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Some notes on the topological centers of module actions

K. Haghnejad Azar[1], A. Bodaghi[2] and A. Jabbari[3]*

[1]*Department of Mathematics, University of Mohaghegh Ardabili, Ardabil, Iran*
[2]*Young Researchers and Elite Club, Islamshahr Branch, Islamic Azad University, Islamshahr, Iran*
[3]*Young Researchers and Elite Club, Ardabil Branch, Islamic Azad University, Ardabil, Iran*
E-mail: Jabbari_al@yahoo.com

Abstract

In this paper, we extend some propositions of Banach algebras into module actions and establish the relationships between topological centers of module actions. We introduce some new concepts as Lw^*w-property and Rw^*w-property for Banach modules and obtain some conclusions in the topological center of module actions and Arens regularity of Banach algebras.

Keywords: Arens regularity; bilinear mappings; factorization; module action; second dual; topological center

1. Introduction

Let A be a Banach algebra. It is well-known that the second dual A^{**} of A endowed with both Arens multiplications is a Banach algebra (Arens, 1951). The constructions of the two Arens multiplications in A^{**} lead us to definition of topological centers for A^{**} with respect to both Arens multiplications.

The topological centers of Banach algebras, module actions and applications of them have been introduced and discussed by many authors in (Baker et al., 1998; Dales, 2000; Duncan and Namioka, 1988; Eshaghi Gordji and Filali, 2007; Esslamzadeh, 2000; Lau and Losert, 1988). In the current work, we introduce some definitions of the topological centers of actions and investigate the relation between these notations.

Let A be a Banach algebra. We say that a net $(e_\alpha)_{\alpha\in I}$ in A is a left approximate identity $(=LAI)$ [resp. right approximate identity $(=RAI)$] if, for each $a\in A$, $e_\alpha a\to a$ [resp. $ae_\alpha\to a$]. For $a\in A$ and $a'\in A^*$, we denote by $a'a$ and aa' respectively, the functionals on A^* defined by

$$\langle a'a,b\rangle=\langle a',ab\rangle=a'(ab)$$

and

$$\langle aa',b\rangle=\langle a',ba\rangle=a'(ba)$$

for all $b\in A$. The Banach algebra A is embedded in its second dual via the identification $\langle a,a'\rangle$ with $\langle a',a\rangle$ for every $a\in A$ and $a'\in A^*$. We denote the set

$$\{a'a: a\in A,\ a'\in A^*\}$$

and

$$\{aa': a\in A,\ a'\in A^*\}$$

by A^*A and AA^*, respectively. Clearly these two sets are subsets of A^*. Suppose that A has a BAI. If the equality $A^*A=A^*$, $(AA^*=A^*)$ holds, then we say that A^* factors on the left (right) and if equalities $A^*A=AA^*=A^*$ hold, then we say that A^* factors on both sides.

Let X, Y and Z normed spaces and

$$m: X\times Y\to Z$$

be a bounded bilinear mapping. Arens in (Arens, 1951) offered two natural extensions m^{***} and m^{t***t} of m from $X^{**}\times Y^{**}$ into Z^{**} as follows:

1. $m^*: Z^*\times X\to Y^*$, given by

*Corresponding author
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$$\langle m^*(z',x),y\rangle = \langle z',m(x,y)\rangle$$

where $x\in X$, $y\in Y$, $z'\in Z^*$.

2. $m^{**}:Y^{**}\times Z^*\to X^*$, given by

$$\langle m^{**}(y'',z'),x\rangle = \langle y'',m^*(z',x)\rangle$$

where $x\in X$, $y''\in Y^{**}$, $z'\in Z^*$.

3. $m^{***}:X^{**}\times Y^{**}\to Z^{**}$, given by

$$\langle m^{***}(x'',y''),z'\rangle = \langle x'',m^{**}(y'',z')\rangle$$

where $x''\in X^{**}$, $y''\in Y^{**}$, $z'\in Z^*$.

The mapping m^{***} is the unique extension of m such that $x''\to m^{***}(x'',y'')$ from X^{**} into Z^{**} is weak*-weak*-continuous for every $y''\in Y^{**}$, but the mapping $y''\to m^{***}(x'',y'')$ is not in general weak*-weak*-continuous from Y^{**} into Z^{**} unless $x''\in X$. Hence the first topological center of m may be defined as:

$$Z_1(m)=\{x''\in X^{**}:\ y''\to m^{***}(x'',y'') \text{ is weak}^*-\text{weak}^*-\text{continuous}\}.$$

Now let $m^t:Y\times X\to Z$ be the transpose of m defined by $m^t(y,x)=m(x,y)$ for every $x\in X$ and $y\in Y$. Then m^t is a continuous bilinear map from $Y\times X$ to Z, and so it may be extended as above to

$$m^{t***}:Y^{**}\times X^{**}\to Z^{**}.$$

The mapping

$$m^{t***t}:X^{**}\times Y^{**}\to Z^{**}$$

in general is not equal to m^{***} (see Arens, 1951). However, m is called Arens regular when

$$m^{***}=m^{t***t}.$$

The mapping $y''\to m^{t***t}(x'',y'')$ is weak*-weak*-continuous for every $y''\in Y^{**}$, but the mapping $x''\to m^{t***t}(x'',y'')$ from X^{**} into Z^{**} is not in general weak*-weak*-continuous for every $y''\in Y^{**}$. So we define the second topological center of m as

$$Z_2(m)=\{y''\in Y^{**}:\ x''\to m^{t***t}(x'',y'') \text{ is weak}^*\text{-weak}^*\text{-continuous}\}.$$

It is clear that m is Arens regular if and only if $Z_1(m)=X^{**}$ or $Z_2(m)=Y^{**}$. Arens regularity of m is equivalent to the following

$$\lim_i\lim_j\langle z',m(x_i,y_j)\rangle=\lim_j\lim_i\langle z',m(x_i,y_j)\rangle,$$

whenever both limits exist for all bounded sequences $(x_i)_i\subseteq X$, $(y_i)_i\subseteq Y$ and $z'\in Z^*$ (for more details see (Arikan, 1982; Dales, 2000; Lau and Ülger, 1996).

The regularity of a normed algebra A is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. Let a'' and b'' be elements of A^{**}. By Goldstein's Theorem (Dales, 2000) there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in A such that

$$a''=weak^*-\lim_\alpha a_\alpha \text{ and } b''=weak^*-\lim_\beta b_\beta.$$

So it is easy to see that for all $a'\in A^*$,

$$\lim_\alpha\lim_\beta\langle a',m(a_\alpha,b_\beta)\rangle=\langle a''b'',a'\rangle$$

and

$$\lim_\beta\lim_\alpha\langle a',m(a_\alpha,b_\beta)\rangle=\langle a''ob'',a'\rangle,$$

where $a''b''$ and $a''ob''$ are the first and second Arens products of A^{**}, respectively (see again Dales, 2000; Eshaghi Gordji and Filali, 2007; Lau and Ülger, 1996).

The mapping m is left strongly Arens irregular if $Z_1(m)=X$ and m is right strongly Arens irregular if $Z_2(m)=Y$.

This paper is organized as follows:

a) In section two, for a Banach A-bimodule B, we restate some topological centers of module actions of A on B ($Z_{A^{**}}(B^{**})$, etc) and show that

1. $a''\in Z_{B^{**}}(A^{**})$ if and only if $\pi_\ell^{****}(b',a'')\in B^*$ for all $b'\in B^*$.

2. $F\in Z_{B^{**}}((A^*A)^*)$ if and only if $\pi_\ell^{****}(g,F)\in B^*$ for all $g\in B^*$.

3. $G\in Z_{(A^*A)^*}(B^{**})$ if and only if $\pi_r^{****}(g,G)\in A^*A$ for all $g\in B^*$.

4. Let B have a BAI $(e_\alpha)_\alpha\subseteq A$ such that $e_\alpha\xrightarrow{w^*}e''$.

Then if $Z^t_{e^{**}}(B^{**})=B^{**}$ [resp. $Z_{e^{**}}(B^{**})=B^{**}$] and B^* factors on the left [resp. right], but not on the right [resp. left], then $Z_{B^{**}}(A^{**})\neq Z^t_{B^{**}}(A^{**})$.

5. $B^*A\subseteq wap_\ell(B)$ if and only if

$$AA^{**}\subseteq Z_{B^{**}}(A^{**}).$$

6. Let $b'\in B^*$. Then $b'\in wap_\ell(B)$ if and only if the adjoint of the mapping $\pi_\ell^*(b',\,):A\to B^*$ is weak*-weak-continuous.

b) In section three, for a Banach A-bimodule B, we define Left weak*-weak-property [Lw^*w-property] and Right weak*-weak-property [Rw^*w-property] for Banach algebra A and we show that

1. If $A^{**}=a_0A^{**}$ [resp. $A^{**}=A^{**}a_0$] for some $a_0\in A$ and a_0 has Rw^*w-property [resp. Lw^*w-property], then $Z_{B^{**}}(A^{**})=A^{**}$.

