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# On the existence of solutions for a class of systems of functional integral equations of Volterra type in two variables

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**Abstract**

The aim of this paper is to show how some measures of noncompactness in the Banach space of continuous functions defined on two variables can be applied to thesolvability ofa general system of functional integral equations. The results obtained generalize and extendseveral equations. An illustrative example is also presented.

***Keywords****:* Measure of noncompactness; modulus of continuity; system of integral equations

## 1. Introduction

Measures of noncompactness are very useful tools in the functional analysis. They are also used in the studies of general functional equations, ordinary and partial differential equations, fractional partial differential equations, integral equations, optimal control theory (Kominek et al., 1974; Kordylewski et al., 1960; Kuczma et al., 1960; Matkowski et al., 1974; O'Regan et al., 1998; O'Regan, 1996; Szep 1971), for example. Recently, several authors have investigated the existence and behavior of solutions of Volterra type integral equations using the technic of measure of noncompactness (Agarwal et al., 2000; Agarwal et al., 2009; Banas et al., 2009; Darwish 2007; Darwish 2008; Darwish 2009; Estrada et al., 1999). Aghajani et al., in (2011), obtained some results on the existence and behavior of solutions of a class of onlinear Voltrra singular integral equations of the form

$$x(t)=f_1(t,x(t),x(a(t)))+(Gx)(t)\int_0^t f_2(t,s)(Qx)(s)ds,$$

and Darwish and Ntouyas in (2011) obtained similar results on quadratic integral equations . Also, Banas and Dhage in (2008), Banas and Rzepka in (2003), Hu and Yan in (2006), Liu and Kang in (2007) and Liu and Guo in (2005) studied the existence and behavior of solutions of integral equation of solutions of one variable integral equation of Volterra type on the unbounded interval. Aghajani and Jalilian in (2010) extended results of Banas and Dhage in (2008) by considering the following general form of integral equation $x(t)=f(t,x(\alpha(t)),\int_0^{\beta(t)} g(t,s,x(\gamma(s)))ds)$.

Moreover, the problem of existence of solutions for a system of integral equation has been studied by many authors, see (Agarwal et al., 2000; Aghajani et al., 2011; Aghajani et al., To appear; Mursaleen et al., 2012; Mursaleen et al., 2012; Olszowy 2009) and references therein. The object of this paper is to discuss the existence of continuous solutions to the system of nonlinear integral equations

$$x_i(t,s)=f_i(t,s,x_1(t,s),\dots,x_n(t,s),\int_0^{\zeta_i(s)}\int_0^{\beta_i(t)} g_i(t,s,v,w,x_1(v,w),\dots,x_n(v,w))\,dvdw),\quad t,s\in R_+, 1\le i\le n, \tag{1}$$

where $f_i, g_i, \zeta_i$ and $\beta_i$, $i=1,\dots,n$, are continuous function which satisfy some certain conditions, specified later. To do this, first we state and prove some existing theorems for a general system of equations involving condensing operators, which extend some results of Aghajani et al., in (2013) and generalize the main result of Rzepecki in (1982). Then using the obtained results, we investigate the problem of existence of solutions for system (1).

## 2. Preliminaries

The concept of measure of noncompactness was initiated by the fundamental paper of Kuratowski in

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(1930). In a metric space $X$, the Kuratowski measure of noncompactness of a subset $S \subset X$ is defined as

$$\alpha(S) := \inf\{\delta > 0 \mid S = \bigcup_{i=1}^{n} S_i \textit{ for some } S_i \textit{ with } diam(S_i) \le \delta \textit{ for } 1 \le i \le n < \infty\}. \quad (2)$$

Here $diam(T)$ denotes the diameter of a set $T \subset X$, namely $diam(T) := sup\{d(x,y) \mid x, y \in T\}$.

Now, we recall some basic facts concerning measures of noncompactness from Banas et al., in (1980). Denote by $\mathbb{R}$ theset of real numbers and put $\mathbb{R}_+ = [0,+\infty)$. Let $(E, \|.\|)$ be a Banach space with zero element 0. The symbol $\overline{X}$, $ConvX$ will denote the closure and closed convex hull of a subset $X$ of $E$, respectively. Moreover, let $\mathfrak{M}_E$ indicate the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ indicate the family of all nonempty and relatively compact sets. We use the following definition of the measure of noncompactness given Banas et al. in (1980).

