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An efficient method for solving strongly nonlinear oscillators: combination of the multi-step homotopy analysis and spectral method

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Abstract

In this study, a new efficient semi-analytical method is introduced to give approximate solutions of strongly nonlinear oscillators. The proposed method is based on combination of two different methods, the multi-step homotopy analysis method and spectral method, called the multi-step spectral homotopy analysis method (MSHAM). In this method, firstly, we propose a new spectral homotopy analysis method and then, we apply it on smaller subintervals and join the resulting solutions that obtained from new spectral homotopy analysis method at the end points of the subintervals. Several significant strongly nonlinear oscillators are tested with the new scheme to demonstrate its accuracy and easy implementation.

***Keywords**:* Homotopy analysis method (HAM); spectral method; spectral HAM (SHAM); multi-step HAM; nonlinear oscillator; Runge-Kutta method

1. Introduction

Many real problems in science and engineering can be modeled by nonlinear ordinary or partial differential equations on large domains. These kind of problems are used in analyzing the problems arising in biology, medicine, economic, physics, thermodynamics, astrophysics, chemical kinetics, population models, thermal behavior of a spherical cloud of gas, fluid mechanics and many other problems. But, in general, most of them do not have analytical solution or a closed form. Thus, development of new efficient numerical algorithms and semi-analytical methods for finding solutions of these problems is an interesting task for many researchers in physics, engineering, as well as in other disciplines.

These known methods are, for example, Runge-Kutta method (Dormand& Prince, 1980), spectral methods (Civalek, 2007), the δ-expansion method (Karmishin, Zhukov & Kolosov, 1990), the Adomian decomposition method (Adomian, 1994), the homotopy perturbation method (He, 2003) and the variational iteration method (He, 1999).

The study of nonlinear oscillators is of crucial importance in all areas of physics, engineering and applied mathematics. One knows that finding the semi-analytical solution of the strongly nonlinear oscillators is difficult. The backgrounds of the strongly nonlinear oscillators with multitudinous references and useful bibliographies have been given by (Mickens, 1996; Nayfeh and Mook, 1979; Agarwal et al., 2003). Several numerical and semi-analytical methods have been applied to solve strongly nonlinear oscillators problems. Among them are the harmonic balance based methods (Lim et al, 2006; Belendez et al, 2006; Ghosh & Roy, 2007; Momani, 2004), Adomian decomposition method (Heydari et al.2011; El-Wakil & Abdou, 2008) differential transform method (He, 2001), variational iteration method (He, 2001; Zhang et al, 2011), bookkeeping parameter perturbation method (Liu et al, 2013), iterative perturbation method (Abbasbabdy, 2007) and max-min approach (Sedighi et al, 2012). The homotopy analysis method is another way for giving analytical approximations of nonlinear oscillators (Ellahi & Vafai, 2012; Ellahi, 2013).

The homotopy analysis method is proposed by Liao, which is general analytical method for studying approximate solutions of nonlinear differential equations (Boyd, 2000; Canuto et al, 1998; Doha et al, 2012). Unlike previous semi-analytic methods, the homotopy analysis method provides us with great freedom to expand solutions of a given nonlinear problem by different basic functions, namely the rule of solution expression. The series solution of homotopy analysis method contains the controlling convergence parameter c_0.

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This parameter plays very important role in controlling the convergence region and rate of series solution.

Spectral methods are very powerful tools for obtaining the approximate solution of many problems

arising in different fields of science and engineering (Xu & Hesthaven, 2014). Convenience of applying these methods and exponential convergence are two useful properties which have persuaded many authors to use them for solving various types of problems. Spectral methods are a class of important tools for obtaining the numerical solutions of fractional differential equations. They have excellent error properties and they offer exponential rates of convergence for smooth problems (Motsa et al, 2010), these are used based on Chebyshev orthogonal polynomials for obtaining the solution of some different types of differential equations, see, for instance (Clenshaw & Curtis, 1960).

