) are integers. The addition and multiplication between any two elements of \( Q_p \) are defined naturally. The norm \( \sum_{k \geq n} a_k p^k \) is a non-Archimedean norm on \( Q_p \) and it makes \( Q_p \) a locally compact field.

**Definition 4.** Let \( X \) be a set. A function \( d : X \times X \to [0, \infty] \) is called a generalized metric on \( X \) if \( d \) satisfies the following conditions:
(i) \( d(x, y) = 0 \) if and only if \( x = y \), for all \( x, y \in X \);
(ii) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);
(iii) \(d(x,z) \leq d(x,y) + d(y,z)\) for all \(x,y,z \in X\)

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

**Theorem 2.** Let \((X,d)\) be a complete generalized metric space and \(J:X \to X\) be a strictly contractive mapping with Lipschitz constant \(L < 1\). Then, for all \(x \in X\); either

\[
\Omega^*_1(x,y) = \lfloor 2^n \rfloor \leq \zeta_0(x,y) 
\]

for all \(x,y \in G\). Then the limit

\[
\mathcal{G}(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}
\]

exists for all \(x \in G\) and \(\mathcal{G}(x):G \to X\) is a mapping satisfying

\[
\|f - \mathcal{G}(x)\| \leq \frac{1}{|2^n|} \Omega(x) 
\]

for all \(x \in G\). Moreover, if

\[
\lim_{j \to \infty} \lim_{n \to \infty} \left\lfloor \zeta_j(2^n x,0), a_j \zeta_j(0,0) \right\rfloor, j \leq k + m \right\rfloor = 0
\]

Then \(\mathcal{G}(x)\) is the unique mapping satisfying (8).

**Proof:** Letting \(x = y = 0\) in (7), we get

\[
\|f(0)\| \leq \frac{\zeta_0(0,0)}{2 + (n - 1)(n - 2)(n - 3) - 2^{n+1}}
\]

Putting \(y = 0\) in (7), we get

\[
\|f(x)\| \leq \frac{\zeta_m(x,0)}{2 + (n - 1)(n - 2)(n - 3) - 2^{n+1}}
\]

for all \(x \in G\). By the above two inequalities, we have

\[
\|f(2x) + (n - 1)(n - 2)(n - 3)f(0) - 2^{n+1} f(x)\| \leq \zeta_m(x,0)
\]

for all \(x \in G\). By the above two inequalities, we have

\[
\|f(2x) - 2^n f(x)\| \leq \max \left\{ \|f(2x) + (n - 1)(n - 2)(n - 3)f(0) - 2^{n+1} f(x)\|, \|f(2x) - 2^n f(x)\| \right\}
\]

for all \(x \in G\). So

\[
\|f(2x) - 2^n f(x)\| \leq \frac{1}{|2^n|} \max \left\{ \zeta_m(x,0), a_m \zeta_m(0,0) \right\}
\]

for all \(x \in G\). Replacing \(x\) by \(2^n x\) and dividing both sides by \(|2^n|\) in (12), we get

\[
\|f(2^n x) - 2^{n+1} f(x)\| \leq \frac{1}{|2^{n+1}|} \max \left\{ \zeta_m(2^n x,0), a_m \zeta_m(0,0) \right\}
\]

for all \(x \in G\). It follows from (5) and (13) that the sequence \(\left\{ \frac{f(2^n x)}{2^n} \right\}_{n>1} \) is a Cauchy sequence in complete non-Archimedean space \(X\), and so is convergent. Set
Using induction on $m$, one can easily see that
\[
\lim_{m \to \infty} \max\{a, b, 0\} = \min\{a, b\}
\]
(14)

By taking $m$ to approach infinity in (14) and using (6) one obtains (8). To show $\mathcal{O}(x)$ satisfies (4), replace $x$ and $y$ by $2^m x$ and $2^m y$, respectively, in (7) and divide by $2^m m$, we obtain
\[
\lim_{m \to \infty} \xi_{2m}^{(x,y)} = 0
\]
for all $x, y \in G$ and all $m \in \mathbb{N}$. Taking the limit as $m \to \infty$, we find that $\mathcal{O}(x)$ satisfies (4) for all $x, y \in G$.

