The uniqueness theorem for discontinuous boundary value problems with aftereffect using the nodal points

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Abstract

In this paper, uniqueness theorem is studied for boundary value problem with "aftereffect" on a finite interval with discontinuity conditions in an interior point. The oscillation of the eigenfunctions corresponding to large modulus eigenvalues is established and an asymptotic of the nodal points is obtained. By using these new spectral parameters, uniqueness theorem is proved.

Keywords: Uniqueness Theorem; nodal Points; discontinuous conditions; eigenvalues; eigenfunctions

1. Introduction

Inverse nodal problems exist in recovering operators from given nodes (zeros) of their eigenfunctions. McLaughlin seems to have been the first to consider this sort of inverse problem for the one-dimensional Schrodinger equations on an interval with Dirichlet boundary conditions [1]. Later on, some remarkable results were obtained. For example, X. F. Yang got the uniqueness for general boundary conditions using the same method as McLaughlin [2], C.K. Law and Ching-Fu Yang [3] have reconstructed the potential function and its derivatives from nodal data. We consider boundary value problem with "aftereffect" on a finite interval with discontinuity conditions in an interior point:

\begin{equation}
ly(x) := -y''(x) + q(x)y(x) + \int_0^x M(x-t)y(t)dt = \lambda y(x), \quad 0 < x < T,
\end{equation}

\begin{equation}
U(y) := y'(0) - hy(0) = 0,
V(y) := y'(T) + hy(T) = 0,
\end{equation}

\begin{equation}
y(\frac{T}{2} + 0) = a_1 y(\frac{T}{2} - 0),
y(\frac{T}{2} + 0) = a_1^{-1} y(\frac{T}{2} - 0) + a_2 y(\frac{T}{2} - 0).
\end{equation}

Here $\lambda$ is the spectral parameter. Let $\lambda = \rho^2$, $\rho = \sigma + i \tau$, $q(x)$, $h$, $H$, $a_1$, $a_2$ be real, $q(x) \in L(0,T)$ and $a_1 > 0$.

Without loss of generality we assume that $\int_0^T q(x)dx = 0$. We denote the boundary value problem (1)-(3) by $L(q,M,h,H)$. Boundary value problems with discontinuities inside the interval often appear in mathematics, mechanics, physics, geophysics and other branches of Natural sciences. For example, discontinuous inverse problems appear in electronics for constructing the parameters of heterogeneous electronic lines with desirable technical characteristics [4], [5]. As a rule, such problems are connected with discontinuous material properties. In [6], the authors considered the inverse nodal problem for the differential equation $-y'' + q(x)y = \lambda y, \quad 0 < x < T$ with discontinuity conditions inside the interval. In the present paper, we investigate uniqueness theorem from given nodes of their eigenfunctions for the boundary value problem $L$. In section 2, the eigenvalues and eigenfunctions corresponding to large modulus eigenvalues are obtained and in section 3, an asymptotic of the nodal points is calculated and the uniqueness theorem is proven.

2. Asymptotic of the eigenvalues and eigenfunctions
Let \( \phi(x, \lambda), C(x, \lambda), S(x, \lambda) \) be the solutions of equation (1) under initial conditions
\[ C(0, \lambda) = S'(0, \lambda) = \phi(0, \lambda) = 1, \]
\[ C'(0, \lambda) = S(0, \lambda) = 0, \quad \phi'(0, \lambda) = h \] and under the jump conditions (3). Then \( U(\phi) = 0 \).

Denote
\[ \Delta(\lambda) = -V(\phi). \] (4)

Let \( C_0(x, \lambda) \) and \( S_0(x, \lambda) \) be smooth solutions of (1) on the initial \([0, T]\) under the initial conditions
\[ C_0(0, \lambda) = 1, \quad C_0'(0, \lambda) = 0, \]
\[ S_0(0, \lambda) = S_0'(0, \lambda) = 0. \] Then, using the jump conditions (3) we get [7]:
\[ C(x, \lambda) = C_0(x, \lambda), \quad S(x, \lambda) = S_0(x, \lambda), \quad x < \frac{T}{2} \quad (5) \]
\[ C(x, \lambda) = A_1 C_0(x, \lambda) + B_1 S_0(x, \lambda), \quad S(x, \lambda) = A_2 C_0(x, \lambda) + B_2 S_0(x, \lambda), \quad x > \frac{T}{2} \quad (6) \]
where
\[ A_1 = a_1 C_0(x, \lambda), \quad B_1 = a_1^{-1} C_0(x, \lambda), \]
\[ A_2 = a_2 C_0(x, \lambda), \quad B_2 = a_2^{-1} C_0(x, \lambda). \]