2. If $B^{**}=a_0B^{**}$ [resp. $B^{**}=B^{**}a_0$] for some $a_0\in A$ and a_0 has Rw^*w-property [resp. Lw^*w-property] with respect to B, then $Z_{A^{**}}(B^{**})=B^{**}$.

3. If B^* factors on the left [resp. right] with respect to A and A has Rw^*w-property [resp. Lw^*w-property], then $Z_{B^{**}}(A^{**})=A^{**}$.

4. If B^* factors on the left [resp. right] with respect to A and A has Rw^*w-property [resp. Lw^*w-property] with respect B, then

$$Z_{A^{**}}(B^{**})=B^{**}.$$

5. If $a_0\in A$ has Rw^*w- property with respect to B, then $a_0A^{**}\subseteq Z_{B^{**}}(A^{**})$ and $a_0B^*\subseteq wap_\ell(B)$.

6. Assume that $AB^*\subseteq wap_\ell B$. If B^* has strong factors on the left [resp. right], then A has Lw^*w-property [resp. Rw^*w-property] with respect to B.

7. Assume that $AB^*\subseteq wap_\ell B$. If B^* has strong factors on the left [resp. right], then A has Lw^*w-property [resp. Rw^*w-property] with respect to B.

2. The topological centers of module actions

Let B be a Banach A-bimodule, and let $\pi_\ell:A\times B\to B$ and $\pi_r:B\times A\to B$ be the left and right module actions of A on B. Then B^{**} is a Banach A^{**}-bimodule with module actions

$$\pi_\ell^{***}:A^{**}\times B^{**}\to B^{**}$$

and

$$\pi_r^{***}:B^{**}\times A^{**}\to B^{**}.$$

Similarly, B^{**} is a Banach A^{**}-bimodule with module actions

$$\pi_\ell^{t***t}:A^{**}\times B^{**}\to B^{**}$$

and

$$\pi_r^{t***t}:B^{**}\times A^{**}\to B^{**}.$$

We may therefore define the topological centers of the right and left module actions of A on B as follows:

$$\begin{aligned}Z_{A^{**}}(B^{**})=Z(\pi_r)=\{b''\in B^{**}:\ &\text{the map } a''\to\pi_r^{***}(b'',a''):A^{**}\to B^{**}\\ &\text{is weak}^*-\text{weak}^*-\text{continuous}\},\\
Z_{B^{**}}(A^{**})=Z(\pi_\ell)=\{a''\in A^{**}:\ &\text{the map } b''\to\pi_\ell^{***}(a'',b''):B^{**}\to B^{**}\\ &\text{is weak}^*-\text{weak}^*-\text{continuous}\},\\
Z^t_{A^{**}}(B^{**})=Z(\pi_\ell^t)=\{b''\in B^{**}:\ &\text{the map } a''\to\pi_\ell^{t***}(b'',a''):A^{**}\to B^{**}\\ &\text{is weak}^*-\text{weak}^*-\text{continuous}\},\\
Z^t_{B^{**}}(A^{**})=Z(\pi_r^t)=\{a''\in A^{**}:\ &\text{the map } b''\to\pi_r^{t***}(a'',b''):B^{**}\to B^{**}\\ &\text{is weak}^*-\text{weak}^*-\text{continuous}\}.\end{aligned}$$

We note also that if B is a left (resp. right) Banach A-module and $\pi_\ell:A\times B\to B$ (resp. $\pi_r:B\times A\to B$) is left (resp. right) module action of A on B, then B^* is a right (resp. left) Banach A-module.

We write $ab=\pi_\ell(a,b)$, $ba=\pi_r(b,a)$, $\pi_\ell(a_1a_2,b)=\pi_\ell(a_1,a_2b)$, $\pi_r(b,a_1a_2)=\pi_r(ba_1,a_2)$, $\pi_\ell^*(a_1b',a_2)=\pi_\ell^*(b',a_2a_1)$, $\pi_r^*(b'a,b)=\pi_r^*(b',ab)$, for all $a_1,a_2,a\in A$, $b\in B$ and $b'\in B^*$ when there is no confusion.

Theorem 2.1. We have the following assertions:
(i) Assume that B is a Banach left A-module. Then, $a'' \in Z_{B^{**}}(A^{**})$ if and only if $\pi_\ell^{*****}(b', a'') \in B^*$ for all $b' \in B^*$;
(ii) Assume that B is a Banach right A-module. Then, $b'' \in Z_{A^{**}}(B^{**})$ if and only if $\pi_r^{*****}(b', b'') \in A^*$ for all $b' \in B^*$.

Proof: (i) Let $b'' \in B^{**}$. Then, for every $a'' \in Z_{B^{**}}(A^{**})$, we have

$$\langle \pi_\ell^{*****}(b', a''), b'' \rangle = \langle b', \pi_\ell^{***}(a'', b'') \rangle = \langle \pi_\ell^{***}(a'', b''), b' \rangle$$
$$= \langle \pi_\ell^{t***t}(a'', b''), b' \rangle$$
$$= \langle \pi_\ell^{t***}(b'', a''), b' \rangle$$
$$= \langle b'', \pi_\ell^{t**}(a'', b') \rangle.$$

The above equalities show that

$$\pi_\ell^{*****}(b', a'') = \pi_\ell^{t**}(a'', b') \in B^*.$$

Conversely, let $a'' \in A^{**}$ and let $\pi_\ell^{*****}(a'', b') \in B^*$ for all $b' \in B^*$. Let $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \xrightarrow{w^*} b''$. Then for every $b' \in B^*$, we have

$$\langle \pi_\ell^{***}(a'', b''_\alpha), b' \rangle = \langle b', \pi_\ell^{***}(a'', b''_\alpha) \rangle$$
$$= \langle \pi_\ell^{*****}(b', a''), b''_\alpha \rangle$$
$$= \langle b''_\alpha, \pi_\ell^{*****}(b', a'') \rangle$$
$$\rightarrow \langle b'', \pi_\ell^{*****}(b', a'') \rangle$$
$$= \langle \pi_\ell^{*****}(b', a''), b'' \rangle$$
$$= \langle \pi_\ell^{***}(a'', b''), b' \rangle.$$

Consequently $a'' \in Z_{B^{**}}(A^{**})$.
(ii) Proof is similar to (i).

Theorem 2.2. Let B be a Banach A-bimodule. Then we have the following assertions:
(i) $F \in Z_{B^{**}}((A^*A)^*)$ if and only if $\pi_\ell^{*****}(g, F) \in B^*$ for all $g \in B^*$;
(ii) $G \in Z_{(A^*A)^*}(B^{**})$ if and only if $\pi_r^{*****}(g, G) \in A^*A$ for all $g \in B^*$.

Proof: (i) Let $F \in Z_{B^{**}}((A^*A)^*)$ and $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \xrightarrow{w^*} b''$. Then for all $g \in B^*$, we have

$$\langle \pi_\ell^{****}(g, F), b''_\alpha \rangle = \langle g, \pi_\ell^{***}(F, b''_\alpha) \rangle = \langle \pi_\ell^{***}(F, b''_\alpha), g \rangle$$
$$\rightarrow \langle \pi_\ell^{***}(F, b''), g \rangle = \langle \pi_\ell^{****}(g, F), b'' \rangle.$$

Thus, we conclude that

$$\pi_\ell^{****}(g, F) \in (B^{**}, weak^*)^* = B^*.$$

For the converse, let $\pi_\ell^{*****}(g, F) \in B^*$ for $F \in (A^*A)^*$ and $g \in B^*$. Assume that $b'' \in B^{**}$ and $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \xrightarrow{w^*} b''$. Then

$$\langle \pi_\ell^{***}(F, b''_\alpha), g \rangle = \langle g, \pi_\ell^{***}(F, b''_\alpha) \rangle$$
$$= \langle \pi_\ell^{****}(g, F), b''_\alpha \rangle$$
$$= \langle b''_\alpha, \pi_\ell^{*****}(g, F) \rangle$$
$$\rightarrow \langle b'', \pi_\ell^{*****}(g, F) \rangle$$
$$= \langle \pi_\ell^{*****}(g, F), b'' \rangle$$
$$= \langle \pi_\ell^{***}(F, b''), g \rangle.$$

It follows that $F \in Z_{B^{**}}((A^*A)^*)$.
(ii) Proof is similar to (i).

