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Derivations with power values on multilinear polynomials

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Abstract

A polynomial $f(X_1, X_2, \cdots, X_n)$ is called multilinear if it is homogeneous and linear in every one of its variables. In the present paper our objective is to prove the following result: Let R be a prime K-algebra over a commutative ring K with unity and let $f(X_1, X_2, \cdots, X_n)$ be a multilinear polynomial over K. Suppose that d is a nonzero derivation on R such that $df(x_1, x_2, \cdots, x_n)^s = f(x_1, x_2, \cdots, x_n)^t$ for all $x_1, x_2, \cdots, x_n \in R$, where s, t are fixed positive integers. Then $f(X_1, X_2, \cdots, X_n)$ is central-valued on R. We also examine the case R which is a semiprime K-algebra.

***Keywords**:* Prime and semiprime rings; ideal; derivation; GPIs

1. Introduction

In all that follows R will be a K-algebra over a commutative ring K with unity, U its Utumi quotient ring and the center of U, denoted by C, is called the extended centroid of R (Beidar et al., 1996) and $f(X_1, X_2, \cdots, X_n)$ will be a multilinear polynomial over K with some coefficients invertible in K. Without loss of generality, we may write

$$f(X_1, X_2, \cdots, X_n) = X_1 X_2 \cdots X_n + \sum_{\sigma \in S_n} \alpha_\sigma X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(n)}$$

where the sum is taken over all permutations in S_n except 1. For any $x, y \in R$, the symbols $[x, y]$ and $x \circ y$ stand for the commutator $xy - yx$ and anti-commutator $xy + yx$. Recall that a ring R is prime if for any $a, b \in R, aRb = 0$ implies that $a = 0$ or $b = 0$, and is semiprime if for any $a \in R, aRa = 0$ implies that $a = 0$. An additive mapping $d: R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$, in particular d is called an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. We denote by $f^d(X_1, X_2, \cdots, X_n)$ the polynomial obtained from $f(X_1, X_2, \cdots, X_n)$ by replacing each coefficient by *d(α.1)*.

Ashraf and Rehman (2002) proved that if R is a prime ring, I a nonzero ideal of R, and d a derivation of R such that $d(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative. Argac and Inceboz (2009) generalized the above result as following: Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer, if R admits a derivation d with the property $(d(x \circ y))^n = x \circ y$ for all $x, y \in I$, then R is commutative. On the other hand, Wong (1996) obtained the following result: Let R be a prime ring, d a nonzero derivation of R and $f(X_1, X_2, \cdots, X_t)$ be a multilinear polynomial not vanishing on R. Suppose that $df(x_1, x_2, \cdots, x_t)^n = 0$ for all $x_1, x_2, \cdots, x_t \in R$, where n is a fixed integer. Then f is central-valued on R. Chuang and Lee (1996) proved that if R is a ring without nonzero nil right ideals and $f(X_1, X_2, \cdots, X_t)$ is

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a multilinear polynomial over K which is nil in R, then f vanishes on R. The present paper is motivated by the previous results and we examine what happens in a prime K-algebra R satisfying the identity

$$df(x_1,x_2,\cdots,x_n)^s = f(x_1,x_2,\cdots,x_n)^t,$$

for all $x_1,x_2,\cdots,x_n \in R$, where s,t are fixed positive integers.

2. Main result

We begin with the simplest case when R is the matrix ring $M_m(F)$ over a field F and d is an inner derivation on R.

Lemma 2.1. Let F be a field and $R=M_m(F)$, the $m\times m$ matrix ring over F. Suppose that $a\in R$ and that $f(X_1,X_2,\cdots,X_n)$ is a multilinear polynomial over F such that

$$[a, f(x_1,x_2,\cdots,x_n)]^s = f(x_1,x_2,\cdots,x_n)^t$$

for all $x_1,x_2,\cdots,x_n \in R$, where s,t are fixed positive integers. Then either $a\in Z(R)$, the center of R, or $f(X_1,X_2,\cdots,X_n)$ is central-valued on R.

