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Some characterizations for Legendre curves in the 3-Dimensional Sasakian space

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Abstract

In this paper, we give some characterizations for Legendre spherical, Legendre normal and Legendre rectifying curves in the 3-dimensional Sasakian space. Furthermore, we show that Legendre spherical curves are also Legendre normal curves. In particular, we prove that the inverse of curvature of a Legendre rectifying curve is a non-constant linear function of the arclength parameter.

Keywords*:* Legendre curve; normal curve; rectifying curve; Sasakian space

1. Introduction

Necessary and sufficient conditions for a curve to be a spherical curve in Euclidean space E^3 have been given by Wong (1963; 1972). The corresponding characterizations for spherical curves in the Minkowski 3-space have been studied by Petrović-Torgašev and Šućurović (2000; 2001). Analogue to the 3-dimensional spherical curves, the authors have given characterizations for 4-dimensional spherical curves in the Minkowski 4-space (Camcı et al., 2003; Önder and Kocayiğit 2007). Moreover, Camcı et al (2008) have considered the concept of spherical curve in the 3-dimensional Sasakian space and have given some conditions for the Legendre spherical curves in this space.

In the Euclidean space E^3, to each regular unit speed curve $\alpha : I \subset IR \to E^3$, with at least four continuous derivatives, it is possible to associate three mutually orthogonal unit vector fields T, N and B, called the unit tangent, the principal normal and the binormal vector fields, respectively. The planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are known as the osculating plane, the rectifying plane and the normal plane, respectively. The curve $\alpha : I \subset IR \to E^3$ for which the position vector α always lies in its normal plane is called normal curve (Chen, 2003). Therefore, for a normal curve, by definition the position vector α satisfies the equation $\alpha(s) = \lambda(s)N(s) + \delta(s)B(s)$ where $\lambda(s)$ and $\delta(s)$ are differentiable functions of arclength parameter s. The characterizations for normal curves in some spaces such as Euclidean 3-space, Minkowski 3-space and dual Minkowski 3-space have been studied by some authors (Chen, 2003; İlarslan, 2005; Önder, 2006). Similar to normal curves, if the position vector α always lies in its rectifying plane, then the curve $\alpha : I \subset IR \to E^3$ is called rectifying curve (Chen, 2003). By this definition, the position vector α of a rectifying curve satisfies the equation $\alpha(s) = \beta(s)T(s) + \mu(s)B(s)$ for some differentiable functions $\beta(s)$ and $\mu(s)$. One of the most interesting characteristics of rectifying curves is that the ratio of their torsion and curvature is a non-constant linear function of arc length parameter s. Rectifying curves lying fully in the Euclidean space E^3 are determined explicitly by Chen (2003).

In this paper, first we give some characterizations for a Legendre curve to be a spherical curve in the 3-dimensional Sasakian space. Later, we define Legendre normal curve and give the characterizations for this curve. We show that Legendre normal curves are spherical curves. Moreover, we give a definition and characterizations of Legendre rectifying curve. In particular, we prove that the inverse of curvature of a Legendre rectifying curve is a non-constant linear function of arclength parameter s. Also, we find some parametrizations for Legendre rectifying curves in the Sasakian 3-space.

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2. Preliminaries

Let M be a smooth manifold. A contact form η on M is a 1-form such that $d(\eta)^n \wedge \eta \neq 0$ on M. A manifold M together with a contact form is called a contact manifold. The distribution D defined by the Phaffian equation $\eta = 0$ is called the contact structure determined by η. That is,

$$D = \{X \in x(M) | \eta : x(M) \to C^\infty(M, \eta), \eta(X) = 0\}$$

The maximum dimension of integral submanifold of D is called a Legendre submanifold of (M, η).

The reel vector field ζ (killing vector field) is defined by

$$\eta(\zeta) = 1, \ d\eta(\zeta, X) = 0$$

(Yano and Kon, 1983; Belkelfa and et al. 2002).