Definition 2.3. Let B be a Banach space and be a Banach A-module. Then
1. B has strongly double limit property (SDLP) from right (left) at $a \in A$ if for each bounded net (b_α) in B and each bounded net (b'_β) in B^*,

$$\lim_\alpha \lim_\beta \langle ab'_\beta, b_\alpha \rangle = \lim_\beta \lim_\alpha \langle ab'_\beta, b_\alpha \rangle$$
$$(\lim_\alpha \lim_\beta \langle b'_\beta a, b_\alpha \rangle = \lim_\beta \lim_\alpha \langle b'_\beta a, b_\alpha \rangle),$$

whenever both iterated limits exist;
2. B has SDLP if for each bounded net (b_α) in B and each bounded net (b'_β) in B^*,

$$\lim_\alpha \lim_\beta \langle b'_\beta, b_\alpha \rangle = \lim_\beta \lim_\alpha \langle b'_\beta, b_\alpha \rangle,$$

whenever both iterated limits exist.

The definition of SDLP for Banach algebras has been introduced by Medghalchi and Yazdanpanah in (Medghalchi and Yazdanpanah, 2005). They showed that every reflexive Banach algebra has (SDLP). Obviously, If B has SDLP, then it has SDLP from left and right in a, for all $a \in A$.

Theorem 2.4. We have the following statments:
(i) If A^* has SDLP, then $Z_{A^{**}}(B^{**}) = B^{**}$;

(ii) If B^* has SDLP, then $Z_{B^{**}}(A^{**}) = A^{**}$.

Proof: (i) Let $b'' \in B^{**}$ and take a bounded net $(b_\beta) \subseteq A^{**}$ such that $b_\beta \xrightarrow{w^*} b''$. Consider $(a''_\alpha) \subseteq A^{**}$ such that $a''_\alpha \xrightarrow{w^*} a''$ in A^{**}. Now, for each $b' \in B^*$, we have

$$\begin{aligned}
\lim_\alpha \langle \pi_r^{*****}(b',b''),a''_\alpha\rangle &= \lim_\alpha \langle b',\pi_r^{***}(b'',a''_\alpha)\rangle \\
&= \lim_\alpha \langle \pi_r^{***}(b'',a''_\alpha),b'\rangle \\
&= \lim_\alpha \lim_\beta \langle \pi_r^{***}(b_\beta,a''_\alpha),b'\rangle \\
&= \lim_\alpha \lim_\beta \langle a''_\alpha,\pi_r^{*}(b',b_\beta)\rangle \\
&= \lim_\beta \lim_\alpha \langle a''_\alpha,\pi_r^{*}(b',b_\beta)\rangle \\
&= \lim_\beta \langle a'',\pi_r^{*}(b',b_\beta)\rangle \\
&= \lim_\beta \langle \pi_r^{***}(b_\beta,a''),b'\rangle \\
&= \langle \pi_r^{*****}(b',b''),a''\rangle.
\end{aligned}$$

Thus the map $\pi_r^{*****}(b',b''): A^{**} \to \mathbb{C}$ is weak*-weak*-continuous and so $\pi_r^{*****}(b',b'') \in A^*$. By Theorem 2.1, $b'' \in Z_{A^{**}}(B^{**})$.

(ii) The proof is similar to part (i).

An element e'' of A^{**} is said to be a mixed unit if e'' is a right unit for the first Arens multiplication and a left unit for the second Arens multiplication. That is, e'' is a mixed unit if and only if for each $a'' \in A^{**}$, $a''e'' = e''oa'' = a''$. By (Bonsall and Duncan, 1973), an element e'' of A^{**} is mixed unit if and only if it is a weak*-cluster point of some BAI $(e_\alpha)_{\alpha\in I}$ in A.

Let B be a Banach A-bimodule and $a'' \in A^{**}$. We define the locally topological center of the left and right module actions of a'' on B, respectively, as follows:

$$\begin{aligned}
Z^t_{a''}(B^{**}) &= Z^t_{a''}(\pi^t_\ell) \\
&= \{b'' \in B^{**}:\ \pi_\ell^{t***t}(a'',b'') = \pi_\ell^{***}(a'',b'')\}, \\
Z_{a''}(B^{**}) &= Z_{a''}(\pi^t_r) \\
&= \{b'' \in B^{**}:\ \pi_r^{t***t}(b'',a'') = \pi_r^{***}(b'',a'')\}.
\end{aligned}$$

Thus we have

$$\bigcap_{a''\in A^{**}} Z^t_{a''}(B^{**}) = Z^t_{A^{**}}(B^{**}) = Z(\pi^t_r),$$
$$\bigcap_{a''\in A^{**}} Z_{a''}(B^{**}) = Z_{A^{**}}(B^{**}) = Z(\pi_r).$$

Let B be a Banach A-bimodule. We say that B is a left [resp. right] factors with respect to A, if $BA = B$ [resp. $AB = B$].

Definition 2.5. Let B be a Banach left A-module and $e'' \in A^{**}$ be a mixed unit for A^{**}. We say that e'' is a left mixed unit for B^{**} if

$$\pi_\ell^{***}(e'',b'') = \pi_\ell^{t***t}(e'',b'') = b'',$$

for all $b'' \in B^{**}$.

The definition of right mixed unit for B^{**} is similar. B^{**} has a mixed unit if it has equal left and right mixed unit.

It is clear that if $e'' \in A^{**}$ is a left (resp. right) unit for B^{**} and $Z_{e''}(B^{**}) = B^{**}$, then e'' is left (resp. right) mixed unit for B^{**}.

Theorem 2.6. Let B be a Banach A-bimodule with a $BAI\ (e_\alpha)_\alpha$ such that $e_\alpha \xrightarrow{w^*} e''$. Then if $Z^t_{e''}(B^{**}) = B^{**}$ [resp. $Z_{e''}(B^{**}) = B^{**}$] and B^* factors on the left [resp. right], but not on the right [resp. left], then $Z_{B^{**}}(A^{**}) \neq Z^t_{B^{**}}(A^{**})$.

Proof: Suppose that B^* factors on the left with respect to A, but not on the right. Let $(e_\alpha)_\alpha \subseteq A$ be a BAI for A such that $e_\alpha \xrightarrow{w^*} e''$. Thus for all $b' \in B^*$ there are $a \in A$ and $x' \in B^*$ such that $x'a = b'$. Then for all $b'' \in B^{**}$ we have

$$\begin{aligned}
\langle \pi_\ell^{***}(e'',b''),b'\rangle &= \langle e'',\pi_\ell^{**}(b'',b')\rangle \\
&= \lim_\alpha \langle \pi_\ell^{**}(b'',b'),e_\alpha\rangle \\
&= \lim_\alpha \langle b'',\pi_\ell^{*}(b',e_\alpha)\rangle \\
&= \lim_\alpha \langle b'',\pi_\ell^{*}(x'a,e_\alpha)\rangle \\
&= \lim_\alpha \langle b'',\pi_\ell^{*}(x',ae_\alpha)\rangle \\
&= \lim_\alpha \langle \pi_\ell^{**}(b'',x'),ae_\alpha\rangle \\
&= \langle \pi_\ell^{**}(b'',x'),a\rangle = \langle b'',b'\rangle.
\end{aligned}$$

Thus $\pi_\ell^{***}(e'',b'')=b''$ consequently B^{**} has left unit A^{**}-module. This follows that $e''\in Z_{B^{**}}(A^{**})$. If we take $Z_{B^{**}}(A^{**})=Z^t_{B^{**}}(A^{**})$, then

$$e''\in Z^t_{B^{**}}(A^{**}).$$

Then the mapping $b''\to\pi_r^{t***t}(b'',e'')$ is weak *-weak*-continuous from B^{**} into B^{**}. Since $e_\alpha\xrightarrow{w^*}e''$, $\pi_r^{t***t}(b'',e_\alpha)\xrightarrow{w^*}\pi_r^{t***t}(b'',e'')$.