To prove the uniqueness of the mapping $\mathcal{O}(x)$, let $\mathcal{O}^\prime(\cdot)$ be another mapping satisfying (8), then for $x \in G$, we get
\[
\mathcal{O}(x) - \mathcal{O}^\prime(x) = \lim_{m \to \infty} \max\{\xi_{2m}^{(x)}, \xi_{2m}^{(x)}\}
\]
(15)

Therefore, $\mathcal{O} = \mathcal{O}^\prime$. This completes the proof.

Corollary 1. For each $n = 1, 2, 3, 4$, let $\eta : [0, \infty) \to [0, \infty)$ be a function satisfying
\[
\eta(2^t) \leq \eta(2) \eta(t) (t \geq 0), \quad \eta([2]) < 2^2.
\]
(17)

Let $\delta > 0$ and $f : G \to X$ be a mapping satisfying
\[
\|\Omega_f^x(x,y)\| \leq \delta \left(\|x\| + \|y\|\right)
\]
for all $x, y \in G$. Then there exists a unique mapping $\mathcal{O} : G \to X$ such that
\[
\|f(x) - \mathcal{O}(x)\| \leq \lim_{n \to \infty} \frac{\delta \eta([2^t])}{[2^t]}
\]
(18)

Proof: Defining $\zeta_n : G^2 \to [0, \infty)$ by
\[
\zeta_n(x,y) = \delta \eta([2^t]) \eta([2]) < 1,
\]
then we obtain that for all $x, y \in G$
\[
\lim_{n \to \infty} \zeta_n(x,y) = 0
\]
Also,
\[
\Omega(x) = \lim_{n \to \infty} \max\{\zeta_n^{(x,y)}; 0 \leq k \leq m\}
\]
(16)

Applying Theorem 3, the desired result is obtained.

Theorem 4. For each $n = 1, 2, 3, 4$, let $\zeta_n : G^2 \to [0, \infty)$ be a function such that
\[
\lim_{n \to \infty} \zeta_n(x,y) = 0
\]
(15)

for all $x, y \in G$. Let for each $x \in G$, the limit
\[
\lim_{n \to \infty} \zeta_n(x,y) = 0
\]
exists. Suppose that $f : G \to X$ be a mapping satisfying the inequality
\[
\Omega_f^x(x,y) = 0
\]
(17)

for all $x, y \in G$. Then the limit
\[
\mathcal{O}(x) = \lim_{n \to \infty} f \left(\frac{x}{2^n}\right)
\]
exists for all $x \in G$ and $\mathcal{O}(x) : G \to X$ is a mapping satisfying
\[
\|f(x) - \mathcal{O}(x)\| \leq \frac{1}{[2] \Omega(x)}
\]
(18)

for all $x \in G$. Moreover, if
\[
\lim_{n \to \infty} \max\{\zeta_n^{(x,y)}; 0 \leq k \leq m\} = 0
\]
Then $\mathcal{O}(x)$ is the unique mapping satisfying (18).

Proof: By (12), we have
Replacing \( x \) by \( \frac{x}{2^n} \) in (19), we obtain

\[
\left\| f\left(\frac{x}{2^n}\right) - 2^{-n} f\left(\frac{x}{2^n}\right) \right\| \leq \frac{1}{m} \max\{\zeta_n(x,0), a, \zeta_n(0,0)\}
\]

for all \( x \in G \) and all non-negative integer \( m \). It follows from (15) and (20) that the sequence

\[
\left\{ 2^m f\left(\frac{x}{2^n}\right) \right\}_{m=1}^{\infty}
\]

is a Cauchy in \( X \) for all \( x \in G \). Since \( X \) is complete, the sequence

\[
\left\{ 2^m f\left(\frac{x}{2^n}\right) \right\}_{m=1}^{\infty}
\]

converges for all \( x \in G \). On the other hand, it follows from (20) that

\[
\left\| f\left(\frac{x}{2^n}\right) - 2^{-n} f\left(\frac{x}{2^n}\right) \right\| \leq \max\{\zeta_n\left(\frac{x}{2^n},0\right), a, \zeta_n(0,0)\}
\]

for all \( x \in G \) and all non-negative integers \( p, q \) with \( q > p \geq 0 \). Letting \( p = 0 \) and passing the limit \( q \to \infty \) in the last inequality and using (16), we obtain (18).