The function \( C_0(x, \lambda) \) satisfies the following integral equation:
\[ C_0(x, \lambda) = \cos \rho x + \frac{\sin \rho x}{\rho} \int_0^x q(t) \, dt + \frac{1}{\rho} \int_0^x M(t-s) C_0(s, \lambda) \, ds \, dt \] (8)
and for \( |\rho| \to \infty \)
\[ C_0(x, \lambda) = \cos \rho x + O\left(\frac{1}{\rho^2} e^{i\rho x}\right). \] (9)

Then (8) implies
\[ C_0(x, \lambda) = \cos \rho x + \frac{\sin \rho x}{2\rho} \int_0^x q(t) \, dt + \frac{1}{2 \rho^2} \int_0^x M(t-s) \cos \rho x \, ds \, dt + O\left(\frac{1}{\rho^2} e^{i\rho x}\right) \] (10)
\[ C_0'(x, \lambda) = -\rho \sin \rho x + \frac{\cos \rho x}{2\rho} \int_0^x q(t) \, dt + \frac{1}{2 \rho^2} \int_0^x q(t) \cos \rho x \, dt + O\left(\frac{1}{\rho^2} e^{i\rho x}\right). \] (11)

Analogously,
\[ S_0(x, \lambda) = \sin \rho x + \frac{\sin \rho x}{\rho} \int_0^x q(t) S_0(t, \lambda) \, dt + \frac{1}{\rho} \int_0^x M(t-s) S_0(s, \lambda) \, ds \, dt. \]
\[ S_0'(x, \lambda) = -\rho \cos \rho x + \frac{\sin \rho x}{2\rho} \int_0^x q(t) \, dt - \frac{1}{2 \rho^2} \int_0^x q(t) \sin \rho x \, dt + O\left(\frac{1}{\rho^2} e^{i\rho x}\right). \] (12)
and for \( |\rho| \to \infty \)
\[ S_0(x, \lambda) = \sin \rho x + O\left(\frac{1}{\rho^2} e^{i\rho x}\right). \] (13)

Then (12) implies
\[ S_0(x, \lambda) = \sin \rho x + \frac{\sin \rho x}{2\rho} \int_0^x q(t) \, dt + \frac{1}{2 \rho^2} \int_0^x q(t) \cos \rho x \, dt + O\left(\frac{1}{\rho^2} e^{i\rho x}\right) \] (14)
\[ S_0'(x, \lambda) = -\rho \cos \rho x + \frac{\sin \rho x}{2\rho} \int_0^x q(t) \, dt - \frac{1}{2 \rho^2} \int_0^x q(t) \sin \rho x \, dt + O\left(\frac{1}{\rho^2} e^{i\rho x}\right). \] (15)

By virtue of (7) and (10)-(15),
\[ A_1 = b_1 + b_2 \cos \rho T + \left(b_1 \int_0^T q(t) \, dt - A_2 \int_0^T q(t) \, dt \right) \sin \rho T \]
\[ + \frac{b_2}{\rho} \int_0^T q(t) \sin \rho (T-t) M(t-s) \, ds \, dt \]
It follows from (20)-(23) that for $|\lambda| \to \infty$

$$\Delta(\lambda) = b_1 \rho \sin \rho T - \omega_1 \cos \rho T - \omega_2 + \kappa(\rho),$$

where

$$\omega_1 = b_1 (H + h + \frac{1}{2} \int_0^T q(t)dt) + \frac{a_2}{2},$$

$$\omega_2 = b_2 (H - h + \frac{1}{2} \int_0^T q(t)dt - \int_0^T \frac{q(t)dt}{2} + \frac{a_2}{2},$$

$$\kappa(\rho) = -b \int_0^T \cos \rho (T - t) \int_0^T M(t - s) cos psdsdt + o(e^{-\rho}).$$