Let $b'\in B^*$ and $(b_\beta)_\beta\subseteq B$ such that $b_\beta\xrightarrow{w^*}b''$. Sinc $Z^t_{e''}(B^{**})=B^{**}$, we have the following equalities

$$\begin{aligned}\langle\pi_r^{t***t}(b'',e''),b'\rangle&=\lim_\alpha\langle\pi_r^{t***t}(b'',e_\alpha),b'\rangle\\&=\lim_\alpha\langle\pi_r^{t***}(e_\alpha,b''),b'\rangle\\&=\lim_\alpha\lim_\beta\langle\pi_r^{t***}(e_\alpha,b_\beta),b'\rangle\\&=\lim_\alpha\lim_\beta\langle\pi_r(b_\beta,e_\alpha),b'\rangle\\&=\lim_\alpha\lim_\beta\langle b',\pi_r(b_\beta,e_\alpha)\rangle\\&=\lim_\beta\lim_\alpha\langle b',\pi_r(b_\beta,e_\alpha)\rangle\\&=\lim_\beta\langle b',b_\beta\rangle=\langle b'',b'\rangle.\end{aligned}$$

So, $\pi_r^{t***t}(b'',e'')=\pi_r^{***}(b'',e'')=b''$. This shows that B^{**} has a right unit. Now, suppose that $b''\in B^{**}$ and $(b_\beta)_\beta\subseteq B$ such that $b_\beta\xrightarrow{w^*}b''$. Then for all $b'\in B^*$ we have

$$\begin{aligned}\langle b'',b'\rangle=\langle\pi_r^{***}(b'',e''),b'\rangle&=\langle b'',\pi_r^{**}(e'',b')\rangle\\&=\lim_\beta\langle\pi_r^{**}(e'',b'),b_\beta\rangle\\&=\lim_\beta\langle e'',\pi_r^{*}(b',b_\beta)\rangle\\&=\lim_\beta\lim_\alpha\langle\pi_r^{*}(b',b_\beta),e_\alpha\rangle\\&=\lim_\beta\lim_\alpha\langle\pi_r^{*}(b',b_\beta),e_\alpha\rangle\\&=\lim_\beta\lim_\alpha\langle b',\pi_r(b_\beta,e_\alpha)\rangle\\&=\lim_\alpha\lim_\beta\langle\pi_r^{***}(b_\beta,e_\alpha),b'\rangle\\&=\lim_\alpha\lim_\beta\langle b_\beta,\pi_r^{**}(e_\alpha,b')\rangle\\&=\lim_\alpha\langle b'',\pi_r^{**}(e_\alpha,b')\rangle.\end{aligned}$$

Hence, weak-$\lim_\alpha\pi_r^{**}(e_\alpha,b')=b'$. So by Cohen factorization theorem, B^* factors on the right that is contradiction.

Corollary 2.7. Let B be a Banach A-bimodule and $e''\in A^{**}$ be a mixed unit for B^{**}. If B^* factors on the left, but not on the right, then $Z_{B^{**}}(A^{**})\neq Z^t_{B^{**}}(A^{**})$.

We recalled that a Banach space B is weakly complete, if for every $(b_\alpha)_\alpha\subseteq B$, such that $b_\alpha\xrightarrow{w}b''$ in B^{**} implies that $b''\in B$.

Theorem 2.8. Suppose that B is a weakly complete Banach space. Then we have the following assertions.

(i) Let B be a Banach left A-module and B^{**} has a left mixed unit $e''\in A^{**}$. If $AB^{**}\subseteq B$, then B is reflexive;

(ii) Let B be a Banach right A-module and B^{**} has a right mixed unit $e''\in A^{**}$. If $Z_{A^{**}}(B^{**})A\subseteq B$, then $Z_{A^{**}}(B^{**})=B$.

Proof: (i) Assume that $b''\in B^{**}$. Since e'' is also mixed unit for A^{**}, there is a BAI $(e_\alpha)_\alpha\subseteq A$ for A such that $e_\alpha\xrightarrow{w^*}e''$.

Then $\pi_\ell^{***}(e_\alpha,b'')\xrightarrow{w^*}\pi_\ell^{***}(e'',b'')=b''$ in B^{**}. It follows from $AB^{**}\subseteq B$ that $\pi_\ell^{***}(e_\alpha,b'')\in B$.

Thus $\pi_\ell^{***}(e_\alpha, b'') \xrightarrow{w} \pi_\ell^{***}(e'', b'') = b''$ in B. Since B is a weakly complete, $b'' \in B$, and so B is reflexive.

(ii) Since $b'' \in Z_{A^{**}}(B^{**})$, we have

$$\pi_r^{***}(b'', e_\alpha) \xrightarrow{w^*} \pi_r^{***}(b'', e'') = b'' \text{ in } B^{**}.$$

Since

$$Z_{A^{**}}(B^{**})A \subseteq B, \pi_r^{***}(b'', e_\alpha) \in B,$$

consequently, we have

$$\pi_r^{***}(b'', e_\alpha) \xrightarrow{w} \pi_r^{***}(b'', e'') = b''$$

in B. Therefore $b'' \in B$.

A functional a' in A^* is said to be wap (weakly almost periodic) on A if the mapping $a \to a'a$ from A into A^* is weakly compact. The preceding definition is equivalent to the following condition (see Dales et al., 2001; Mohamadzadeh and Vishki, 2008).

For any two net $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\{a \in A : \|a\| \le 1\}$, we have

$$\lim_\alpha \lim_\beta \langle a', a_\alpha b_\beta \rangle = \lim_\beta \lim_\alpha \langle a', a_\alpha b_\beta \rangle,$$

whenever both iterated limits exist. The collection of all wap functionals on A is denoted by $wap(A)$. Also we have $a' \in wap(A)$ if and only if $\langle a''b'', a' \rangle = \langle a'' o b'', a' \rangle$ for every $a'', b'' \in A^{**}$.

Definition 2.9. Let B be a Banach left A-module. Then, $b' \in B^*$ is said to be left weakly almost periodic functional if the set $\{\pi_\ell^*(b', a) : a \in A, \|a\| \le 1\}$ is relatively weakly compact. We denote by $wap_\ell(B)$ the closed subspace of B^* consisting of all the left weakly almost periodic functionals in B^*.

The definition of the right weakly almost periodic functional ($wap_r(B)$) is the same. By (Ülger, 1999), the definition of $wap_\ell(B)$ is equivalent to the following equality.

$$\langle \pi_\ell^{***}(a'', b''), b' \rangle = \langle \pi_\ell^{t***t}(a'', b''), b' \rangle$$

for all $a'' \in A^{**}$ and $b'' \in B^{**}$. Thus, we can write

$$wap_\ell(B) = \{b' \in B^* : \langle \pi_\ell^{***}(a'', b''), b' \rangle = \langle \pi_\ell^{t***t}(a'', b''), b' \rangle \; for \; all \; a'' \in A^{**}, b'' \in B^{**}\}.$$

Theorem 2.10. Suppose B is a Banach left A-module. Consider the following statements:

(i) $B^*A \subseteq wap_\ell(B)$; If $AB^{**} \subseteq B$, then B is reflexive;

(ii) $AA^{**} \subseteq Z_{B^{**}}(A^{**})$;

(iii) $AA^{**} \subseteq AZ_{B^{**}}((A^*A)^*)$.

Then, we have (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii).

Proof: (i)$\Rightarrow$ (ii): Let $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \xrightarrow{w^*} b''$. Then for all $a \in A$ and $a'' \in A^{**}$, we have

$$\begin{aligned}\langle \pi_\ell^{***}(aa'', b''_\alpha), b' \rangle &= \langle aa'', \pi_\ell^{**}(b''_\alpha, b') \rangle \\ &= \langle a'', \pi_\ell^{**}(b''_\alpha, b')a \rangle \\ &= \langle a'', \pi_\ell^{**}(b''_\alpha, b'a) \rangle \\ &= \langle \pi_\ell^{***}(a'', b''_\alpha), b'a \rangle \\ &\to \langle \pi_\ell^{***}(a'', b''), b'a \rangle \\ &= \langle \pi_\ell^{***}(aa'', b''), b' \rangle.\end{aligned}$$

Hence, $aa'' \in Z_{B^{**}}(A^{**})$.

(ii)$\Rightarrow$(i): Let $a \in A$ and $b' \in B^*$. Then

$$\begin{aligned}\langle \pi_\ell^{***}(a'', b''), b'a \rangle &= \langle a\pi_\ell^{***}(a'', b''), b' \rangle \\ &= \langle \pi_\ell^{***}(aa'', b''), b' \rangle \\ &= \langle \pi_\ell^{t***t}(aa'', b''), b' \rangle \\ &= \langle \pi_\ell^{t***t}(a'', b''), b'a \rangle.\end{aligned}$$

Therefore $b'a \in wap_\ell(B)$.

(iii)$\Rightarrow$ (ii): Since $AZ_{B^{**}}((A^*A)^*) \subseteq Z_{B^{**}}(A^{**})$, proof is held.

Corollary 2.11. Let B be a Banach A-bimodule. Then if A is a left [resp. right] ideal in A^{**}, then $B^*A \subseteq wap_\ell(B)$ [resp. $AB^* \subseteq wap_r(B)$].

Theorem 2.12. We have the following assertions:

(i) Suppose that B is a Banach left $\mathrm{A}-\mathrm{module}$

and $b' \in B^*$. Then $b' \in wap_\ell(B)$ if and only if the adjoint of the mapping $\pi_\ell^*(b',.): A \to B^*$ is weak*-weak-continuous;

(ii) Suppose that B is a Banach right $A-$module and $b' \in B^*$. Then $b' \in wap_r(B)$ if and only if the adjoint of the mapping $\pi_r^*(b',.): B \to A^*$ is weak*-weak-continuous.