On a contact manifold (M, η), there exists an endomorphism field ϕ and a metric g satisfying

$$\begin{aligned} \phi^2 &= -X + \eta(X)\zeta, \\ g(\phi X, \phi Y) &= g(X, Y) - \varepsilon\eta(X)\eta(Y), \\ d\eta(X, Y) &= \varepsilon g(X, \phi Y) \end{aligned}$$

for all vector fields X and Y on M where $g(\zeta, \zeta) = \varepsilon = \pm 1$. When $\varepsilon = 1$ and $\varepsilon = -1$, then g is Riemannian and Lorentzian metric, respectively. The structure tensors (ζ, ϕ, g) are called the associated almost contact structure of η (Belkelfa et al. 2002).

A contact manifold $(M, \eta; \zeta, \phi, g, \varepsilon)$ is said to be a Sasaki manifold if M satisfies

$$(\nabla_X \phi)Y = \varepsilon g(X, Y)\zeta - \eta(Y)X, \quad X, Y \in \chi(M).$$

(Yano and Kon, 1983; Belkelfa and et al. 2002).

Now let $M^3 = (M, \eta, \zeta, \phi, g, \varepsilon)$ be a contact 3-manifold with an associated metric g. A curve $\alpha = \alpha(s) : I \to M$ parameterized by arclength parameter s is said to be a Legendre curve if α is tangent to contact distribution D of M. It is obvious that α is a Legendre curve if and only if $\eta(\alpha') = 0$.

Let α be a Legendre curve in M^3. Then the Frenet frame of α is given by $\{T, N, B\}$ where $T = \alpha'$, $N = \phi(\alpha')$ and $B = \zeta$. Then, the Frenet formulae of α are given explicitly by

$$\begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B \end{bmatrix} = \begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \varepsilon \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

where the function κ is the curvature of α. Namely, every Legendre curve has constant torsion 1 (Baikoussis and Blair, 1994; Belkelfa et al. 2002). In a 3-dimensional Sasakian space M, the sphere is defined by $H_1^2 \cup S_1^2$ where

$$H_1^2 = \{P \in M : \ g(P, P) = -r^2\}$$

and

$$S_1^2 = \{P \in M : \ g(P, P) = r^2\}$$

(Camci et al. 2008).

3. Legendre Spherical Curves

In this section we give the characterizations for Legendre spherical curves in the 3-dimensional Sasakian space.

Theorem 3.1. Let α be a Legendre curve with curvature $\kappa > 0$ in the 3-dimensional Sasakian space. Then α is a Legendre spherical curve if and only if

$$\left(\frac{1}{\kappa}\right)^2 + \varepsilon\left[\left(\frac{1}{\kappa}\right)'\right]^2 = \pm r^2, \tag{1}$$

holds, where r is the radius of the sphere.

Proof: Let m be the center of sphere on which α lies. Then, we have

$$g(\alpha - m, \alpha - m) = \pm r^2. \tag{2}$$

If we derive this equation with respect to s, the arc-length of α, then we have

$$g(\alpha - m, T) = 0, \tag{3}$$

where T is unit tangent vector of α. If we repeat the derivation we get

$$g(\alpha - m, N) = -\frac{1}{\kappa}, \tag{4}$$

and the derivation of the last equality gives us

$$g(\alpha - m, B) = -\varepsilon \left(\frac{1}{\kappa}\right)'. \quad (5)$$

Then, from (4) and (5) we can write

$$\alpha - m = -\frac{1}{\kappa} N - \left(\frac{1}{\kappa}\right)' B. \quad (6)$$

Since we have that the radius of the sphere is $g(\alpha - m, \alpha - m) = \pm r$, we obtain that

$$\left(\frac{1}{\kappa}\right)^2 + \varepsilon \left[\left(\frac{1}{\kappa}\right)'\right]^2 = \pm r^2,$$

which completes the proof.