Proof: If $m=1$, then R is a field and there is nothing to prove; so we assume that $m\geq 2$ and proceed to show that $a\in Z(R)$ if $f(X_1,X_2,\cdots,X_n)$ is not central-valued on R. Denote by e_{ij} the usual matrix unit with 1 in the (i,j)-entry and zero elsewhere. Write $a=\sum \alpha_{ij}e_{ij}$ where $\alpha_{ij}\in F$. We claim first that a is a diagonal matrix, namely, $\alpha_{kh}=0$ for $h\neq k$. Since $f(X_1,X_2,\cdots,X_n)$ is assumed to be non-central on R, by Lee (1993) and Leron (1975), there exists an odd sequence $r=(r_1,r_2,\cdots,r_n)$ from R such that $0\neq f(r)=f(r_1,r_2,\cdots,r_n)=\beta e_{pq}$ for some $\beta\neq 0, p\neq q$. For distinct h,k, let σ be a permutation in the symmetric group S_n such that $\sigma(p)=h, \sigma(q)=k$, and let φ be the automorphism of R defined by $(\sum \xi_{ij}e_{ij})^{\varphi}=\sum \xi_{ij}e_{\sigma(i)\sigma(j)}$.
Then $f(r^{\varphi})=f(r^{\varphi}{}_1,r_2{}^{\varphi},\cdots,r_n{}^{\varphi})=\beta e_{hk}$ and $[a,f(r^{\varphi})]=\beta(\sum \sigma_{ih}e_{ik}-\sum \sigma_{kj}e_{hj})$.

By hypothesis, $[a, f(r^{\varphi})]^s = f(r^{\varphi})^t$. Note that $[a, f(r^{\varphi})]$ has zero (i,j)-entries for $i\neq h, j\neq k$, and so does any power $[a, f(r^{\varphi})]^s$. Also, the (k,k)-entry of $[a, f(r^{\varphi})]$ is $e_{hk}\beta$ and that of $[a, f(r^{\varphi})]^s$ is $(e_{hk}\beta)^s$. On the other hand, if $t=1$ then $f(r^{\varphi})^t=\beta e_{hk}$ and if $t\geq 2$ then $f(r^{\varphi})^t=0$, in both cases the (k,k)-entry of $f(r^{\varphi})^t$ is zero. It follows from $[a, f(r^{\varphi})]^s = f(r^{\varphi})^t$ that $(e_{hk}\beta)^s=0$, whence the fact $e_{hk}=0$ follows. Next we show that $a=\sum \alpha_{ii}e_{ii}$ is a scalar matrix, that is, $\alpha_{hh}=\alpha_{kk}$ for distinct h,k. For any automorphism θ of R, a^{θ} enjoys the same property as a does, namely, $[a^{\theta}, f(x^{\varphi})]^s = f(x^{\varphi})^t$ for all $x\in R$. It is easy to check that

$$\theta(x)=(1+e_{hk})x(1-e_{hk})$$

is an automorphism of R and hence $a^{\theta}=a+(\alpha_{hh}-\alpha_{kk})e_{hk}$ is a diagonal matrix, which implies that $\alpha_{hh}=\alpha_{kk}$. Hence a is a scalar matrix, proving the lemma.

We are now in a position to prove our main theorem.

Theorem 2.2. Let R be a prime K-algebra over a commutative ring K with unity and let $f(X_1,X_2,\cdots,X_n)$ be a multilinear polynomial over K. Suppose that d is a nonzero derivation on R such that

$$df(x_1,x_2,\cdots,x_n)^s = f(x_1,x_2,\cdots,x_n)^t$$

for all $x_1,x_2,\cdots,x_n \in R$, where s,t are fixed positive integers.
Then $f(X_1,X_2,\cdots,X_n)$ is central-valued on R.

Proof: Using Kharchenko's result (1978), we can divide the proof into two cases.