The rest of the proof is similar to the proof of Theorem 3.

**Corollary 2.** For each \( n = 1, 2, 3, 4 \), let \( \eta : [0, \infty) \to [0, \infty) \) be a function satisfying

\[
\eta\left(\|f\|^r\right) \leq \eta\left(\|f\|^{r+1}\right) \eta(t) \quad (t \geq 0), \quad \eta\left(\|f\|^r\right) < \|f\|^r
\]

Let \( \delta > 0 \) and \( f : G \to X \) is a mapping satisfying

\[
\|f(x) - f(y)\| \leq \delta(|x| + |y|)
\]

for all \( x, y \in G \). Then there is a unique mapping \( \delta : G \to X \) such that

\[
\left\| f\left(\frac{x}{2^n}\right) - \delta\left(\frac{x}{2^n}\right) \right\| \leq \frac{\delta\eta\left(\|f\|^r\right)}{|2^n|^r}
\]

Proof: Defining \( \zeta_n : G \to [0, \infty) \) by

\[
\zeta_n(x, y) = \delta(|x| + |y|),
\]

then we obtain

\[
\lim_{n \to \infty} 2^n \zeta_n\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0.
\]

Also,

\[
\Omega(x) = \lim_{n \to \infty} \max\left\{ \|f^n - \zeta_n\left(\frac{x}{2^n},0\right)\|, \|a, \zeta_n(0,0)\|, 0 \leq k < m \right\}
\]

\[
\geq \zeta_n\left(\frac{x}{2^n},0\right)
\]

\[
\leq \|f^n\| \delta t\left(\|f\|\right)
\]

And

\[
\lim_{n \to \infty} \max\left\{ \|f^n - \zeta_n\left(\frac{x}{2^n},0\right)\|, \|a, \zeta_n(0,0)\|, j \leq k + m + j \right\} = 0.
\]


Throughout this section, assume that \( X \) is a non-Archimedean normed vector space and that \( Y \) is a non-Archimedean Banach space. In the rest of the present paper, let \( |2| \neq 1 \).

**Theorem 5.** For \( n = 1, 2, 3, 4 \), \( \zeta_n : X \to [0, \infty) \) be a function such that there exists an \( L < 1 \) with

\[
\zeta_n(2x, 2y) \leq \|f^n\| \zeta_n(x, y)
\]

for all \( x, y \in X \). Let \( f : X \to Y \) be a mapping satisfying

\[
\|f(x) - C(x)\| \leq \frac{\max\{\zeta_n(x,0), a, \zeta_n(0,0)\}}{|2^n|^{r+1}(1 - L)}
\]

**Proof:** By (12), we have

\[
\|f(2x) - 2^{n+1} f(x)\| \leq \max\{\zeta_n(x,0), a, \zeta_n(0,0)\}
\]

for all \( x \in X \). Consider the set

\[
S := \{ g : X \to Y \}
\]

and the generalized metric \( d \) in \( S \) defined by

\[
d(f, g) = \inf \{ u \in \mathbb{R} : \|u| f(x) - h(x)\| \leq u \zeta_n(x,0), a, \zeta_n(0,0), \forall x \in X \},
\]

where \( \inf \varphi = +\infty \). It is easy to show that \((S, d)\) is complete. Now, we consider a linear mapping \( J : S \to S \) such that