Conversely, assume that (1) holds. If we define the vector

$$m = \alpha + \frac{1}{\kappa} N + \left(\frac{1}{\kappa}\right)' B, \quad (7)$$

and consider (1), we have that $m' = 0$, i.e., m is a constant. Moreover (7) gives that

$$g(m - \alpha, m - \alpha) = \left(\frac{1}{\kappa}\right)^2 + \varepsilon \left[\left(\frac{1}{\kappa}\right)'\right]^2, \quad (8)$$

and from hypothesis

$$g(m - \alpha, m - \alpha) = \pm r = constant, \quad (9)$$

we get that the Legendre curve α lies on the sphere whose center is m and radius is r.

Let now give the integral characterization for Legendre spherical curves in the 3-dimensional Sasakian space.

Theorem 3.2. In 3-dimensional Sasakian space, a unit speed Legendre curve $\alpha(s)$ with curvature $\kappa > 0$ is a spherical curve if and only if there are constants $A, B \in IR$ such that

$$\frac{1}{\kappa} = A\mu(s) + B\eta(s). \quad (10)$$

where $\mu(s) = \sin(s),\ \eta(s) = \cos(s)$ if $\varepsilon = 1$; and $\mu(s) = \sinh(s),\ \eta(s) = \cosh(s)$ if $\varepsilon = -1$.

Proof: Assume that the Legendre curve α lies on the sphere with center m and radius r. Then, from Theorem 3.1 we have

$$\left(\frac{1}{\kappa}\right)^2 + \varepsilon \left[\left(\frac{1}{\kappa}\right)'\right]^2 = \pm r^2. \quad (11)$$

If we take $y = \frac{1}{\kappa}$ in (11) we get

$$y^2 + \varepsilon (y')^2 = \pm r^2. \quad (12)$$

Then we can write that $\frac{dy}{ds} = \sqrt{r^2 - \varepsilon y^2}$ and $\frac{dy}{ds} = -\sqrt{r^2 - \varepsilon y^2}$. These differential equations can be put into their variables, so we can write

$$\frac{dy}{\sqrt{r^2 - \varepsilon y^2}} = ds, \quad \frac{dy}{\sqrt{r^2 - \varepsilon y^2}} = -ds, \quad (13)$$

respectively. If $\varepsilon = 1$, from the last equalities we get

$$\frac{dy}{\sqrt{r^2 - y^2}} = ds, \quad \frac{dy}{\sqrt{r^2 - y^2}} = -ds. \quad (14)$$

By integrating the equations in (14), we have

$$\begin{cases} \int_0^s \frac{dy}{\sqrt{r^2 - y^2}} = \int_0^s ds = s, \\ \int_0^s \frac{dy}{\sqrt{r^2 - y^2}} = -\int_0^s ds = -s, \end{cases} \quad (15)$$

and then the particular solutions of these equalities are

$$y_1 = r \sin s, \quad y_2 = r \cos s, \quad (16)$$

respectively. So, the general solution of differential equation (12) with $\varepsilon = 1$ is

$$y = c_1 y_1 + c_2 y_2 = c_1 r \sin s + c_2 r \cos s, \quad (17)$$

where $c_1, c_2 \in IR$. If we write $y = \frac{1}{\kappa}$, $c_1 r = A$, $c_2 r = B$, the integration of (12) will be in the form

$$\frac{1}{\kappa} = A\sin(s) + B\cos(s), \quad (18)$$

that finishes the proof.
If $\varepsilon = -1$, then from (13) we have

$$\begin{cases} \int_0^s \frac{dy}{\sqrt{r^2+y^2}} = \int_0^s ds = s, \\ \int_0^s \frac{dy}{\sqrt{r^2+y^2}} = -\int_0^s ds = -s. \end{cases} \quad (19)$$