**Definition 1.** A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the followingconditions:

$1^\circ$. The family $ker\,\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu = \mathfrak{N}_E$.

$2^\circ$. $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.

$3^\circ$. $\mu(\overline{X}) = \mu(X)$.

$4^\circ$. $\mu(ConvX) = \mu(X)$.

$5^\circ$. $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.

$6^\circ$. If $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1,2,\ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then $X_\infty = \cap_{n=1}^{\infty} X_n \ne \varnothing$.

We need the following theorem that proved Aghajani et al., in (2013), which guarantees the existence of a fixed point for condensing operators (i.e. mappings under which the image of any set is in a certain sense more compact than the set itself) on bounded, closed and convex subsets of a Banach space E.

**Theorem 1. (**Aghajani et al., 2013) Let $\Omega$ be a nonempty, bounded, closed and convex subset of a Banach space E and let $F : \Omega \to \Omega$ be a continuous mapping such that

$$\mu(FX) \le \varphi(\mu(X)) \quad (3)$$

for any nonempty subset X of $\Omega$ where $\mu$ is an arbitrary measure of noncompactness and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing functions such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \ge 0$. Then F has at least one fixed point in the set $\Omega$.

The following theorems and examples are basic to all the results of this work.

**Theorem 2. (**Banas et al., 1980) Suppose $\mu_1, \mu_2, \ldots, \mu_n$ are measures of noncompactness in Banach spaces $E_1, E_2, \ldots, E_n$ respectively. Moreover assume that the function $F : \mathbb{R}_+^n \to \mathbb{R}_+$ is convex and $F(x_1, \ldots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1,2,\ldots,n$. Then

$$\mu(X) = F(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n))$$

defines a measure of noncompactness in $E_1 \times E_2 \times \ldots \times E_n$ where $X_i$ denotes the natural projection of X into $E_i$ for $i = 1,2,\ldots,n$.

As a result of Theorem 2 we present the following example.

**Example 1.** Let $\mu_i$ $(i = 1,2,\ldots,n)$ be measures of noncompactness in Banach spaces $E_1, E_2, \ldots, E_n$ respectively, considering $F_1(x_1, \ldots, x_n) = k \max_{1\le i\le n} x_i$ and $F_2(x_1, \ldots, x_n) = k(x_1 + \ldots + x_n)$, $k \in R_+$ for any $(x_1, \ldots, x_n) \in \mathbb{R}_+^n$, then all the conditions of Theorem 2.2 are satisfied. Therefore, $\mu_1 := k \max_{1\le i\le n} \mu(X_i)$ and $\mu_2 := k(\mu(X_1) + \ldots + \mu(X_n))$ define measures of noncompactness in the space $E_1 \times E_2 \times \ldots \times E_n$ where $X_i$, $i = 1,2,\ldots,n$ denote the natural projections of X into $E_i$.

## 3. Main results

In this section, we state and prove an existence theorem of solutions for a system of equations involving condensing operators in Banach spaces which will be used in section 4 to study the system of nonlinear integral equations (1).

**Theorem 3.** Let $C_i$ be a nonempty, bounded, convex and closed subset of a Banach space $E_i$ $(i = 1,2,\ldots,n)$, and let $F_i : C_1 \times C_2 \times \ldots \times C_n \to C_i$ $(i = 1,2,\ldots,n)$ be a continuous operator such that for any subset $X_i$ of $C_i$

$$\mu(Fi_1(X_1 \times X_2 \times \ldots \times X_n)) \le \varphi(\max_i \mu(Xj)) \quad (4)$$

where $\mu_i$ is an arbitrary measure of noncompactness on $E_i$ $(i = 1,2,\ldots,n)$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \ge 0$. Then there exist $(x_1^*, x_2^*, \ldots, x_n^*) \in C_1 \times C_2 \times \ldots \times C_n$ such that for all $1 \le i \le n$

$$F_i(x_1^*, x_2^*, \ldots, x_n^*) = x_i^*. \quad (5)$$

**Proof:** Define $F : C_1 \times C_2 \times \ldots C_n \to C_1 \times C_2 \times \ldots \times C_n$ as follows

$$F(x_1, x_2, \ldots, x_n) = (F_1(x_1, x_2, \ldots, x_n), F_2(x_1, x_2, \ldots, x_n), \ldots, F_n(x_1, x_2, \ldots, x_n)).$$