2. Homotopy analysis method for solving a general strongly nonlinear oscillator

In this section, the homotopy analysis method is used to give approximate solution of the general strongly nonlinear oscillator. Let us consider the general nonlinear oscillator equation as follows:

$$u''(t) + \alpha u(t) + F(t, u(t), u'(t), u''(t)) = 0, \quad 0 \le t \le T, \tag{1}$$

subject to the initial conditions

$$u(0) = A, \quad u'(0) = B, \tag{2}$$

where $\alpha \ge 0$ and $F(t, u(t), u'(t), u''(t))$ is a nonlinear function. According to the initial conditions (2), it is natural to select $u_0(t) = A + Bt$ as the initial approximations of u. We define auxiliary linear operator L by

$$L[\tilde{u}(t,q)] = \frac{\partial^2}{\partial t^2}\tilde{u}(t,q) + \alpha\tilde{u}(t,q) \tag{3}$$

with the properties

$$L[C_1\cos(\sqrt{\alpha}t) + C_2\sin(\sqrt{\alpha}t)] = 0,$$

where $q \in [0,1]$ is an embedding parameter, $\tilde{u}(t,q)$ a kind of mapping function for $u(t,q)$ and C_1, C_2 are constants. We define non-linear operators N_1, N_2, N_3 and N_4 in the following forms:

$$N[\tilde{u}(t,q)] = \frac{\partial^2}{\partial t^2}\tilde{u}(t,q) + \alpha\tilde{u}(t,q) + F\left(t, \tilde{u}(t,q), \frac{\partial}{\partial t}\tilde{u}(t,q), \frac{\partial^2}{\partial t^2}\tilde{u}(t,q)\right).$$

Using these operators, we can construct the zero th-order deformation equation as

$$(1-q)L[\tilde{u}(t,q) - u_0(t)] = qc_0 N[\tilde{u}(t,q)], \tag{4}$$

where $c_0 \ne 0$ is an auxiliary parameter. The initial conditions for Eq. (1) are

$$\tilde{u}(0,q) = A, \quad \tilde{u}'(0,q) = B. \tag{5}$$

Obviously, when the parameter q increases from 0 to 1, the solutions $\tilde{u}(t,q)$ varies from $u_0(t)$ to $u(t)$. If this continuous variation is smooth enough, the Maclaurin series with respect to q can be constructed for $\tilde{u}(t,q) = u_0(t) + \sum_{m=1}^{\infty} u_m(t)q^m$, and further, if this series is convergent at $q = 1$, we have

$$u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t),$$

where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \tilde{u}(t,q)}{\partial q^m}\Big|_{q=0}. \tag{6}$$

For the m th-order deformation equations, we differentiate Eqs. (4)-(5) m times with respect to q, divide by $m!$ and then set $q = 0$. The resulting deformation equations are

$$L[u_m(t) - \chi_m u_{m-1}(t)] = c_0 R_m(t), \tag{7}$$

with the following initial conditions

$$u_m(0) = 0, \quad u_m'(0) = 0, \tag{8}$$

where

$$R_m(t) = u''_{m-1} + \alpha u_{m-1} + \frac{\partial^{m-1}}{\partial q^{m-1}} F\left(t, \tilde{u}(t,q), \frac{\partial}{\partial t}\tilde{u}(t,q), \frac{\partial^2}{\partial t^2}\tilde{u}(t,q)\right)\Big|_{q=0}, \tag{9}$$

and

$$\chi_m = \begin{cases} 0 & \mathrm{m} \le 1 \\ 1 & \mathrm{m} > 1 \end{cases} \tag{10}$$

Let u_m^* denote the particular solutions of Eq. (7). From Eq. (4), their general solutions have the following forms:

$$u_m = u_m^* + C_1\cos(\sqrt{\alpha}t) + C_2\sin(\sqrt{\alpha}t), \tag{11}$$

where the coefficients C_1, C_2, are determined by the initial conditions (8). According to the auxiliary linear operator Eq. (3) and Eqs. (7) and (8), the Eq. (11) is reduced to:

$$u_m(t) = \chi_m u_{m-1}(t) + \frac{c_0}{\sqrt{\alpha}}\left(\sin(\sqrt{\alpha}t)\int_0^t R_m(x)\cos(\sqrt{\alpha}x)dx - \cos(\sqrt{\alpha}t)\int_0^t R_m(x)\sin(\sqrt{\alpha}x)dx\right), \quad m = 1,2,\cdots. \tag{12}$$