Proof: (i) Assume that $b' \in wap_\ell(B)$ and $\pi_\ell^*(b',.)^*: B^{**} \to A^*$ is the adjoint of $\pi_\ell^*(b',)$. Then for every $b'' \in B^{**}$ and $a \in A$, we have

$$\langle \pi_\ell^*(b',.)^* b'', a\rangle = \langle b'', \pi_\ell^*(b', a)\rangle.$$

Suppose $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \xrightarrow{w^*} b''$ and $a'' \in A^{**}$ and $(a_\beta)_\beta \subseteq A$ such that $a_\beta \xrightarrow{w^*} a''$. By an easy calculation, for all $y'' \in B^{**}$ and $y' \in B^*$, we have

$$\langle \pi_\ell^*(y',.)^*, y''\rangle = \pi_\ell^{**}(y'', y').$$

Since $b' \in wap_\ell(B)$,

$$\langle \pi_\ell^{***}(a'', b''_\alpha), b'\rangle \to \langle \pi_\ell^{***}(a'', b''), b'\rangle.$$

Then we have

$$\begin{aligned}\lim_\alpha \langle a'', \pi_\ell^*(b',)^* b''_\alpha\rangle &= \lim_\alpha \langle a'', \pi_\ell^{**}(b''_\alpha, b')\rangle \\ &= \lim_\alpha \langle \pi_\ell^{***}(a'', b''_\alpha), b'\rangle \\ &= \langle \pi_\ell^{***}(a'', b''), b'\rangle \\ &= \langle a'', \pi_\ell^*(b',)^* b''\rangle.\end{aligned}$$

Hence, the adjoint of the mapping $\pi_\ell^*(b',.): A \to B^*$ is weak*-weak-continuous.

Conversely, assume that the adjoint of the mapping $\pi_\ell^*(b',.): A \to B^*$ is weak*-weak-continuous. Suppose $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \xrightarrow{w^*} b''$ and $b' \in B^*$. Then for every $a'' \in A^{**}$, we have

$$\begin{aligned}\lim_\alpha \langle \pi_\ell^{***}(a'', b''_\alpha), b'\rangle &= \lim_\alpha \langle a'', \pi_\ell^{**}(b''_\alpha, b')\rangle \\ &= \lim_\alpha \langle a'', \pi_\ell^*(b',)^* b''_\alpha\rangle \\ &= \langle a'', \pi_\ell^*(b',)^* b''\rangle \\ &= \langle \pi_\ell^{***}(a'', b''), b'\rangle.\end{aligned}$$

The above equalities show that $b' \in wap_\ell(B)$.

(ii) proof is similar to (i).

Corollary 2.13. Let A be a Banach algebra. Assume that $a' \in A^*$ and $T_{a'}$ is the linear operator from A into A^* defined by $T_{a'} a = a' a$. Then, $a' \in wap(A)$ if and only if the adjoint of $T_{a'}$ is weak*-weak-continuous. So A is Arens regular if and only if the adjoint of the mapping $T_{a'} a = a' a$ is weak*-weak-continuous for every $a' \in A^*$. We consider the following special sets:

$Z_{B^{**}}(A) = \{a \in A : \text{the map } B^* \to B^*;$

$b' \mapsto \pi_l^*(b', a)$ is weak* $-$ weak $-$ continuous$\}$

$Z_{A^{**}}(B) = \{b \in B : \text{the map } B^* \to A^*;$

$b' \mapsto \pi_r^*(b', b)$ is weak* $-$ weak $-$ continuous$\}$.

Obviously, $\mathcal{Z}_{B^{**}}(A)$ is a left ideal of A and $\mathcal{Z}_{A^{**}}(B)$ is a left ideal of B when B is a Banach algebra.

Theorem 2.14. We have the following statements:

(i) Suppose that B is a Banach left A-module. Then $a \in \mathcal{Z}_{B^{**}}(A)$ if and only if $\pi_l^{***}(a, b'') \in B$ for all $b'' \in B^{**}$;

(ii) Suppose that B is a Banach right A-module. Then $b \in \mathcal{Z}_{A^{**}}(B)$ if and only if $\pi_r^{****}(b, a'') \in B$ for all $a'' \in A^{**}$.

Proof: (i) Let $\pi_l^{***}(a, b'') \in B$ for all $b'' \in B^{**}$. Assume that (b'_α) is a net in B^* such that $b'_\alpha \xrightarrow{w^*} b'$. Thus

$$\lim_\alpha \langle b'', \pi_l^*(b'_\alpha, a)\rangle = \lim_\alpha \langle \pi_l^{**}(b'', b'_\alpha), a\rangle$$
$$= \lim_\alpha \langle \pi_l^{***}(a, b''), b'_\alpha\rangle$$
$$= \langle \pi_l^{***}(a, b''), b'\rangle$$
$$= \langle b'', \pi_l^*(b', a)\rangle.$$

This shows that $\pi_l^*(b'_\alpha, a) \xrightarrow{w} \pi_l^*(b', a)$, and so $a \in \mathcal{Z}_{B^{**}}(A)$. Now, suppose that $a \in \mathcal{Z}_{B^{**}}(A)$ and $b'' \in B^{**}$. Take $(b'_\alpha) \subseteq B^*$ such that $b'_\alpha \xrightarrow{w^*} b'$. Since $\pi_l^*(b'_\alpha, a) \xrightarrow{w} \pi_l^*(b', a)$,

$$\lim_\alpha \langle \pi_l^{***}(a, b''), b'_\alpha\rangle = \lim_\alpha \langle b'', \pi_l^*(b'_\alpha, a)\rangle$$
$$= \langle b'', \pi_l^*(b', a)\rangle$$
$$= \langle \pi_l^{***}(a, b''), b'\rangle.$$

Hence, the map $\pi_l^{***}(a, b''): B^* \to \mathbb{C}$ is weak*-weak*-continuous, so $\pi_l^{***}(a, b'') \in B$.

(ii) Let $b \in \mathcal{Z}_{A^{**}}(B), a'' \in A^{**}$ and $b'_\alpha \xrightarrow{w^*} b'$ in B^*. Then $\pi_r^*(b'_\alpha, b) \xrightarrow{w} \pi_r^*(b', b)$ in A^*, and we have

$$\lim_\alpha \langle \pi_r^{***}(b, a''), b'_\alpha\rangle = \lim_\alpha \langle \pi_r^{**}(a'', b'_\alpha), b\rangle$$
$$= \lim_\alpha \langle a'', \pi_r^*(b'_\alpha, b)\rangle$$
$$= \langle a'', \pi_r^*(b', b)\rangle$$
$$= \langle \pi_r^{***}(b, a''), b'\rangle.$$

It follows from the above equalities that the map $\pi_r^{***}(b, a''): B^* \to \mathbb{C}$ is weak*-weak*-continuous, and so $\pi_r^{***}(b, a'') \in B$.

Assume that $\pi_r^{***}(b, a'') \in B$ for all $a'' \in A^{**}$. Take a net (b'_α) in B^* such that $b'_\alpha \xrightarrow{w^*} b'$. Then

$$\lim_\alpha \langle a'', \pi_r^*(b'_\alpha, b)\rangle = \lim_\alpha \langle \pi_r^{**}(a'', b'_\alpha), b\rangle$$
$$= \lim_\alpha \langle \pi_r^{***}(b, a''), b'_\alpha\rangle$$
$$= \langle \pi_r^{***}(b, a''), b'\rangle$$
$$= \langle a'', \pi_r^*(b', b)\rangle.$$

Thus $\pi_l^*(b'_\alpha, a) \xrightarrow{w} \pi_l^*(b', a)$.

Therefore $b \in \mathcal{Z}_{A^{**}}(B)$. Consider the following set:

$$Z_{a''}(B) = \{b \in B : \pi_r^{***}(b, a'') \in B\},$$

then we have the following lemma.

Lemma 2.15. Let A be a Banach algebra with a BAI $(e_\alpha)_\alpha$ such that $e_\alpha \xrightarrow{w^*} e''$, and let B be a Banach A-bimodule. If B factors on the left, then $\mathcal{Z}_{e''}(B) = B$.