Also, consider the measure of noncompactness $\mu$ on $E_1 \times E_2 \times \ldots \times E_n$ defined by $\mu(X) = \max_i \mu(X_i)$, for any bounded subset $X \subset E_1 \times E_2 \times \ldots \times E_n$, where $X_i$ $(i = 1,2,\ldots,n)$ denote the natural projections of $X$ into $E_i$ (see Example 2.1). It is obvious that $F$ is continuous. Now we show that $F$ satisfies (3). To prove this, let $X$ be any nonempty and bounded subset of $C_1 \times C_2 \times \ldots \times C_n$. Then by $(2^\circ)$ and (4), we obtain

$$\begin{aligned}\mu(F(X)) &\le \mu(F_1(X_1 \times X_2 \times \ldots \times X_n) \times F_2(X_1 \times X_2 \times \ldots \times X_n) \times \ldots \times F_n(X_1 \times X_2 \times \ldots \times X_n)) \\ &= \max_k \mu(F_k(X_1 \times X_2 \times \ldots \times X_n) \\ &\le \max_k \varphi(\max_i \mu(X_i)) \\ &\le \varphi(\mu(X))\end{aligned}$$

Therefore, all the conditions of Theorem 1 are satisfied, hence by that theorem $F$ has a fixed point, i.e., there exist $(x_1^*, x_2^*, \ldots, x_n^*) \in C_1 \times C_2 \times \ldots \times C_n$ such that

$$\begin{aligned}(x_1^*, x_2^*, \ldots, x_n^*) &= F\ (x_1^*, x_2^*, \ldots, x_n^*) \\ &= (F_1(x_1^*, x_2^*, \ldots, x_n^*), F_2(x_1^*, x_2^*, \ldots, x_n^*), \ldots, F_n(x_1^*, x_2^*, \ldots, x_n^*))\end{aligned}$$

which gives (5) and the proof is complete.

In (Aghajani et al., 2013, Lemma 2.1) Aghajani et al. proved that for every nondecreasing and upper semicontinuous function $\varphi : R_+ \to \mathbb{R}_+$, the following two conditions are equivalent:

(I) $\lim_{n\to\infty} \varphi^n(t) = 0$ forany $t > 0$.

(II) $\varphi(t) < t$ for any $t > 0$.

So the results of Theorem 3 remain true if (I) is replaced by (II). The following result is a generalization of similar results by Aghajaniet al., and Rzepecki in (1982).

**Corollary 1.** Let $C_i$ be a nonempty, bounded, convex and closed subset of a Banach space $E_i$ $(i = 1,2,\ldots,n)$, and let $F_i : C_1 \times C_2 \times \ldots \times C_n \to C_i$ $(i = 1,2,\ldots,n)$ be a continuous operator such that $\mu(F_i(X_1 \times X_2 \times \ldots \times X_n)) \le k \max_j \mu(X_j)$ for any subset $X_i$ of $C_i$, where $\mu_i$ is an arbitrary measure of noncompactness on $E_i$ and $k \in [0,1)$. Then there exist $(x_1^*, x_2^*, \ldots, x_n^*) \in C_1 \times C_2 \times \ldots \times C_n$ such that for all $1 \le i \le n$
$F_i(x_1^*, x_2^*, \ldots, x_n^*) = x_i^*$.

**Proof:** Take $\varphi(t) = kt$ in Theorem 3.

As a consequence of Theorem 3 we obtain the following corollary, which plays an important role in the next section.