3. Hybrid spectral-homotopy analysis method

Homotopy analysis method is a capable method for solving nonlinear equation. This method has some limitations, which barricade its application. It is clear that in computation of integrals in the right hand side of (12), three difficulties may arise:
i) The function $R_m(x)$ may be ill-conditioned such that the integration became very complicated.
ii) By increasing m the number of terms of approximate solution may increase so rapidly that the integration become both complicated and time consuming.
iii) As you know, in framework of the standard homotopy analysis method, we are faced with the rule of solution expression and the rule of ergoticity. These rules may make difficulties in computation of integral in the right hand of (12).
In order to increase the rate of convergence and decrease computation and time consumption, various modifications were suggested, of which the spectral homotopy analysis method was presented by Motsa, et. al (Yabushita et al, 2007; Liao, 2010) which is the most important. Spectral homotopy analysis method is a power tool for solving nonlinear differential equations. To overcome the above three difficulties, we propose a new hybrid spectral-homotopy analysis method, which is a combination of homotopy analysis method and Chebyshev pseudo-spectral method. The presented method does not require the solution of any linear and nonlinear system of algebraic equations unlike spectral homotopy analysis method and spectral methods.

3.1. Chebyshev polynomials

The Chebyshev polynomials $\{T_n\}_{n=0}^{+\infty}$, which are eigenfunctions of the singular Sturm-Liouville problem:

$$\left(\sqrt{1-t^2}T_n'(t)\right)' + \frac{n^2}{\sqrt{1-t^2}}T_n(t) = 0,$$

on $[-1,1]$, are defined by the recurrence formula

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t), \quad n = 1,2,\cdots,$$

where $T_0(t) = 1$ and $T_1(t) = t$. Chebyshev polynomials $\{T_n(t)\}_{n=0}^{\infty}$ are orthogonal basis with weight function $w(t) = \frac{1}{\sqrt{1-t^2}}$ and orthogonality property:

$$\int_{-1}^{1} T_n(t)T_m(t)w(t)dt = \frac{\pi}{2}c_n\delta_{n,m},$$

where $c_0 = 2$, $c_n = 1, n \geq 1$ and $\delta_{n,m}$ is the Kronecker delta function. A function $u(t) \in L_w^2(-1,1)$, may be expanded in terms of Chebyshev polynomials as:

$$u(t) = \sum_{j=0}^{\infty} u_j T_j(t),$$

where the coefficients u_j are given by $u_j = \frac{2}{\pi c_j}\int_{-1}^{1} u(t)T_j(t)w(t)dt$.

Then the $u(t)$ can be approximated by a truncated series of Chebyshev polynomials:

$$u^M(t) = \sum_{j=0}^{M} \tilde{u}_j T_j(t), \qquad (13)$$

where the Chebyshev coefficients $\tilde{u}_j$ can be approximated using Cleanshaw-Curtis scheme (Heydari et al, 2015). Then we have the following formulation:

$$\tilde{u}_j = \frac{2(-1)^j}{M\tilde{c}_j}\sum_{i=0}^{M} \frac{1}{\tilde{c}_i}u(t_i)\cos\left(\frac{\pi ij}{M}\right), \quad j = 0,1,2,\cdots,M \quad (14)$$

where $\tilde{c}_0 = \tilde{c}_N = 2$ and $\hat{c}_i = 1$ for $1 \leq i \leq N-1$, and the t_i are the Chebyshev-Gauss-Lobatto points:

$$t_i = -\cos\left(\frac{\pi i}{M}\right), \quad i = 0,1,2,\cdots,M.$$

Now, we define the $B_w^m(-1,1)$ weighted space:

$$B_w^m(-1,1) = \{u \in L_w^2(-1,1) : \partial_t^k \in L_{w^{k-1/2}}^2(-1,1), 0 \leq k \leq m\},$$

where $w^k = (1-t^2)^k$.

Theorem 3.1. *(Liao & Chwang, 1998), p.137. Estimates for the Chebyshev-Gusse-Lobato interpolation error.) Let* $u(t) \in B_w^m(-1,1)$ *and* $u^M(t) = \sum_{j=0}^{M} \tilde{u}_j T_j(t)$, *the following estimates hold*

$$\| \partial_t^l(u(t) - u^M(t)) \|_{w^{l-1/2}} \leq c\sqrt{\frac{(M-m+1)!}{M!}}M^{l-(m+1)/2} \| \partial_t^m u(t) \|_{w^{m-1/2}},$$

for $0 \leq l \leq m$ and $0 \leq m \leq M+1$ where c is a positive constant independent of m, M and $u(t)$.