Then we obtain the particular solutions of these equalities as follows

$$y_1 = r\sinh s, \; y_2 = r\cosh s. \quad (20)$$

So, the general solution of differential equation (12) with $\varepsilon = -1$ is

$$y = c_1 y_1 + c_2 y_2 = c_1 r\sinh s + c_2 r\cosh s, \quad (21)$$

where $c_1, c_2 \in IR$. If we write $y = \frac{1}{\kappa}$, $c_1 r = A,\; c_2 r = B$, the integration of (12) will be in the form

$$\frac{1}{\kappa} = A\sinh(s) + B\cosh(s). \quad (22)$$

Using (18) and (22) we can write

$$\frac{1}{\kappa} = A\mu(s) + B\eta(s). \quad (23)$$

where $\mu(s) = \sin(s),\; \eta(s) = \cos(s)$ if $\varepsilon = 1$; and $\mu(s) = \sinh(s),\; \eta(s) = \cosh(s)$ if $\varepsilon = -1$ and $A, B \in IR$.

Conversely, if (23) holds for the Legendre curve $\alpha(s)$, we have

$$\left(\frac{1}{\kappa}\right)^2 + \varepsilon\left[\left(\frac{1}{\kappa}\right)'\right]^2 = \pm r^2.$$

Then, from Theorem 3.1, we say that the curve $\alpha(s)$ is a Legendre spherical curve in 3-dimensional Sasakian space.

4. The Legendre Normal Curves

In this section, we give the definition and the characterization of Legendre normal curves and show that Legendre normal curves are also Legendre spherical curves in the 3-dimensional Sasakian space.

Definition 4.1. Let $\alpha(s)$ be a unit speed Legendre curve in the 3-dimensional Sasakian space with Frenet frame $\{T, N, B\}$ and curvature $\kappa > 0$. The curve $\alpha(s)$ is called Legendre normal curve if the position vector α always lies on the normal plane of the Legendre curve $\alpha(s)$.

By definition, for a Legendre normal curve the position vector α satisfies the equation $\alpha(s) = \lambda(s)N(s) + \delta(s)B(s)$ for some differentiable functions $\lambda(s)$ and $\delta(s)$. Then, we can give the following characterizations.

Theorem 4.1. Let $\alpha(s)$ be a unit speed Legendre normal curve with curvature $\kappa(s) > 0$ in the 3-dimensional Sasakian space. Then the following statements hold:

i) The curvature $\kappa(s)$ satisfies the following equality

$$\frac{1}{\kappa} = d_1\mu(s) + d_2\eta(s), \quad (24)$$

where $\mu(s) = \sin(s),\; \eta(s) = \cos(s)$ if $\varepsilon = 1$; and $\mu(s) = \sinh(s),\; \eta(s) = \cosh(s)$ if $\varepsilon = -1$ and $d_1, d_2 \in IR$.

ii) The principal normal and binormal components of the position vector of the Legendre curve $\alpha(s)$ are given by

$$\begin{cases} g(\alpha(s), N) = -\left(d_1\mu(s) + d_2\eta(s)\right), \\ g(\alpha(s), B) = -\left(\varepsilon d_1\eta(s) - d_2\mu(s)\right), \end{cases} \quad (25)$$

respectively.

Proof: Suppose that $\alpha(s)$ is a unit speed Legendre normal curve. Then by Definition 4.1, we have

$$\alpha(s) = \lambda(s)N(s) + \delta(s)B(s). \quad (26)$$

Differentiating (26) with respect to s and using the Frenet equations, we find

$$\begin{cases} -\lambda(s)\kappa(s) = 1, \\ \lambda'(s) - \delta(s) = 0, \\ \varepsilon\lambda(s) + \delta'(s) = 0. \end{cases} \quad (27)$$

From the first and second equations of (27) we get

$$\lambda = -\frac{1}{\kappa}, \quad \delta = -\left(\frac{1}{\kappa}\right)'. \quad (28)$$

Thus,

$$\alpha = -\frac{1}{\kappa} N - \left(\frac{1}{\kappa}\right)' B. \quad (29)$$

Further, from the third equation in (27) and using (28) we find the following differential equation

$$\left(\frac{1}{\kappa}\right)'' + \varepsilon\left(\frac{1}{\kappa}\right) = 0. \quad (30)$$