\[
Jh(x) := \frac{1}{2} h(2x)
\]
for all \( x \in X \). Let \( g, h \in S \) be such that \( d(g, h) = \varepsilon \). Then
\[
\|g(x) - h(x)\| \leq \varepsilon \max \{\zeta_{\varepsilon}(x, 0), a_{\varepsilon} \zeta_{\varepsilon}(0, 0)\}
\]
for all \( x \in X \). So
\[
\|g(x) - f_{\varepsilon}(x)\| = \frac{1}{2^n} g(2^n x) - \frac{1}{2^n} h(2^n x) \leq \frac{\varepsilon}{2^n} \max \{\zeta_{\varepsilon}(2^n x, 0), a_{\varepsilon} \zeta_{\varepsilon}(0, 0)\},
\]
for \( x \in X \). Thus \( d(g, h) = \varepsilon \) implies that \( d(Jg, Jh) \leq L \varepsilon \), this means that \( d(Jg, Jh) \leq L d(g, h) \) for all \( g, h \in S \). It follows from (24) that \( d(f^{(m)} J f) \leq \frac{1}{2^{m+1}} \).

By Theorem 2, there exists a mapping \( C : X \to Y \) satisfying the following:

(i) \( C \) is a fixed point of \( J \), that is, for all \( x \in X \), \( C(2^n x) = 2^n C(x) \) (25)

(ii) the mapping \( C \) is a unique fixed point of \( J \) in the set \( \Omega = \{ h \in S : d(g, h) < \infty \} \). This implies that \( C \) is a unique mapping satisfying (25) such that there exists \( \mu \in (0, \infty) \) satisfying
\[
\| f(x) - C(x) \| \leq \mu \max \{\zeta_{\varepsilon}(x, 0), a_{\varepsilon} \zeta_{\varepsilon}(0, 0)\},
\]
for all \( x \in X \).

(iii) \( d(J^n f, C) \to 0 \) as \( m \to \infty \). This implies the equality,
\[
\lim_{n \to \infty} \frac{f(2^n x)}{2^n} = C(x), \quad \text{for all } x \in X.
\]

(iv) \( d(f^{(m)} J f, C) \leq \frac{d(f^{(m)} J f)}{1 - L} \) with \( f \in \Omega \), which implies the inequality \( d(f^{(m)} J f, C) \leq \frac{1}{2^{m+1} (1 - L)} \).

This implies that the inequality (23) holds.

**Corollary 3.** Let \( \theta \geq 0 \) and \( p \) be a real number with \( 0 < p < 1 \). Let \( f : X \to Y \) be a mapping satisfying
\[
\|f^p(x, y)\| \leq \theta (\|x\|^{p} + \|y\|^{p})
\]
for all \( x, y \in X \). Then, the limit
\[
C(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}
\]
exists for all \( x \in X \) and \( C : X \to Y \) is a unique mapping such that
\[
\|f(x) - C(x)\| \leq \frac{L \max \{\zeta_{\varepsilon}(x, 0), a_{\varepsilon} \zeta_{\varepsilon}(0, 0)\}}{2^{m+1} (1 - L)}
\]
for all \( x \in X \).

**Proof:** The proof follows from Theorem 5 by taking \( \zeta_{\varepsilon}(x, y) = \theta (\|x\|^{p} + \|y\|^{p}) \), for all \( x, y \in X \). In fact, if we choose \( L = \frac{2^n}{2^{m+1}} \) we get the desired result.

**Theorem 6.** For \( n = 1, 2, 3, 4 \), let \( \zeta_n : X \to [0, \infty) \) be a function such that there exists an \( L < 1 \) with
\[
\zeta_n(x, y) \leq \frac{L}{2^{n+1}} \zeta_n(2^n x, 2^n y)
\]
for all \( x, y \in X \). Let \( f : X \to Y \) be a mapping satisfying
\[
\|f^p(x, y)\| \leq \zeta_n(x, y)
\]
for all \( x, y \in X \). Then there is a unique mapping \( C : X \to Y \) such that
\[
\|f(x) - C(x)\| \leq \frac{L \max \{\zeta_n(x, 0), a_{\varepsilon} \zeta_n(0, 0)\}}{2^n (1 - L)}
\]
(26)