3.2. The methodology

Let us consider the general nonlinear oscillator Eq. (1) with initial conditions (2). According to HAM, we obtain the iteration formula (12) for problem (1). Under the initial conditions (2), it is natural to select the initial approximation $u_0(t) = A + Bt$. At first, by using iteration formula (12), we have

$$u_1(t) = \frac{c_0}{\sqrt{\alpha}}(\sin(\sqrt{\alpha}t)\int_0^t R_1(x)\cos(\sqrt{\alpha}x)dx -$$

$\cos(\sqrt{\alpha}t)\int_0^t R_1(x)\sin(\sqrt{\alpha}x)dx.$ (15)

From (13) and (14), the function $u_1(t)$ on $[0,T]$ can be approximated as:

$$u_1^M(t)=\sum_{j=0}^{M}\tilde{u}_{1,j}T_j\left(\frac{2}{T}t-1\right), \quad (16)$$

where

$$\tilde{u}_{1,j}=\frac{2(-1)^j}{M\tilde{c}_j}\sum_{i=0}^{M}\frac{1}{\tilde{c}_i}u_1(\tilde{t}_i)\cos\left(\frac{\pi ij}{M}\right),\quad j=0,1,2,\cdots,M, \quad (17)$$

and $\tilde{t}_i=\frac{T}{2}(t_i+1),\quad i=0,1,\cdots,M$. By substituting the grid points $\tilde{t}_i,\quad i=0,1,\cdots,M$ in (15), we can find the unknown coefficients $u_1(\tilde{t}_i),\ i=0,1,\cdots,M$, as

$$u_1(\tilde{t}_i)=\frac{c_0}{\sqrt{\alpha}}(\sin(\sqrt{\alpha}\tilde{t}_i)\int_0^{\tilde{t}_i}R_1(x)\cos(\sqrt{\alpha}x)dx-\cos(\sqrt{\alpha}\tilde{t}_i)\int_0^{\tilde{t}_i}R_1(x)\sin(\sqrt{\alpha}x)dx. \quad (18)$$

By applying a quadrature rule, we can approximate the definite integrals in the right hand of (18). In this paper, we use the Gauss-Legendre-Lobatto rule to approximate the definite integrals in the right hand of (18). In a similar way, the function $u_2(t)$ on $[0,T]$ can be approximated as

$$u_2^M(t)=\sum_{j=0}^{M}\tilde{u}_{2,j}T_j\left(\frac{2}{T}t-1\right), \quad (19)$$

where

$$\tilde{u}_{2,j}=\frac{2(-1)^j}{M\tilde{c}_j}\sum_{i=0}^{M}\frac{1}{\tilde{c}_i}u_2(\tilde{t}_i)\cos\left(\frac{\pi ij}{M}\right),\quad j=0,1,2,\cdots,M. \quad (20)$$

By substituting the grid points $\tilde{t}_i,\quad i=0,1,\cdots,M$ in (15), we can find the unknown coefficients $u_2(\tilde{t}_i),\quad i=0,1,\cdots,M$, as

$$u_2(\tilde{t}_i)=u_1^M(\tilde{t}_i)+\frac{c_0}{\sqrt{\alpha}}(\sin(\sqrt{\alpha}\tilde{t}_i)\int_0^{\tilde{t}_i}R_2(x)\cos(\sqrt{\alpha}x)dx-\cos(\sqrt{\alpha}\tilde{t}_i)\int_0^{\tilde{t}_i}R_2(x)\sin(\sqrt{\alpha}x)dx. \quad (21)$$

Similarly, the definite integrals in the right hand of (21) are approximated by the Gauss-Legendre-Lobatto rule. Generally, for $m\geq 2$, according to the above method, we can obtain the approximation of $u_m(t)$ in $[0,T]$ as follows:

$$u_m^M(t)=\sum_{j=0}^{M}\tilde{u}_{m,j}T_j\left(\frac{2}{T}t-1\right), \quad (22)$$

where

$$\tilde{u}_{m,j}=\frac{2(-1)^j}{M\tilde{c}_j}\sum_{i=0}^{M}\frac{1}{\tilde{c}_i}u_m(\tilde{t}_i)\cos\left(\frac{\pi ij}{M}\right),\quad j=0,1,2,\cdots,M, \quad (23)$$

and

$$u_m(\tilde{t}_i)=u_{m-1}^M(\tilde{t}_i)+\frac{c_0}{\sqrt{\alpha}}(\sin(\sqrt{\alpha}\tilde{t}_i)\int_0^{\tilde{t}_i}R_m(x)\cos(\sqrt{\alpha}x)dx-\cos(\sqrt{\alpha}\tilde{t}_i)\int_0^{\tilde{t}_i}R_m(x)\sin(\sqrt{\alpha}x)dx. \quad (24)$$