Proof: Take $b \in B$ such that $b = b_1 \cdot a$ for some $b_1 \in B$ and $a \in A$. We have

$$\langle \pi_r^{***}(b, e''), b'\rangle = \langle \pi_r^{**}(e'', b'), b\rangle = \langle e'', \pi_r^*(b', b)\rangle$$
$$= \lim_\alpha \langle \pi_r^*(b', b), e_\alpha\rangle$$
$$= \lim_\alpha \langle b', \pi_r(b_1 \cdot a, e_\alpha)\rangle$$
$$= \lim_\alpha \langle \pi_r^*(b', b_1), ae_\alpha\rangle$$
$$= \langle \pi_r^*(b', b_1), a\rangle$$
$$= \langle b', b_1 \cdot a\rangle = \langle b', b\rangle.$$

Hence, $\pi_r^{***}(b, e'') = b$. This shows that

$$B \subseteq \mathcal{Z}_{e''}(B).$$

Corollary 2.16. Let B be a Banach A-bimodule and $e'' \in A^{**}$ be a mixed unit for B^{**}. If B factors on the left, then $\mathcal{Z}_{e''}(B) = B$.

Lemma 2.17. Under one of the following conditions, we have $Z_{A^{**}}(B) = B$.

1. B has (SDLP);
2. $\mathcal{Z}_{A^{**}}(B)A = B$.

Proof: (i) Let (b'_α) is a bounded net in B^* such that $b'_\alpha \xrightarrow{w^*} b'$. It is enough to show that $\pi_r^*(b'_\alpha, b) \xrightarrow{w} \pi_r^*(b', b)$ in A^*.

Assume that $a'' \in A^{**}$ and takes a bounded net in $(a_\beta) \subseteq A$ such that $a_\beta \xrightarrow{w^*} a''$. Thus

$$\begin{aligned}\lim_{\alpha}\langle a'',\pi_r^*(b'_\alpha,b)\rangle &= \lim_{\alpha}\lim_{\beta}\langle \pi_r^*(b'_\alpha,b),a_\beta\rangle\\ &= \lim_{\alpha}\lim_{\beta}\langle b'_\alpha,b\cdot a_\beta\rangle\\ &= \lim_{\beta}\lim_{\alpha}\langle b'_\alpha,b\cdot a_\beta\rangle\\ &= \lim_{\beta}\langle b',b\cdot a_\beta\rangle\\ &= \lim_{\beta}\langle \pi_r^*(b',b),a_\beta\rangle\\ &= \langle a'',\pi_r^*(b',b)\rangle.\end{aligned}$$

Therefore $\mathcal{Z}_{A^{**}}(B)=B$.

(ii) Take $b\in B$. By the assumption $b=b_1a$ where $b_1\in\mathcal{Z}_{A^{**}}(B)$ and $a\in A$. We have

$$\begin{aligned}\pi_r^{***}(b,a'') &= \pi_r^{***}(b_1a,a'')\\ &= \pi_r^{***}(b_1,aa'')\in B,\end{aligned}$$

for all $a''\in A^{**}$, and thus $\mathcal{Z}_{A^{**}}(B)=B$.

Example 2.18. Let G be a locally compact group. Then the group algebra $L^1(G)$ with convolution product is isometrically embedded as an ideal with in the measure algebra $M(G)$, via $f\mapsto fd\lambda$, where λ denotes a left Haar measure on G. Since $L^1(G)$ and $M(G)$ are w*-dense in the second adjoint $L^1(G)^{**}$ and $M(G)^{**}$ respectively, $L^1(G)^{**}$ is an ideal in $M(G)^{**}$. Civin in (Civin, 1962) proved that $L^1(G)$ is an ideal in $L^1(G)^{**}$ if and only if G is commutative. In the case that G is compact it has been shown by Watanabe in (Watanabe, 1976). Since $L^1(G)$ has a bounded approximate identity, $L^1(G)$ is an ideal in $M(G)^{**}$ when G is compact. Now, if $1\le p\le\infty$ and q is conjugate of p, by Theorems 2.10 and 2.14 we have the following statements:

1. For any locally compact group G,

$$\mathcal{Z}_{M(G)^{**}}(L^1(G)^{**})=L^1(G)^{**};$$

2. If G is compact, then

$$\begin{aligned}L^q(G)*L^1(G)\subseteq wap_\ell(L^p(G)),L^1(G)*L^q(G)\\ \subseteq wap_r(L^p(G))\end{aligned}$$

and

$$Z_{L^1(G)^{**}}(L^1(G))=Z_{M(G)^{**}}(L^1(G))=L^1(G);$$

3. If G is commutative, then

$$\mathcal{Z}_{L^1(G)^{**}}(L^1(G))=L^1(G);$$

4. If G is finite, then

$$L^q(G)\subseteq wap_\ell(L^p(G))\cap wap_r(L^p(G)),$$

and so

$$\mathrm{Z}_{\mathrm{L}^1(\mathrm{G})^{**}}(\mathrm{L}^p(\mathrm{G})^{**})=\mathrm{L}^p(\mathrm{G})$$

and

$$Z_{L^p(G)^{**}}(L^1(G)^{**})=L^1(G).$$

Example 2.19. Let S be an infinite, regular (i.e., for each $s\in S$, there exists $s'\in S$ with $ss's=s$ and $s'ss'=s'$) semigroup with finitely many idempotents. Then $\ell^1(S)$ is not Arens regular (Esslamzadeh, 2000, Theorem 5.1). An example within Example 7.14 of (Dales, et al., 2006) exhibits an infinite, regular semigroup S with $E(S)$ finite for which $\ell^1(S)$ is not strongly Arens irregular.

Let S be an inverse semigroup (i.e., S is regular and the element s' specified above is unique) with the set of idempotents E. Since E is a commutative subsemigroup of S (Howie, 1976, Theorem V.1.2), actually a semilattice, $\ell^1(E)$ could be regarded as a commutative subalgebra of $\ell^1(S)$, and thereby $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module with the following actions

$$\begin{aligned}&\delta_e\cdot\delta_s=\delta_e*\delta_s=\delta_{es},\\ &\delta_s\cdot\delta_e=\delta_{se}=\delta_s*\delta_e \qquad (s\in S,e\in E).\end{aligned}$$

Let $k\in\mathbb{N}$. Recall that E satisfies condition D_k (Duncan and Namioka, 1988) if given $f_1,f_2,\dots,f_{k+1}\in E$ there exist $e\in E$ and i,j

such that

$$1 \le i < j \le k+1, \quad f_i e = f_i, f_j e = f_j.$$

Duncan and Namioka in (Duncan and Namioka, 1988, Lemma 15) proved that for any inverse semigroup S, $\ell^1(E)$ has a bounded approximate identity if and only if E satisfies condition D_k for some k. It follows from the proof of (Dales, et al., 2001, Lemma 13) that each BAI for $\ell^1(E)$ is a BAI for $\ell^1(S)$. Now, if E satisfies condition D_k for some k, by (Howie, 1976, Corollary 2.4),

$$Z_{\ell^1(E)^{**}}(\ell^\infty(S)^{**}) = Z_{\ell^1(S)^{**}}(\ell^\infty(S)^{**}) = \ell^\infty(S);$$

3. Lw^*w-property and Rw^*w-property

In this section, we introduce the new definition as Left weak *-weak property and Right weak *-weak property for Banach algebra A and make some relations between these concepts and topological centers of module actions. As some conclude, for locally compact group G, if $a \in M(G)$ has Lw^*w-property [resp. Rw^*w-property], then we have

$L^1(G)^{**} * a \neq L^1(G)^{**}$ [resp. $a * L^1(G)^{**} \neq L^1(G)^{**}$],

and also we have

$$Z_{L^1(G)^{**}}(M(G)^{**}) \neq M(G)^{**}$$

and

$$Z_{M(G)^{**}}(L^1(G)^{**}) = L^1(G).$$

For a finite group G, we have

$$Z_{M(G)^{**}}(L^1(G)^{**}) = L^1(G)^{**}$$

and

$$Z_{L^1(G)^{**}}(M(G)^{**}) = M(G)^{**}.$$

Definition 3.1. Let B be a Banach left $A-module$. We say that $a \in A$ has Left weak *-weak-property ($= Lw^*w$-property) with respect to B, if for all $(b'_\alpha)_\alpha \subseteq B^*$, $ab'_\alpha \xrightarrow{w^*} 0$ implies $ab'_\alpha \xrightarrow{w} 0$. If every $a \in A$ has Lw^*w-property with respect to B, then we say that A has Lw^*w-property with respect to B. The definition of the Right-weak *-weak property ($= Rw^*w$-property) is the same.

We say that $a \in A$ has weak *-weak-property ($= \mathrm{w}^*\mathrm{w}$-property) with respect to B if it has Lw^*w-property and Rw^*w-property with respect to B. If $a \in A$ has Lw^*w-property with respect to itself, then we say that $a \in A$ has Lw^*w-property.

For preceding definition, we have some examples and remarks as follows.

a) If B is a Banach A-bimodule and reflexive, then A has w^*w-property with respect to B. Then we have the following statements for group algebras.

(i) $L^1(G)$, $M(G)$ and $A(G)$ have w^*w-property when G is finite.