**Corollary 2.** Let $C_i$ be a nonempty, bounded, convex and closed subset of a Banach space $E_i$ $(i = 1,2,\ldots,n)$ and let $F_i, G_i : C_1 \times C_2 \times \ldots \times C_n \to E_i$ and $T_i : C_1 \times C_2 \times \ldots \times C_n \to C_i$ be operators such that

$$\|F_i(x_1, x_2, \ldots, x_n) - F_i(y_1, y_2, \ldots, y_n)\| \le \varphi(\max_j \|x_j - y_j\|)$$

and

$$\|T_i(x_1,x_2,\ldots,x_n)-T_i(y_1,y_2,\ldots,y_n)\| \le \|F_i(x_1,x_2,\ldots,x_n)-F_i(y_1,y_2,\ldots,y_n)\| + \Phi(\|G_i(x_1,x_2,\ldots,x_n)-G_i(y_1,y_2,\ldots,y_n)\|) \quad (6)$$

for any $x_i, y_i \in C_i$ $(i=1,2,\ldots,n)$, where $\varphi, \Phi : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing and right continuous functions such that $\lim_{n\to\infty}\varphi^n(t)=0$ for each $t\ge 0$ and $\Phi(0)=0$. Assume that $G_i$ are compact, continuous operators for $i=1,2,\ldots,n$. Then there exists $(x_1^*,x_2^*,\ldots,x_n^*)\in C_1\times C_2\times\ldots\times C_n$ such that for all $1\le i\le n$ $T_i(x_1^*,x_2^*,\ldots,x_n^*)=x_i^*$.

**Proof:** Let $X_j$ be an arbitrary subset of $C_j$ $(j=1,2,\ldots,n)$ and fixed $1\le i\le n$. By the definition of Kuratowski measure of noncompactness, for every $\varepsilon>0$ there exist $S_1,\ldots,S_m$ such that $X_1\times X_2\times\ldots\times X_n \subseteq \bigcup_{k=1}^m S_k$, $diam\,(F_i(S_k)) < \alpha(F_i(X_1\times X_2\times\ldots\times X_n))+\varepsilon$ and $diam(G_i(S_k))<\varepsilon.$

Let us fix arbitrary $1\le k\le m$. Then for every $p,q\in S_k$ we have

$$\|T_i(p)-T_i(q)\| \le \|F_i(p)-F_i(q)\| + \Phi(\|G_i(p)-G_i(q)\|).$$

Thus, by properties of $\Phi$ we obtain

$$diam\,(T_i(S_k)) \le diam\,(F_i(S_k)) + \Phi(diam\,(G_i(S_k))),$$

$$diam\,(T_i(S_k)) \le \alpha(F_i(X_1\times\ldots\times X_n)) + \varepsilon + \Phi(\varepsilon)$$

and since $\varepsilon$ was arbitrarily and $\Phi$ and $\varphi$ are nondecreasing and right continuous functions, the following estimate holds

$$\alpha(T_i(X_1\times X_2\times\ldots\times X_n)) \le \alpha(F_i(X_1\times X_2\times\ldots\times X_n)). \quad (7)$$

Now we show that $T_i$ satisfies (4) for $(i=1,2,\ldots,n)$. To do this fix arbitrary $x_j, y_j\in X_j$ $(j=1,2,\ldots,n)$. Then

$$\|F_i(x_1,x_2,\ldots,x_n)-F_i(y_1,y_2,\ldots,y_n)\| \le \varphi(\max_j \|x_j-y_j\|) \le \varphi(\max_j diam\ X_j)$$

so

$$diam F_i(X_1\times X_2\times\ldots\times X_n) \le \phi(\max_j diam\ X_j)$$

Therefore from the definition of Kuratowski measure of noncompactness we get

$$\alpha(F_i(X_1\times X_2\times\ldots\times X_n)) \le \phi(\max_j \alpha(X_j)). \quad (8)$$

Using (8) in (7) we deduce $\alpha(T_i(X_1\times X_2\times\ldots\times X_n)) \le \varphi(\max_j \alpha(X_j)).$

Also, from condition (6), $T_i$ is a continuous operator, now an application of Theorem 3 completes the proof.

## 4. Application

In this section, as an application of Theorem 3 we prove the existence of solutions for a large class of systems of functional integral equations of Volterra type in two variables.