Also, the definite integrals in the right hand of (24) are approximated by the Gauss-Legendre-Lobatto rule. The nth-order approximation of $u(t)$ is given by

$$U_{n,M}(t)=\sum_{m=0}^{n}u_m^M(t). \quad (25)$$

3.3. Determination of the controlling convergence parameter

The series solution (25) contains the controlling convergence parameter c_0. This parameter plays a very important role in controlling the convergence region and rate of series solution which is one of the important advantages of homotopy analysis method with respect to other semi analytical methods. So far, several techniques have been applied for determining proper value of c_0. Liao, suggested c_0-curve to determine the valid region of c_0 and to accelerate the convergence of series solution. So, this technique can not determine the optimal value of c_0. In 2007, Yabushita et al. (Clenshaw & Curtis, 1960) obtained an optimal value of c_0 by minimizing the square of residual error

$$\Delta(c_0)=\int_0^T\left(U''_{n,M}(t)+\alpha U_{n,M}(t)+F\left(t,U_{n,M}(t),U'_{n,M}(t),U''_{n,M}(t)\right)\right)^2dt. \quad (26)$$

This method is named the "optimal homotopy analysis method". Obviously, if the order n of approximation increases, the computing of Eq. (26) is difficult. Liao in (Shen et al, 2011) suggested a method for finding optimal value of c_0 by minimizing the discretized Eq. (26) by numerical quadrature rules like Trapezoidal or Clenshaw-Curtis.

4. Multi-step spectral homotopy analysis method

It is clear that, the hybrid spectral homotopy analysis method is ideally suited for solving differential equations whose solutions do not change rapidly or oscillate over small parts of the domain of the governing problem. For solving strongly nonlinear oscillators on large domains, we introduce the main idea of the multi-step spectral homotopy analysis method, to overcome the proposed problem. We first divide the interval $[0,T]$ into subintervals $\Omega_r=[T_{r-1},T_r]$ where

$r = 1,2,\cdots,m$ and $\Delta = T_r - T_{r-1}$. Moreover, we define the linear mappings $\psi_r : \Omega_r \to [-1,1]$ by

$$\psi_r(t) = \frac{2(t-T_{r-1})}{\Delta_r} - 1, r = 1,2,\cdots,m, \quad (27)$$

and choose grid points $\tilde{t}_i^r$ as:

$$\tilde{t}_i^r = \psi_r^{-1}(t_i) = \frac{\Delta_r}{2}(t_i+1) + T_{r-1}, r = 1,2,\cdots,m, i = 0,1,\cdots,M, \quad (28)$$

where $\psi_r^{-1}(t)$ is inverse map of $\psi_r(t)$. On $\Omega_1 = [T_0, T_1]$, let $u_{1,0}(t) = u(T_0) + u'(T_0)(t - T_0) = A + Bt$ and for $m \ge 1$

$$u_{1,m}(t) \simeq u_{1,m}^M(t) = \sum_{j=0}^M \tilde{u}_{mj}^{(1)} T_j(\psi_1(t)), \quad (29)$$

where

$$\tilde{u}_{mj}^{(1)} = \frac{2(-1)^j}{M\tilde{c}_j} \sum_{i=0}^M \frac{1}{\tilde{c}_i} u_{1,m}(\tilde{t}_i^1) \cos\left(\frac{\pi i j}{M}\right), j = 0,1,\cdots,M, \quad (30)$$

$$u_{1,m}(\tilde{t}_i^1) \simeq \chi_m u_{1,m-1}(\tilde{t}_i^1) + \frac{c_0^1(\tilde{t}_i^1 - T_0)}{2\sqrt{\alpha}} (\sin(\sqrt{\alpha}\tilde{t}_i^1) \int_{-1}^{1} R_m(x)\cos(\sqrt{\alpha}x)dx - \cos(\sqrt{\alpha}\tilde{t}_i^1) \int_{-1}^{1} R_m(x)\sin(\sqrt{\alpha}x)dx, \quad ,m = 1,2,\cdots,n, \quad (31)$$

where, the definite integrals in the right hand of (31) are approximated by the Gauss-Legendre-Lobatto rule. Then we can obtain the n-order approximation $U_{1,n}(t) = \sum_{m=0}^n u_{1,m}(t)$ on Ω_1. Then, we obtain an optimal value of c_0^1 by minimizing the square residual error

$$\Delta(c_0^1) = \int_{\Omega_1} \left(U_{1,n}''(t) + \alpha U_{1,n}(t) + F\left(t, U_{1,n}(t), U_{1,n}'(t), U_{1,n}''(t)\right)\right)^2 dt. \quad (32)$$