**Proof:** By (11), we have
\[
\|f(x) - 2^n f\left(\frac{x}{2^n}\right)\| \leq \frac{1}{2^n} \max \{\zeta_n(x, 0), a_{\varepsilon} \zeta_n(0, 0)\}
\]
(27)
for all \( x \in X \). Let \( (S, d) \) be the generalized metric space defined as in the proof of Theorem 5, we consider a linear mapping \( f : S \to S \) such that \( Jf(x) = 2^n f\left(\frac{x}{2^n}\right) \) for all \( x \in X \). Let \( g, h \in S \) be such that \( d(g, h) = \varepsilon \). Then
\[
\|f(x) - h(x)\| \leq \varepsilon \max \{\zeta_n(x, 0), a_{\varepsilon} \zeta_n(0, 0)\}
\]
for all \( x \in X \). So
\[
\begin{aligned}
\|f(x) - J(x)\| &= \left\| 2^n g \left(\frac{x}{2}\right) - 2^n h \left(\frac{x}{2}\right) \right\| \\
&\leq \|f\| \mu \max \left\{ \zeta_n(x,0), \alpha_n, \zeta_n(0,0) \right\} \\
&\leq \|f\| \frac{L}{2^n} \max \left\{ \zeta_n(x,0), \alpha_n, \zeta_n(0,0) \right\}
\end{aligned}
\]

for all \( x \in X \). Thus \( d(g,h) = \varepsilon \) implies that 
\( d(Jg,Jh) \leq L \varepsilon \), this means that 
\( d(Jg,Jh) \leq Ld(g,h) \) for all \( g,h \in S \). It follows from (27) that 
\( d(f,Jf) \leq \frac{L}{2^n} \).

By Theorem 2, there exists a mapping 
\( C : X \rightarrow Y \) satisfying the following:

(a) \( C\left(\frac{x}{2}\right) = \frac{1}{2^n} C(x) \) for all \( x \in X \).

(b) The mapping \( C \) is a unique fixed point of \( J \) in the set \( \Omega = \{ h \in S : d(g,h) < \infty \} \). This implies \( C \) is a unique mapping satisfying (28) such that \( \mu \in (0, \infty) \) satisfying
\[
\|f(x) - C(x)\| \leq \mu \max \left\{ \zeta_n(x,0), \alpha_n, \zeta_n(0,0) \right\} ,
\]
for all \( x \in X \).

(c) \( d(J^m f, C) \rightarrow 0 \) as \( m \rightarrow \infty \), this implies the equality
\[
\lim_{m \rightarrow \infty} 2^m f \left(\frac{x}{2^n}\right) = C(x) \quad \text{for all } x \in X.
\]

(d) \( d(f,C) \leq \frac{d(f,Jf)}{1-L} \) with \( f \in \Omega \), which implies the inequality
\[
d(f,C) \leq \frac{L}{2^n} \left(\frac{1}{1-L} \right).
\]

This implies that the inequality (26) holds.

The rest of the proof is similar to the proof of Theorem 5.

**Corollary 4.** Let \( \theta \geq 0 \) and \( p \) be a real number with \( p > 1 \). Let \( f : X \rightarrow Y \) be a mapping satisfying
\[
\|f(x,y)\| \leq \theta \|x\| + \|y\| + \theta \|x\| \|y\|
\]
for all \( x, y \in X \). Then, the limit
\[
C(x) = \lim_{m \rightarrow \infty} 2^m f \left(\frac{x}{2^n}\right)
\]
exists for all \( x \in X \), and
\( C : X \rightarrow Y \) is a mapping such that
\[
\|f(x) - C(x)\| \leq \frac{\|x\| \theta}{2^n} \left(\frac{\|y\|}{2^n} \right)
\]
for all \( x \in X \).

**Proof:** The proof follows from Theorem 6 by taking \( \zeta_n(x,y) = \theta \left(\|x\| + \|y\| + \theta \right) \)
for all \( x,y \in X \). In fact, if we choose \( L = \frac{2^n}{2^n} \), we get the desired result.

**Acknowledgement**

The second author was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2009-0070788).

**References**