Using (24) by the well-known method (see, for example, [7]) one has that for $n \to \infty$,

$$\rho^2 = \frac{n \pi}{T} = \frac{1}{2} \int_0^T q(t)dt + \frac{a_2}{2},$$

$$b_1 = \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

The eigenfunctions of the boundary value problem $L$ have the form $y_n(x) = \phi(x, \lambda_n)$. Substituting (25) into (20) and (21) we obtain the following asymptotic formulae for $n \to \infty$ uniformly in $x$ (see [10]):

$$y_n(x) = \cos \frac{n \pi x}{T} (b_1 \sin x + b_2 \cos x),$$

$$y_n(x) = \cos \frac{n \pi x}{T} \left(b_1 \sin x + b_2 \cos x\right).$$

3. Computation the nodal points

The eigenfunction $y_n(x)$ has exactly $n$ (simple) zeros inside the interval $(0, T)$ namely:

$$0 < x_1 < \ldots < x_n < T.$$ The set $X_B = \{x_i\}_{i=1, \ldots, n}$ is called the set of nodal points of the boundary value problem $L$. Denote $X_B^k = \{x_i\}_{i=1, \ldots, n}$, $k = 0, 1$. Clearly, $X_B \cup X_B^1 = X_B$. Inverse nodal problems consist in recovering the $M(x)$ and coefficients $h$ and $H$ from the given set $X_B$ of nodal points. Denote $\alpha'_n = \left(j - \frac{1}{2}\right).$ Taking

$$\text{(26)-(27)}$$

into account, we obtain the following
asymptotic formulae for nodal points as \( n \to \infty \) uniformly in \( j \):

\[
 x_\ell^j = \alpha_\ell^j + \frac{T}{2n\pi^2} - [T(2h + \int_0^T q(t)dt) - \frac{2}{b_1}(\omega_1 + (-1)^{\ell-2}\omega_2)\alpha_\ell^j] + 2T\int_0^T [M(t-s)ds]dt \quad \ell = 0, 1, 2, \ldots \tag{28}
\]

\[
 x_\ell^j = \alpha_\ell^j + \frac{T}{2n\pi^2} - [T\alpha_\ell + \frac{T}{2n\pi^2} - \frac{2}{b_1}(\omega_1 + (-1)^{\ell-2}\omega_2)\alpha_\ell^j] + 2T\int_0^T [M(t-s)ds]dt + C
\]

\[
 + \frac{b_1}{b_2} \int_0^T T(2h + \int_0^T q(t)dt) - \frac{2}{b_1}(\omega_1 + (-1)^{\ell-2}\omega_2)\alpha_\ell^j + \frac{C}{T} \tag{29}
\]

where

\[
 C = \frac{1}{b_1 + (-1)^{\ell-2}b_2} (2Th + (-1)^{\ell-1}b_2) + T\alpha_\ell + 1 + (-1)^{\ell-1} + 2(-1)^{\ell-1}Tb_2 \int_0^T q(t)dt + 2(-1)^{\ell-1}b_2 \int_0^T T(\omega_1 + (-1)^{\ell-2}\omega_2)) \tag{30}
\]

We note that the sets \( X^k_B \), \( k = 0, 1 \) are dense on \( (0, T) \). Using these formulae we arrive at the following assertion.

**Theorem 1.** Fix \( k = 0 \lor 1 \) and \( x \in [0, T] \). Let \( \{x_n^j \} \in X^k_B \) be chosen such that

\[
 \lim_{n \to \infty} x_n^j = x.
\]

Then there exists a finite limit

\[
 g_k(x) := \lim_{n \to \infty} \frac{2\pi^2n}{T^2} (nx_n^j - (j - \frac{1}{2})T), \tag{31}
\]

and

\[
 g_k(x) = 2h + \int_0^x q(t)dt - \frac{2}{b_1T}(\omega_1 + (-1)^{k-2}\omega_2)x + 2\int_0^x [M(t-s)ds]dt \tag{32}
\]

\[
 g_k(x) = \int_0^x q(t)dt - \frac{2}{b_1T}(\omega_1 + (-1)^{k-2}\omega_2)x + 2\frac{b_1}{b_1 + (-1)^k b_2} \int_0^x [M(t-s)ds]dt + \frac{C}{T} \tag{33}
\]

where \( C \) are defined by (30).

Let us now prove uniqueness theorem.

**Theorem 2.** Fix \( k = 0 \lor 1 \). Let \( X \subset X^k_B \) be a subset of nodal points which is dense on \( (0, T) \).

Let \( X = \tilde{X} \) then \( M(x) = \tilde{M}(x) \) a.e. on \( (0, T) \), \( h = \tilde{h} \), \( H = \tilde{H} \).

**Proof:** If \( X = \tilde{X} \) then (31) yields \( g_k(x) = \tilde{g}_k(x) \), \( x \in [0, T] \). By virtue of (32)-(33) we get a.e. on \( M(x) = \tilde{M}(x) \). From \( h = \frac{g_k(0)}{2} \), we have \( h = \tilde{h} \). Similarly, we can derive \( H = \tilde{H} \).

**References**