(ii) Let G be locally compact group. $L^1(G)$ [resp. $M(G)$] has w^*w-property [resp. Lw^*w-property] with respect to $L^p(G)$ whenever $p > 1$.

b) Suppose that B is a Banach left A-module and e is left unit element of A such that $eb = b$ for all $b \in B$. If e has Lw^*w-property, then B is reflexive.

c) If S is a compact semigroup, then

$$C^+(S) = \{f \in C(S) : f > 0\}$$

has w^*w-property.

Theorem 3.2. Suppose that B is a Banach A-bimodule. Then we have the following assertions:

(i) If $A^{**} = a_0 A^{**}$ [resp. $A^{**} = A^{**} a_0$] for some $a_0 \in A$ and a_0 has Rw^*w-property [resp. Lw^*w-property], then $Z_{B^{**}}(A^{**}) = A^{**}$;

(ii) If $B^{**} = a_0 B^{**}$ [resp. $B^{**} = B^{**} a_0$] for some $a_0 \in A$ and a_0 has Rw^*w-property [resp. Lw^*w-property] with respect to B, then $Z_{A^{**}}(B^{**}) = B^{**}$.

Proof: (i) Suppose that $A^{**} = a_0 A^{**}$ for some $a_0 \in A$ and a_0 has Rw^*w-property. Let

$(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \xrightarrow{w^*} b''$. Then for all $a \in A$ and $b' \in B^*$, we have

$$\begin{aligned}\langle \pi_\ell^{**}(b''_\alpha, b'), a\rangle &= \langle b''_\alpha, \pi_\ell^*(b', a)\rangle \\ &\to \langle b'', \pi_\ell^*(b', a)\rangle \\ &= \langle \pi_\ell^{**}(b'', b'), a\rangle,\end{aligned}$$

and so $\pi_\ell^{**}(b''_\alpha, b') \xrightarrow{w^*} \pi_\ell^{**}(b'', b')$. Also we can write $\pi_\ell^{**}(b''_\alpha, b')a_0 \xrightarrow{w^*} \pi_\ell^{**}(b'', b')a_0$. Since a_0 has Rw^*w-property, $\pi_\ell^{**}(b''_\alpha, b')a_0 \xrightarrow{w} \pi_\ell^{**}(b'', b')a_0$. Now let $a'' \in A^{**}$. Then there is $x'' \in A^{**}$ such that $a'' = a_0 x''$ and consequently

$$\begin{aligned}\langle \pi_\ell^{***}(a'', b''_\alpha), b'\rangle &= \langle a'', \pi_\ell^{**}(b''_\alpha, b')\rangle \\ &= \langle x'', \pi_\ell^{**}(b''_\alpha, b')a_0\rangle \\ &\to \langle x'', \pi_\ell^{**}(b'', b')a_0\rangle \\ &= \langle \pi_\ell^{***}(a'', b''_\alpha), b'\rangle.\end{aligned}$$

The above statements show that $a'' \in Z_{B^{**}}(A^{**})$. Proof of the next part is the same as the preceding.

(ii) Let $B^{**} = a_0 B^{**}$ for some $a_0 \in A$ and a_0 has Rw^*w-property with respect to B. Assume that $(a''_\alpha)_\alpha \subseteq A^{**}$ such that $a''_\alpha \xrightarrow{w^*} a''$. Then for all $b \in B$, we have

$$\begin{aligned}\langle \pi_r^{**}(a''_\alpha, b'), b\rangle &= \langle a''_\alpha, \pi_r^*(b', b)\rangle \\ &\to \langle a'', \pi_r^*(b', b)\rangle \\ &= \langle \pi_r^{**}(a'', b'), b\rangle.\end{aligned}$$

We conclude that $\pi_r^{**}(a''_\alpha, b') \xrightarrow{w^*} \pi_r^{**}(a'', b')$ then we have $\pi_r^{**}(a''_\alpha, b')a_0 \xrightarrow{w^*} \pi_r^{**}(a'', b')a_0$. Since a_0 has Rw^*w-property with respect to B, $\pi_r^{**}(a''_\alpha, b')a_0 \xrightarrow{w} \pi_r^{**}(a'', b')a_0$. Let $b'' \in B^{**}$. Thus there exists $x'' \in B^{**}$ such that $b'' = a_0 x''$. Hence, we have

$$\begin{aligned}\langle \pi_r^{***}(b'', a''_\alpha), b'\rangle &= \langle b'', \pi_r^{**}(a''_\alpha, b')\rangle \\ &= \langle a_0 x'', \pi_r^{**}(a''_\alpha, b')\rangle \\ &= \langle x'', \pi_r^{**}(a''_\alpha, b')a_0\rangle \\ &\to \langle x'', \pi_r^{**}(a'', b')a_0\rangle \\ &= \langle b'', \pi_r^{**}(a'', b')\rangle \\ &= \langle \pi_r^{***}(b'', a''), b'\rangle.\end{aligned}$$

Therefore $b'' \in Z_{A^{**}}(B^{**})$. The next part is similar to the preceding proof.

Example 3.3.

1. Using Theorem 3.2, for locally compact group G, if $a \in M(G)$ has Lw^*w-property [resp. Rw^*w-property], then we have $L^1(G)^{**} * a \neq L^1(G)^{**}$ [resp. $a * L^1(G)^{**} \neq L^1(G)^{**}$].
2. If G is finite, then by Theorem 3.2, we have

$$Z_{M(G)^{**}}(L^1(G)^{**}) = L^1(G)^{**}$$

and

$$Z_{L^1(G)^{**}}(M(G)^{**}) = M(G)^{**}.$$

Theorem 3.4. Suppose that B is a Banach A-bimodule and A has a BAI. Then we have the following assertions:

(i) If B^* factors on the left [resp. right] with respect to A and A has Rw^*w-property [resp. Lw^*w-property], then $Z_{B^{**}}(A^{**}) = A^{**}$;

(ii) If B^* factors on the left [resp. right] with respect to A and A has Rw^*w-property [resp. Lw^*w-property] with respect B, then

$$Z_{A^{**}}(B^{**}) = B^{**}.$$

Proof: (i) Assume that B^* factors on the left and A has Rw^*w-property. Let $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \xrightarrow{w^*} b''$. Since $B^*A = B^*$, for all $b' \in B^*$ there are $x \in A$ and $y' \in B^*$ such that $b' = y'x$. Then for all $a \in A$, we have

$$\begin{aligned}\langle \pi_\ell^{**}(b''_\alpha, y')x, a\rangle &= \langle b''_\alpha, \pi_\ell^{*}(y', a)x\rangle \\ &= \langle \pi_\ell^{**}(b''_\alpha, b'), a\rangle \\ &= \langle b''_\alpha, \pi_\ell^{*}(b', a)\rangle \\ &\to \langle b'', \pi_\ell^{*}(b', a)\rangle \\ &= \langle \pi_\ell^{**}(b'', y')x, a\rangle.\end{aligned}$$

Thus, we conclude that

$$\pi_\ell^{**}(b''_\alpha, y')x \xrightarrow{w^*} \pi_\ell^{**}(b'', y')x.$$

Since A has Rw^*w-property,

$$\pi_\ell^{**}(b''_\alpha, y')x \xrightarrow{w} \pi_\ell^{**}(b'', y')x.$$

Now, let $a'' \in A^{**}$. Then

$$\begin{aligned}\langle \pi_\ell^{***}(a'', b''_\alpha), b'\rangle &= \langle a'', \pi_\ell^{**}(b''_\alpha, b')\rangle \\ &= \langle a'', \pi_\ell^{**}(b''_\alpha, y')x\rangle \\ &\to \langle a'', \pi_\ell^{**}(b'', y')x\rangle \\ &= \langle \pi_\ell^{***}(a'', b''), b'\rangle.\end{aligned}$$

So, it follows that $a'' \in Z_{B^{**}}(A^{**})$. If B^* factors on the right with respect to A, and assume that A has Lw^*w-property, then proof is similar to the preceding proof.

(ii) Let B^* factors on the left with respect to A and A have Rw^*w-property with respect to B. Assume that $(a''_\alpha)_\alpha \subseteq A^{**}$ such that $a''_\alpha \xrightarrow{w^*} a''$.