Putting $y(s) = \frac{1}{\kappa}$, equation (30) can be written as

$$y'' + \varepsilon y = 0. \quad (31)$$

The solution of (31) is

$$\frac{1}{\kappa} = d_1\mu(s) + d_2\eta(s), \quad (32)$$

where $\mu(s) = \sin(s),\ \eta(s) = \cos(s)$ if $\varepsilon = 1$; and $\mu(s) = \sinh(s),\ \eta(s) = \cosh(s)$ if $\varepsilon = -1$ and $d_1, d_2 \in IR$. Thus we have proved statement (i).

By Theorem 3.2, we see that the Legendre normal curve $\alpha(s)$ is a spherical curve. So we can give the following corollary.

Corollary 4.1. Every Legendre normal curve $\alpha(s)$ is also a Legendre spherical curve in the 3-dimensional Sasakian space.

Furthermore, Camcı et al. (2008) have shown that there are no Legendre spherical curves in the 3-dimensional Sasakian space $IR^3(-3\varepsilon)$. Then, by considering Corollary 4.1, we can give the following corollary.

Corollary 4.2. There are no Legendre normal curves $\alpha(s)$ in the 3-dimensional Sasakian space $IR^3(-3\varepsilon)$.

Let us now prove statement (ii). Substituting (32) into (28) and (29), we get

$$\lambda(s) = -\left(d_1\mu(s) + d_2\eta(s)\right), \quad (33)$$

$$\delta(s) = -\left(d_1\eta(s) - \varepsilon d_2\mu(s)\right), \quad (34)$$

$$\begin{aligned} \alpha = &-\left(d_1\mu(s) + d_2\eta(s)\right) N(s) \\ &-\left(d_1\eta(s) - \varepsilon d_2\mu(s)\right) B(s) \end{aligned} \quad (35)$$

Therefore, since $g(B,B) = \varepsilon$, from (35) we easily find that

$$g(\alpha,\alpha) = d_2^2 + \varepsilon d_1^2, \quad (36)$$

$$g(\alpha(s), N) = -\left(d_1\mu(s) + d_2\eta(s)\right), \quad (37)$$

$$g(\alpha(s), B) = -\left(\varepsilon d_1\eta(s) - d_2\mu(s)\right). \quad (38)$$

Consequently we have proved (ii).

Conversely, suppose that statement (i) holds. Then we have

$$\frac{1}{\kappa} = d_1\mu(s) + d_2\eta(s), \quad (39)$$

where $\mu(s) = \sin(s),\ \eta(s) = \cos(s)$ if $\varepsilon = 1$; and $\mu(s) = \sinh(s),\ \eta(s) = \cosh(s)$ if $\varepsilon = -1$ and $d_1, d_2 \in IR$. Since $\mu'(s) = \eta(s)$ and $\eta'(s) = -\varepsilon\mu(s)$, by differentiating (39) two times with respect to s we find

$$\left(\frac{1}{\kappa(s)}\right)'' = -\varepsilon\left(d_1\mu(s) + d_2\eta(s)\right).$$

(40)

Then from (39) and (40) we have

$$\left(\frac{1}{\kappa}\right)'' + \varepsilon\left(\frac{1}{\kappa}\right) = 0. \quad (41)$$

Equation (41) shows that

$$\left[\left(\left(\frac{1}{\kappa}\right)'\right)\right]^2+\varepsilon\left(\frac{1}{\kappa}\right)^2=constant=r^2.$$

It also means that $\alpha(s)$ is a Legendre spherical curve. So, from (6), by applying Frenet equations we obtain

$$\frac{d}{ds}\left[\alpha(s)+\frac{1}{\kappa(s)}N(s)+\left(\frac{1}{\kappa(s)}\right)'B(s)\right]=0. \quad (42)$$

Consequently, $\alpha(s)$ is congruent to a Legendre normal curve. Next, assume that statement (ii) holds. Then the equations (36) and (37) are satisfied. Differentiating (36) with respect to s and using (37), we find $g(\alpha,T)=0$, which means that $\alpha(s)$ is a Legendre normal curve.