Case 1. If d is Q-inner, that is, $d(x)=[a,x]$ for all $x\in R$, where a is a non-central element in the symmetric quotient ring Q (Beidar et al., 1996), then

$$[a, f(x_1,x_2,\cdots,x_n)]^s = f(x_1,x_2,\cdots,x_n)^t$$

for all $x_1,x_2,\cdots,x_n \in R$. By a theorem due to Chuang (1988), this generalized polynomial identity is also satisfied by Q. In case the center C of Q is infinite, we have

$$[a, f(x_1,x_2,\cdots,x_n)]^s = f(x_1,x_2,\cdots,x_n)^t$$

for all $x_1,x_2,\cdots,x_n \in Q\otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q\otimes_C \overline{C}$ are prime and centrally closed (Erickson et al., 1975), we may replace R by Q or $Q\otimes_C \overline{C}$ according to whether C is finite or infinite. Thus we may assume that R is centrally closed over C (i.e., $RC=R$) which is either finite or algebraically closed and

$$[a, f(x_1,x_2,\cdots,x_n)]^s = f(x_1,x_2,\cdots,x_n)^t$$

for all $x_1,x_2,\cdots,x_n \in R$. From Martindale's result (1969), RC (and so R) is a primitive ring having nonzero socle H with C as the associated division ring. In view of Jacobson (1969), R is isomorphic to a dense ring of linear transformations of some vector space V over C and H consists of the finite rank linear transformations in R. If V is finite-dimensional over C, then $R=M_m(C)$, where $m=\dim_C V$, and so $f(X_1,X_2,\cdots,X_n)$ is central-valued on R by Lemma 2.1. Suppose that V is infinite-dimensional over C, we claim that v, av are linearly C-dependent for all $v\in V$. Since if $av=0$ then v, av are C-dependent. Suppose that $av\neq 0$. Assume that v and av are C-independent, since $\dim_C V=\infty$, then there exist $w_3,\cdots,w_n \in V$ such that

$$v=w_1, av=w_2, w_3\cdots,w_n$$

are also C-independent. By density of R, there exist $r_1,\cdots,r_n \in R$ such that

$$r_1w_1=w_1; r_2w_n=w_1; r_3w_3=w_3,\ r_iw_i=w_{i-1}$$

for all $4\le i\le n-1$ and $r_iw_j=0$ for all other possible choices of i,j, and $r_nw_2=w_{n-1}$. Therefore, we obtain the contradiction

$$\begin{aligned}(-1)^s v &= [a, f(x_1,x_2,\cdots,x_n)]^s v\\ &= f(x_1,x_2,\cdots,x_n)^t v = 0.\end{aligned}$$

So we conclude that v and av are linearly C-dependent for all $v\in V$. Our next goal is to show that there exists $\lambda\in C$ such that $av=v\lambda$ for all $v\in V$. In fact, $v,w\in V$ is chosen to be linearly independent. Since $\dim_C V=\infty$, then there exists $u\in V$ such that u,v,w are linearly independent, and so there exists $\lambda_u,\lambda_v,\lambda_w\in C$ such that $au=u\lambda_u$ and also $av=v\lambda_v, aw=w\lambda_w$, that is, $a(u+v+w)=u\lambda_u+v\lambda_v+w\lambda_w$. Moreover $a(u+v+w)=(u+v+w)\lambda_{u+v+w}$ for a suitable $\lambda_{u+v+w}\in C$. Then we have

$$\begin{aligned}(\lambda_{u+v+w}-\lambda_u)u+(\lambda_{u+v+w}-\lambda_v)v\\ +(\lambda_{u+v+w}-\lambda_w)w=0\end{aligned}$$

and because u,v,w are linearly independent, $\lambda_{u+v+w}=\lambda_u=\lambda_v=\lambda_w$, that is, λ does not depend on the choice of v. Hence we have $av=v\lambda$ for all $v\in V$. Now for $r\in R, v\in V$, we have

$$(ra)v=r(av)=r(v\lambda)=(rv)\lambda=a(rv),$$

that is, $[a,R]V=0$. Since V is a left faithful irreducible R-module, $[a,R]=0$, i.e., $a\in Z(R)$ and so $d=0$, contradicting the hypothesis.