On $\Omega_2 = [T_1, T_2]$, let $u_{2,0}(t) = U_{1,n}(T_1) + U_{1,n}'(T_1)(t - T_1)$ and for $m \ge 1$

$$u_{2,m}(t) \simeq u_{2,m}^M(t) = \sum_{j=0}^M \tilde{u}_{mj}^{(2)} T_j(\psi_2(t)), \quad (33)$$

where

$$\tilde{u}_{mj}^{(2)} = \frac{2(-1)^j}{M\tilde{c}_j} \sum_{i=0}^M \frac{1}{\tilde{c}_i} u_{2,m}(\tilde{t}_i^2) \cos\left(\frac{\pi i j}{M}\right), j = 0,1,\cdots,M, \quad (34)$$

$$u_{2,m}(\tilde{t}_i^2 \simeq \chi_m u_{2,m-1}(\tilde{t}_i^2) + \frac{c_0^2(\tilde{t}_i^2 - T_0)}{2\sqrt{\alpha}} (\sin(\sqrt{\alpha}\tilde{t}_i^2) \int_{-1}^{1} R_m(x)\cos(\sqrt{\alpha}x)dx - \cos(\sqrt{\alpha}\tilde{t}_i^2) \int_{-1}^{1} R_m(x)\sin(\sqrt{\alpha}x)dx, \quad ,m = 1,2,\cdots,n, \quad (35)$$

similarly, the definite integrals in the right hand of (35) are approximated by the Gauss-Legendre-Lobatto rule. Then we can obtain the n-order approximation $U_{2,n}(t) = \sum_{m=0}^n u_{2,m}(t)$ on Ω_2. Again, we obtain an optimal value of c_0^2 by minimizing the square residual error

$$\Delta(c_0^2) = \int_{\Omega_2} \left(U_{2,n}''(t) + \alpha U_{2,n}(t) + F\left(t, U_{2,n}(t), U_{2,n}'(t), U_{2,n}''(t)\right)\right)^2 dt. \quad (36)$$

In a similar way, we can obtain the n-order approximation $U_{s,n}(t) = \sum_{m=0}^n u_{2,m}(t)$ on $\Omega_s[T_{s-1}, T_s], s = 3,4,\cdots,m$. Finally, the approximate solution $u(t)$ in the entire interval $[0,T]$ is given by

$$u(t) \simeq U_n^M(t) = \begin{cases} U_{1,n}^M(t), & t \in \Omega_1, .2cm \\ U_{2,n}^M(t), & t \in \Omega_2, .2cm \\ \vdots & .2cm \\ U_{m,n}^M(t), & t \in \Omega_m. \end{cases} \quad (37)$$

5. Numerical results

In order to verify the efficiency and accuracy of the multi-step spectral homotopy analysis method (MSHAM) as an appropriate tool for solving strongly nonlinear oscillators, three examples are examined in this section. To check the accuracy of the presented method, a comparison is made with the numerical solutions obtained by using the 4th order Runge-Kutta method (RK4). The description of the physical problem and the values of parameters closely follows that of Heydari et al. (Heydari, et al., 2015).

Example 5.1. ***(Rotational pendulum system)***
Consider a simple pendulum system attached to a rotating base about its neutral axis for large amplitudes of oscillation as shown in Fig. 5.1. The equation of motion for this system is (Liao & Chwang, 1998)

$$\ddot{u} + \omega_0^2(1 - \Lambda\cos u)\sin u = 0, u(0) = A \in (0°, 180°), \dot{u}(0) = 0, \quad (38)$$

where u is the angular displacement in relation to the temporal coordinate t, A is the initial amplitude of oscillation, $\omega_0^2 = g/l$ and $\Lambda = \Omega^2 g/l$ where g, l and Ω are the acceleration of gravity, the length of weightless rod and the constant angular velocity of revolution, respectively. The range of the positive constant Λ is assumed to be $0 < \Lambda < 1$(Liao & Chwang, 1998). The time range studied in this example is [0,10]. In order to assess the validity and accuracy of the obtained results, we compare the approximate results given by MSHAM ($\Delta_r =$