5. The Legendre Rectifying Curves

In this section, we give the definition and characterizations of Legendre rectifying curve in the 3-dimensional Sasakian space.

Definition 5.1. Let $\alpha(s)$ be a unit speed Legendre curve in the 3-dimensional Sasakian space with Frenet frame $\{T,N,B\}$ and curvature $\kappa>0$. The curve $\alpha(s)$ is called Legendre spherical curve if the position vector α always lies on the normal plane of the Legendre curve $\alpha(s)$.

By definition, for a Legendre normal curve the position vector α satisfies the equation $\alpha(s)=\beta(s)T(s)+\mu(s)B(s)$ for some differentiable functions $\beta(s)$ and $\mu(s)$.

Theorem 5. 1. Let $\alpha=\alpha(s)$ be a unit speed Legendre rectifying curve in Sasakian 3-space with curvature $\kappa(s)>0$. Then the following statements hold:

(i) The distance function $\rho=\|\alpha\|$ satisfies $\rho^2=(s+n_1)^2+\varepsilon n_2^2$, for some $n_1,n_2\in IR$.

(ii) The tangential component of the position vector of α is given by $g(\alpha,T)=s+n_1$, where $n_1\in IR$.

(iii) The normal component α^N of the position vector of the curve has a constant length and the distance function ρ is non-constant.

(iv) The binormal component of the position vector of the curve is constant, i.e., $g(\alpha,B)$ is constant.

Conversely, if $\alpha(s)$ is a unit speed Legendre curve in the Sasakian 3-space with curvature $\kappa(s)>0$ and one of the statements (i), (ii), (iii) and (iv) holds, then α is a rectifying curve.

Proof: Let us suppose that $\alpha=\alpha(s)$ is a unit speed Legendre rectifying curve. Then the position vector α of the curve satisfies the equation

$$\alpha(s)=\beta(s)T(s)+\mu(s)B(s), \quad (43)$$

where $\beta(s)$ and $\mu(s)$ are some differentiable functions of arclength parameter s. Differentiating (43) with respect to s and applying the Frenet-Serret equations gives

$$\beta'=1,\ \beta\kappa-\mu=0,\ \mu'=0. \quad (44)$$

Therefore, it follows that

$$\begin{cases}\beta(s)=s+n_1,\ \ \mu(s)=n_2,\\ \beta\kappa=\mu\neq 0,\ n_1,n_2\in IR.\end{cases}$$
(45)

From the equations (43) and (45), we easiliy find $\rho^2=\|\alpha\|^2=(s+n_1)^2+\varepsilon n_2^2$,
and so (i) holds. Further, from (43) we obtain $g(\alpha,T)=\beta$ which together with (45), $g(\alpha,T)=s+n_1$, where $n_1\in IR$ and (ii) holds. Next, from the relation (43) it follows that the normal component α^N of the position vector α is given by $\alpha^N=\mu B$. Therefore $\|\alpha^N\|=|\varepsilon\mu|=|n_2|\neq 0$. Thus we proved statement (iii). Finally, from (43) we easily get $g(\alpha,B)=\varepsilon n_2=\text{constant}$ and so the statement (iv) is proved.

Conversely, assume that statement (i) or statement (ii) holds. Then, there holds the equation $g(\alpha,T)=s+n_1$, where $n_1\in IR$. Differentiating this equation with respect to s, we get $\kappa(s)g\left(\alpha(s),N(s)\right)=0$. Since $\kappa(s)>0$, it follows that $g(\alpha,N)=0$. Hence α is a Legendre rectifying curve.