Case 2. If d is Q-outer, then we have

$$\begin{aligned}&df(x_1,x_2,\cdots,x_n)^s\\ &=(f^d(x_1,x_2,\cdots,x_n)+\sum f(x_1,\cdots,d(x_i)\cdots,x_n))^s\\ &=f(x_1,x_2,\cdots,x_n)^t.\end{aligned}$$

Applying Kharchenko technique (1978), we arrive at

$$(f^d(x_1,x_2,\cdots,x_n)+\sum f(x_1,\cdots,y_i\cdots,x_n))^s$$
$$=f(x_1,x_2,\cdots,x_n)^t,$$

for all $x_1,x_2,\cdots,x_n;y_1,y_2,\cdots,y_n\in R$. In particular, $f(y_1,y_2,\cdots,y_n)^s=0$ for all $y_1,y_2,\cdots,y_n\in R$ by setting $x_1=0$. Thus, $f(X_1,X_2,\cdots,X_n)$ vanishes on R by Leron (1975) and so f is central-valued on R. This completes the proof.

Our next goal is to prove the same result is also valid for a semiprime K-algebra.

Theorem 2.3. Let R be a semiprime K-algebra over a commutative ring K with unity and let $f(X_1,X_2,\cdots,X_n)$ be a multilinear polynomial over K. Suppose that d is a derivation on R such that

$$df(x_1,x_2,\cdots,x_n)^s=f(x_1,x_2,\cdots,x_n)^t$$

for all $x_1,x_2,\cdots,x_n\in R$, where s,t are fixed positive integers. Then there exists a central idempotent e of U such that d vanishes identically on eU and $f(X_1,X_2,\cdots,X_n)$ is a central polynomial for $(1-e)U$.

Proof: By a result of Beidar et al. (1996), the derivation d can be uniquely extended to U. Since U and R satisfy the same differential identities (Lee, 1992), then

$$df(x_1,x_2,\cdots,x_n)^s=f(x_1,x_2,\cdots,x_n)^t$$

for all $x_1,x_2,\cdots,x_n\in U$. Let B be the complete boolean algebra of idempotents in C and M be any maximal ideal of B. Since U is an orthogonal complete B-algebra (Chuang, 1994) and MU is a prime ideal of U, which is d-invariant, denote $\overline{U}=U/MU$ and $\overline{d}$ the derivation induced by d on U, i.e., $\overline{d}(\overline{u})=\overline{d(u)}$ for all $u\in U$. Then $\overline{d}$ is satisfied in $\overline{U}$ the same property of d on U. In particular, U is prime and so, by Theorem 2.2, one has $df(x_1,x_2,\cdots,x_n)^s=f(x_1,x_2,\cdots,x_n)^t$ for all $x_1,x_2,\cdots,x_n\in U$. For all maximal ideals M of B we obtain that either $\overline{d}$ is the zero derivation on $\overline{U}$, that is, $d(U)\subseteq MU$, or $f(X_1,X_2,\cdots,X_n)$ is central-valued on U, that is, $[f(x_1,x_2,\cdots,x_n),x]\in MU$ for all $x_1,x_2,\cdots,x_n\in U$. In any case we have

$$[f(x_1,x_2,\cdots,x_n),x]d(U)\subseteq MU,$$

and hence

$$[f(x_1,x_2,\cdots,x_n),x]d(U)\subseteq\bigcap MU=0.$$

Now using the theory of orthogonal completion for semiprime rings (Beidar et al., 1996), there exists a central idempotent e of U such that $U=eU\oplus(1-e)U$ with $d=0$ on eU and $f(X_1,X_2,\cdots,X_n)$ is central-valued on $(1-e)U$. This completes the proof of the theorem.

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