Since $B^*A = B^*$, for all $b' \in B^*$ there are $x \in A$ and $y' \in B^*$ such that $b' = y'x$. Then for all $b \in B$, we have

$$\begin{aligned}\langle \pi_r^{**}(a''_\alpha, y')x, b\rangle &= \langle \pi_r^{**}(a''_\alpha, b'), b\rangle \\ &= \langle a''_\alpha, \pi_r^{*}(b', b)\rangle \\ &= \langle a'', \pi_r^{*}(b', b)\rangle \\ &= \langle \pi_r^{**}(a'', y')x, b\rangle.\end{aligned}$$

Consequently, $\pi_r^{**}(a''_\alpha, y')x \xrightarrow{w^*} \pi_r^{**}(a'', y')x$. Since A has Rw^*w-property with respect to B, $\pi_r^{**}(a''_\alpha, y')x \xrightarrow{w} \pi_r^{**}(a'', y')x.$ Thus, for all $b'' \in B^{**}$

$$\begin{aligned}\langle \pi_r^{***}(b'', a''_\alpha), b'\rangle &= \langle b'', \pi_r^{**}(a''_\alpha, y')x\rangle \\ &\to \langle b'', \pi_r^{**}(a'', y')x\rangle \\ &= \langle \pi_r^{***}(b'', a''), b'\rangle.\end{aligned}$$

Therefore $b'' \in Z_{A^{**}}(B^{**})$.

The proof of the next assertions is the same as the preceding proof.

Theorem 3.5. Suppose that B is a Banach A-bimodule. Then we have the following assertions:

(i) If $a_0 \in A$ has Rw^*w-property with respect to B, then

$$a_0A^{**} \subseteq Z_{B^{**}}(A^{**}) \text{ and } a_0B^* \subseteq wap_\ell(B);$$

(ii) If $a_0 \in A$ has Lw^*w-property with respect to B, then $A^{**}a_0 \subseteq Z_{B^{**}}(A^{**})$ and $B^*a_0 \subseteq wap_\ell(B)$;

(iii) If $a_0 \in A$ has Rw^*w-property with respect to B, then

$$a_0B^{**} \subseteq Z_{A^{**}}(B^{**}) \text{ and } B^*a_0 \subseteq wap_r(B);$$

(iv) If $a_0 \in A$ has Lw^*w-property with respect to B, then $B^{**}a_0 \subseteq Z_{A^{**}}(B^{**})$ and $a_0B^* \subseteq wap_r(B)$.

Proof: (i) Let $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \xrightarrow{w^*} b''$. Then for all $a \in A$ and $b' \in B^*$, we have

$$\begin{aligned}\langle \pi_\ell^{**}(b''_\alpha, b')a_0, a\rangle &= \langle \pi_\ell^{**}(b''_\alpha, b'), a_0a\rangle \\ &= \langle b''_\alpha, \pi_\ell^{*}(b', a_0a)\rangle \\ &\to \langle b'', \pi_\ell^{*}(b', a_0a)\rangle \\ &= \langle \pi_\ell^{**}(b'', b')a_0, a\rangle.\end{aligned}$$

Thus $\pi_\ell^{**}(b''_\alpha, b')a_0 \xrightarrow{w^*} \pi_\ell^{**}(b'', b')a_0$. Since a_0 has Rw^*w-property with respect to B, $\pi_\ell^{**}(b''_\alpha, b')a_0 \xrightarrow{w} \pi_\ell^{**}(b'', b')a_0$. We conclude that $a_0a'' \in Z_{B^{**}}(A^{**})$ so that $a_0A^{**} \in Z_{B^{**}}(A^{**})$. Now, the result follows from

$$\pi_\ell^{**}(b'', b')a_0 = \pi_\ell^{**}(b'', b'a_0).$$

(ii) proof is similar to (i).

(iii) Assume that $(a''_\alpha)_\alpha \subseteq A^{**}$ such that $a''_\alpha \xrightarrow{w^*} a''$.

Let $b \in B$ and $b' \in B^*$. Then we have

$$\begin{aligned}\langle \pi_r^{**}(a''_\alpha, b')a_0, b\rangle &= \langle \pi_r^{**}(a''_\alpha, b'), a_0 b\rangle \\ &= \langle a''_\alpha, \pi_r^{*}(b', a_0 b)\rangle \\ &\to \langle a'', \pi_r^{*}(b', a_0 b)\rangle \\ &= \langle \pi_r^{**}(a'', b')a_0, b\rangle.\end{aligned}$$

Hence $\pi_r^{**}(a''_\alpha, b')a_0 \xrightarrow{w^*} \pi_r^{**}(a'', b')a_0$. Since a_0 has Rw^*w-property with respect to B, $\pi_r^{**}(a''_\alpha, b')a_0 \xrightarrow{w} \pi_r^{**}(a'', b')a_0$. If $b'' \in B^{**}$, then we have

$$\begin{aligned}\langle \pi_r^{***}(a_0 b'', a''_\alpha), b'\rangle &= \langle a_0 b'', \pi_r^{***}(a''_\alpha, b')\rangle \\ &= \langle b'', \pi_r^{**}(a''_\alpha, b')a_0\rangle \\ &= \langle b'', \pi_r^{**}(a''_\alpha, b')a_0\rangle \\ &= \langle \pi_r^{***}(a_0 b'', a''), b'\rangle.\end{aligned}$$

Therefore $a_0 b'' \in Z_{A^{**}}(B^{**})$. Consequently we have $a_0 B^{**} \in Z_{A^{**}}(B^{**})$.

The proof of the next assertion is clear.
(iv) Proof is similar to (iii).

Theorem 3.6. Let B be a Banach A-bimodule. Then we have the following assertions:
(i) If B has SDLP, then A has Lw^*w-property and Rw^*w-property with respect to B;
(ii) If B has SDLP from right (left) at $a \in A$, then a has Rw^*w-property (Lw^*w-property) with respect to B.

Proof: (i) Assume that $a \in A$ such that $ab'_\beta \xrightarrow{w^*} 0$ where $(b'_\beta)_\beta \subseteq B^*$. Let $b'' \in B^{**}$ and $(b_\alpha)_\alpha \subseteq B$ such that $b_\alpha \xrightarrow{w^*} b''$. Then

$$\begin{aligned}\lim_\beta \langle b'', ab'_\beta\rangle &= \lim_\beta \lim_\alpha \langle b_\alpha, ab'_\beta\rangle \\ &= \lim_\beta \lim_\alpha \langle ab'_\beta, b_\alpha\rangle \\ &= \lim_\alpha \lim_\beta \langle ab'_\beta, b_\alpha\rangle = 0.\end{aligned}$$

We have $ab'_\beta \xrightarrow{w} 0$, so A has Lw^*w-property. It is also easy for A to have Rw^*w-property.

(ii) Proof is easy and is the same as (i).

Definition 3.7. Let B be a Banach left A-module. We say that B^* strong factors on the left [resp. right] if for all $(b'_\alpha)_\alpha \subseteq B^*$ there are $(a_\alpha)_\alpha \subseteq A$ and $b' \in B^*$ such that $b'_\alpha = b' a_\alpha$ [resp. $b'_\alpha = a_\alpha b'$] where $(a_\alpha)_\alpha$ has limited the weak *-topology in A^{**}.

If B^* strong factors on the left and right, then we say that B^* strong factors on both sides.

It is clear that if B^* strong factors on the left [resp. right], then B^* factors on the left [resp. right].

Theorem 3.8. Let B be a Banach A-bimodule. Assume that $AB^* \subseteq wap_\ell B$. If B^* strong factors on the left [resp. right], then A has Lw^*w-property [resp. Rw^*w-property] with respect to B.

Proof: Let $(b'_\alpha)_\alpha \subseteq B^*$ such that $ab'_\alpha \xrightarrow{w^*} 0$. Since B^* strong factors on the left, there are $(a_\alpha)_\alpha \subseteq A$ and $b' \in B^*$ such that $b'_\alpha = b' a_\alpha$. Let $b'' \in B^{**}$ and $(b_\beta)_\beta \subseteq B$ such that $b_\beta \xrightarrow{w^*} b''$. Then we have

$$\begin{aligned}\lim_\alpha \langle b'', ab'_\alpha\rangle &= \lim_\alpha \lim_\beta \langle b_\beta, ab'_\alpha\rangle \\ &= \lim_\alpha \lim_\beta \langle ab'_\alpha, b_\beta\rangle \\ &= \lim_\alpha \lim_\beta \langle ab' a_\alpha, b_\beta\rangle \\ &= \lim_\alpha \lim_\beta \langle ab', a_\alpha b_\beta\rangle \\ &= \lim_\beta \lim_\alpha \langle ab', a_\alpha b_\beta\rangle \\ &= \lim_\beta \lim_\alpha \langle ab'_\alpha, b_\beta\rangle = 0.\end{aligned}$$

Now, it follows from the above equalities that $ab'_\alpha \xrightarrow{w} 0$.

We finish this section with the following open problems:

Problems:
1. Suppose that B is a Banach A-bimodule. If B

is left or right factors with respect to A, does A have Lw^*w-property or Rw^*w-property, respectively?

2. Suppose that B is a Banach A-bimodule. Let A have Lw^*w-property with respect to B. Dose $Z_{B^{**}}(A^{**}) = A^{**}$?

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