Next, suppose that statement (iii) holds. Let us put $\alpha(s)=m(s)T(s)+\alpha^N$, $m(s)\in IR$. Then we easily find that $g(\alpha^N,\alpha^N)=C=\text{constant}=g(\alpha,\alpha)-g(\alpha,T)^2$. Differentiating this equation with respect to s we have

$$\kappa g(\alpha, N)=0. \tag{46}$$

Since $\rho \neq 0$, we have $g(\alpha,T)\neq 0$. Moreover, since $\kappa(s)>0$ from (46) we obtain $g(\alpha,N)=0$, which means that α is a Legendre rectifying curve.

Finally, if statement (iv) holds, then by applying Frenet equations, we easily obtain that the curve α is a Legendre rectifying curve.

In the next theorem, we prove that the inverse of the curvature of a Legendre rectifying curve is a non-constant linear function of arc length parameter s.

Theorem 5.2. Let $\alpha=\alpha(s)$ be a unit speed Legendre rectifying curve with curvature $\kappa(s)>0$ in the Sasakian 3-space. Then, up to isometries of Sasakian 3-space, the curve α is a Legendre rectifying curve if and only if $\frac{1}{\kappa(s)}=n_3 s+n_4$ holds, where $n_3, n_4 \in IR$.

Proof: Let us first suppose that $\alpha(s)$ is Legendre rectifying curve. By the proof of Theorem 5.1 and by relations (44) and (45), it follows that

$$\frac{1}{\kappa(s)}=\frac{\beta}{\mu}=\frac{s+n_1}{n_2}, \tag{47}$$

where $n_1, n_2 \in IR$. Consequently, $1/\kappa(s)=n_3 s+n_4$ where $n_3=1/n_2$, $n_4=n_1/n_2$ are real constants. Conversely, let us suppose that $1/\kappa(s)=n_3 s+n_4$ and $n_3,n_4\in IR$. Then, we may choose $n_3=1/n_2$, $n_4=n_1/n_2$ where $n_1,n_2\in IR$. Hence, $\frac{1}{\kappa(s)}=\frac{s+n_1}{n_2}$. Applying the Frenet equations, we easily find that

$$\frac{d}{ds}\big(\alpha(s)-(s+n_1)T-n_2 B\big)=0,$$

which means that up to isometries of Sasakian space, the Legendre curve $\alpha=\alpha(s)$ is a rectifying curve.

Theorem 5.3. Let $\alpha=\alpha(s)$ be a unit speed Legendre curve in the Sasakian 3-space. Then α is a Legendre rectifying curve if and only if, up to a parametrization, α is given by

$$\begin{cases} \alpha(t)=y(t)\dfrac{n_2}{\cos t},\ n_2\in IR^+, & if\ \varepsilon=1 \\ \alpha(t)=y(t)\dfrac{n_2}{\sinh t},\ n_2\in IR^+, & if\ \varepsilon=-1 \end{cases} \tag{48}$$

where $y(t)$ is a unit speed Legendre curve lying on the sphere $H_1^2(1)\cup S_1^2(1)$.

Proof: Let us first assume that $\alpha(s)$ is a unit speed Legendre rectifying curve. Since $g(T,T)=1,\ g(B,B)=\varepsilon$, by the proof of Theorem 5.1, it follows that $\rho^2=\|\alpha\|^2=(s+n_1)^2+\varepsilon n_2^2$, $n_1,n_2\in IR$. We may choose $n_2\in IR^+$. Also, we may apply a translation with respect to s, such that $\rho^2=s^2+\varepsilon n_2^2$. Next, we define a curve by

$$y(s)=\frac{\alpha(s)}{\rho(s)}. \tag{49}$$

and assume that the curve y is lying on the sphere $H_1^2(1)\cup S_1^2(1)$. Then we have

$$\alpha(s)=y(s)\sqrt{s^2+\varepsilon n_2^2}. \tag{50}$$

Differentiating (50) with respect to s, we get

$$T(s)=y(s)\frac{s}{\sqrt{s^2+\varepsilon n_2^2}}+y'(s)\sqrt{s^2+\varepsilon n_2^2}. \tag{51}$$