$\Delta = 0.5, M = 15, n = 5)$ with numerical solution obtained by 4th order Runge-Kutta method $(\Delta t = 0.001)$ in the following three cases:

Case1. $(A, \Lambda, \omega_0) = (170°, 0.9, \sqrt{2})$,

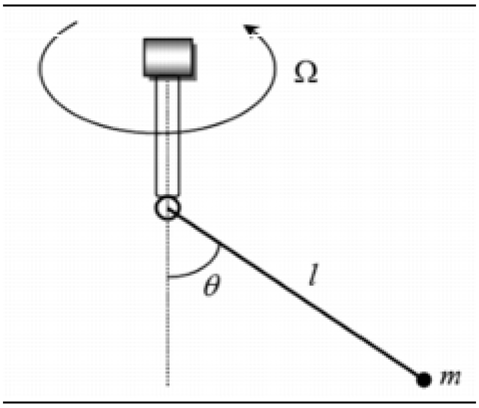

Fig. 1. Rotational pendulum system

Case 2.$(A, \Lambda, \omega_0) = (140°, 0.5, \sqrt{1.5})$,

Case 3.$(A, \Lambda, \omega_0) = (110°, 0.1, 1)$.

Figures 5.1(a-c) and 5.1(d-f) show the displacement and phase diagram, respectively, in the above cases. It can be seen from Fig. 5.1 that the solutions obtained by the proposed procedure are in good agreement with the RK4 based solutions.

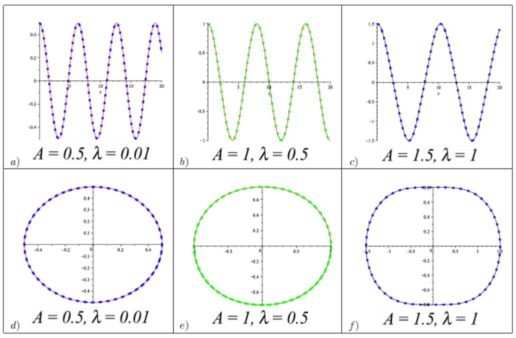

Fig. 2. Plots of displacement u versus time t (a-c) and phase plane (d-f). Solid line: MSHAM; Solid circle: RK4

Example 5.2. ***(Oscillations of a mass attached to a stretched elastic wire)***

Consider the motion of a particle of mass m attached to the center of a stretched elastic wire (Lai et al., 2011) of stiffness coefficient which is equals to k. The length of elastic wire when a force is applied to it is 2a. We assume that the movement of particle is one-dimensional and this is constrained to move only in the horizontal x direction. The governing differential equation of motion and the associated initial conditions for a mass attached to a stretched elastic wire (Lai, et al., 2011), shown in Fig. 5.2 are:

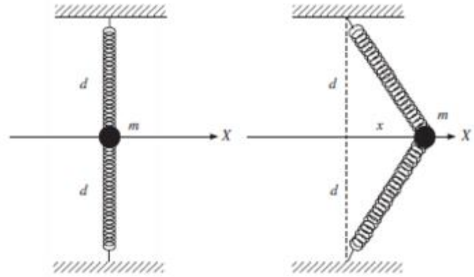

Fig. 3. Mass attached to a stretched elastic wire

$$m\frac{d^2x}{d\tau^2} + 2kx - \frac{2kax}{\sqrt{d^2+x^2}} = 0, x(0) = B, \frac{dx}{d\tau}(0) = 0. \quad (39)$$

Two dimensionless variables u and t can be constructed as follows:

$$u = \frac{x}{d}, t = \sqrt{\frac{2k}{m}}\tau. \quad (40)$$

Substituting these dimensionless variables into (39) gives

$$\ddot{u} + u - \frac{\lambda u}{\sqrt{1+u^2}} = 0, u(0) = A, \dot{u}(0) = 0, \quad (41)$$

where $A = \frac{B}{d}, \lambda = \frac{a}{d}$ and $0 < \lambda \leq 1$. In order to illustrate the remarkable accuracy of the presented method, we compare the approximate results given by MSHAM $(\Delta_r = \Delta = 1, M = 15, n = 5)$ with numerical solution obtained by 4th order Runge-Kutta method$(\Delta t = 0.001)$ on interval $[0,20]$ in the cases of $(A, \lambda) = (0.5, 0.01), (1, 0.5)$ and $(1.5, 1)$. Figs. 5.2(a-c) and 5.2(d-f) show the displacement and phase diagram, respectively, in the above cases. We can clearly observe from Fig. 5.2 that the solutions obtained by the proposed method are in good agreement with the RK4 based solutions.