Since $g(y,\ y)=1$, it follows that $g(y,\ y')=0$. From (51) we obtain

$$1=g(T,T)=g(y',y')(s^2+\varepsilon n_2^2)+\frac{s^2}{s^2+\varepsilon n_2^2}$$

and hence

$$g(y',y')=\frac{\varepsilon n_2^2}{(s^2+\varepsilon n_2^2)^2}. \tag{52}$$

From (52), we get $\|y'(s)\|=n_2/(s^2+\varepsilon n_2^2)$. Let $t=\int_0^s\|y'(u)\|du$ be the arclength parameter of the curve y. Then we have

$$t=\int_0^s \frac{n_2}{u^2+\varepsilon n_2^2}du\,.$$

If $\varepsilon=1$, then we get $s=n_2\tan t$ and if $\varepsilon=-1$, then we get $s=n_2\coth t$. Substituting these into (50), respectively, we obtain the parametrizations given in (48).

Conversely, assume that α is a curve defined by (48) where $y(t)$ is a unit speed Legendre curve lying on the sphere $H_1^2\cup S_1^2$ with radius 1. Differentiating the equation (48) with respect to s, we get

$$\begin{cases}\alpha'(t)=\dfrac{n_2}{\cos^2 t}\left(y(t)\sin t+y'(t)\cos t\right), & if\ \varepsilon=1\\ \alpha'(t)=\dfrac{n_2}{\sinh^2 t}\left(-y(t)\cosh t+y'(t)\sinh t\right), & if\ \varepsilon=-1\end{cases}$$

By assumption, we have $g(y,\,y)=1$, $g(y',\,y')=1$ and consequently $g(y,\,y')=0$. Therefore, it follows that

$$\begin{cases}g(\alpha,\alpha')=\dfrac{n_2^2\sin t}{\cos^3 t},\ g(\alpha',\alpha')=\dfrac{n_2^2}{\cos^4 t}, & if\ \varepsilon=1\\ g(\alpha,\alpha')=-\dfrac{n_2^2\cosh t}{\sinh^3 t},\ g(\alpha',\alpha')=-\dfrac{n_2^2}{\sinh^4 t}, & if\ \varepsilon=-1\end{cases}\qquad(53)$$

and consequently

$$\begin{cases}\|\alpha'(t)\|=n_2/\cos^2 t, & if\ \varepsilon=1\\ \|\alpha'(t)\|=n_2/\sinh^2 t, & if\ \varepsilon=-1\end{cases}$$

Let us put $\alpha(t)=m(t)\alpha'(t)+\alpha^N$, where $m(t)\in IR$ and α^N is normal component of the position vector α. Then we easily find that $m=g(\alpha,\alpha')/g(\alpha',\alpha')$, and therefore

$$g(\alpha^N,\alpha^N)=g(\alpha,\alpha)-\frac{g(\alpha,\alpha')^2}{g(\alpha',\alpha')}\,.$$

Since

$$\begin{cases}g(\alpha,\alpha)=\dfrac{n_2^2}{\cos^2 t}, & if\ \varepsilon=1\\ g(\alpha,\alpha)=\dfrac{n_2^2}{\sinh^2 t}, & if\ \varepsilon=-1\end{cases}$$

by using (53), the last equation becomes $g(\alpha^N,\alpha^N)=n_2^2=\text{constant}$. It follows that $\|\alpha^N\|=\text{constant}$ and since $\rho=\|\alpha\|=n_2/\cos t\neq\text{constant}$, Theorem 5.1 implies that α is a Legendre rectifying curve.

6. Conclusion

In the study of contact manifolds, Legendre curves have an important role. In the contact manifolds, a diffeomorphism is a contact transformation if and only if any Legendre curves in a domain of it go to Legendre curves (Baikoussis and Blair, 1994). Then the study of special Legendre curves is fascinating. By considering the importance of this, Legendre spherical, Legendre normal and Legendre rectifying curves in Lorentzian Sasakian space have been introduced. It is shown that Legendre normal curves are also Legendre spherical curves.

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