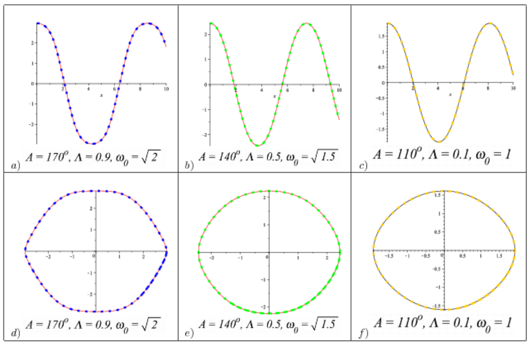

Fig. 4. Plots of displacement u versus time t (a-c) and phase plane (d-f). Solid line: MSHAM; Solid circle: RK4

Example 5.3. ***(Van der Pol equation)***

In this example, we study the Van der Pol equation and solve it by means of MSHAM. This equation is a mathematical model of self-sustained oscillations of a triode electric circuit (Van der Pol, 1926). It was introduced in the 1920s (Van der Pol, 1920) by the physicist Balthasar Van der Pol. The Van der Pol oscillator is given by a self-excited differential equation and its standard form is as shown below

$$\ddot{u} - \mu(1 - u^2)\dot{u} + u = 0, u(0) = A, \dot{u}(0) = B, \qquad (42)$$

where $\mu \geq 0$ is a scalar parameter indicating the degree of nonlinearity and the strength of the damping. If $\mu = 0$, the equation reduces to the equation of simple harmonic motion $\ddot{u} + u = 0$. When $u > 1$, $-\mu(1 - u^2)$ is positive and the system behaves as a damped (energy dissipating) system, and when $u < 1$, $-\mu(1 - u^2)$ is negative and the system behaves as a self excited (energy absorbing) system. To demonstrate the validity and applicability of the MSHAM, we compare the approximate results given by presented method in the three cases $\mu = 0.1, \mu = 1 and \mu = 10 (\Delta_r = \Delta = 0.5, M = 15, n = 7)$ with numerical solution obtained by 4th order Runge-Kutta method $(\Delta t = 0.001)$ on interval $[0,30]$. In all cases, we take $A = B = 0.01$. Figs. 5.3(a-c) and 5.3(d-f) show the displacement and phase diagram, respectively, in the above cases. From Fig. 5.3, we find that MSHAM results are close to the numerical solutions obtained using RK4.

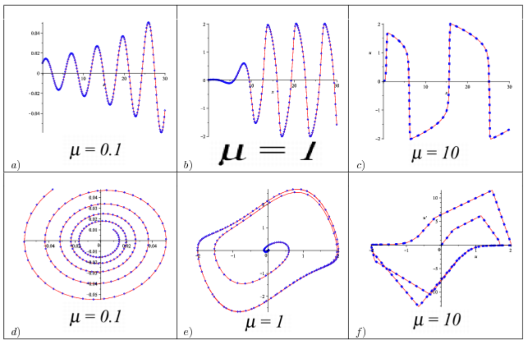

Fig. 5. Plots of displacement u versus time t (a-c) and phase plane (d-f). Solid line: MSHAM; Solid circle: RK4

6. Conclusion

In this work, a new modification of the multi-step homotopy analysis method is presented for solving strongly nonlinear oscillators. This method, named multi-step spectral homotopy analysis method (MSHAM), was successfully used to solve the rotational pendulum system, oscillations of a mass attached to a stretched elastic wire and Van der Pol equation.By checking and probing the modus operandi used in MSHAM, HAM and spectral method, we obtain the following advantages:

1. The new approach overcome the difficulties (***i, ii*** and iii) that arise in calculating complicated and time consuming integrals and terms that are not needed in the standard HAM.
2. The introduced technique does not require the solution of any linear or nonlinear system of equations unlike spectral method.
3. The obtained results showed that the proposed method can solve the rotational pendulum system, oscillations of a mass attached to a stretched elastic wire and Van der Pol equation effectively and the comparison showed that the proposed method is in good agreement with the numerical results obtained using RK4.

In future work, other Jacobi polynomials can be used instead of Chebyshev polynomial in MSHAM.

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