Let $BC(\mathbb{R}_+\times\mathbb{R}_+)$ be the Banach space of all bounded and continuous functions on $\mathbb{R}_+\times\mathbb{R}_+$ equipped with the standard norm $\|x\| = \sup\{|x(t,s)| : t,s\ge 0\}.$

For any nonempty bounded subset $X$ of $BC(\mathbb{R}_+\times\mathbb{R}_+)$, $x\in X$, $L>0$ and $\varepsilon>0$ let

$$\omega^L(x,\varepsilon) = \sup\{|x(t,s)-x(u,v)| : t,s, u,v\in[0,L], |t-u|\le\varepsilon, |s-v|\le\varepsilon\},$$

$$\omega^L(X,\varepsilon) = \sup\{\omega^L(x,\varepsilon) : x\in X\},$$

$$\omega_0^L(X) = \lim_{\varepsilon\to 0}\omega^L(X,\varepsilon),$$

$$\omega_0(X) = \lim_{L\to\infty}\omega_0^L(X),$$

$$X(t,s) = \{x(t,s) : x\in X\}$$

and

$$\mu(X) = \omega_0(X) + \limsup_{\|(t,s)\|\to\infty} diam X(t,s) \quad (9)$$

where $\|(t,s)\| = \max(t,s)$. Similar to Banas et al., (1980) (cf. also Banas et al., (2003)), it can be shown that the function $\mu$ is a measure of oncompactness in the space $BC(\mathbb{R}_+\times\mathbb{R}_+)$ (in the sense of Definition 1).

**Theorem 4.** Assume that the following conditions are satisfied:

(i) $\beta_i, \zeta_i : R_+ \to R_+$ (i=1,2) are continuous functions.

(ii) $f_i : R_+ \times R_+ \times R^{n+1} \to R$ $(i = 1,2,\ldots,n)$ is continuous. Moreover there exist nondecreasing and right continuous functions $\varphi, \Phi_i : R_+ \to R_+$ such that $\varphi(t) < t$ for all $t \geq 0$, $\Phi_i(0) = 0$ $(i = 1,2,\ldots,n)$ and

$$|f_i(t,s,x_1,\ldots,x_{n+1}) - f_i(t,s,y_1,\ldots,y_{n+1})| \leq \varphi(\max_{1\leq j\leq n} |x_j - y_j|) + \Phi_i(m_i(t,s)|x_{n+1} - y_{n+1}|) \quad (10)$$

where $m_i : R_+ \times R_+ \to R_+$ is a continuous function for $i = 1,2,\ldots,n$.

(iii) $M := \sup\{|f_i(t,s,0,\ldots,0)| : t,s \in R_+, 1 \leq i \leq n\} < \infty.$

(iv) $g_i : R_+ \times R_+ \times R_+ \times R_+ \times R^n \to R$ are continuous functions for $i = 1,2,\ldots,n$ and

$$D := \sup\{m_i(t,s) | \int_0^{\zeta_i(s)} \int_0^{\beta_i(t)} g_i(t,s,v,w,x_1(v,w),\ldots,x_n(v,w))\, dvdw | : t,s \in R_+, 1 \leq i \leq n, x_1, x_2, \ldots, x_n \in BC(R_+ \times R_+)\} < \infty. \quad (11)$$

Moreover,

$$\lim_{\|(t,s)\|\to\infty} m_i(t,s) | \int_0^{\zeta_i(s)} \int_0^{\beta_i(t)} [g_i(t,s,v,w,x_1(v,w),\ldots,x_n(v,w)) - g_i(t,s,v,w,y_1(v,w),\ldots,y_n(v,w))] dvdw | = 0$$

uniformly with respect to $x_1,\ldots,x_n, y_1,\ldots,y_n \in BC(R_+ \times R_+)$ for all $1 \leq i \leq n$.

(v) There exists a positive solution $r_0$ to the inequality $\varphi(r) + M + \max_i \{\Phi_i(D)\} \leq r.$

Then the system of functional integral equations (1) has at least one solution in the space $BC(R_+ \times R_+)^n$.

The proof relies on the following useful observation.

**Lemma 1.** Assume that $g_i$ satisfy the hypothesis (iv) for $i = 1,2,\ldots,n$, then $G_i : BC(R_+ \times R_+)^n \to BC(R_+ \times R_+)$ defined by

$$G_i((x_j)_{j=1}^n)(t,s) = m_i(t,s) \int_0^{\zeta_i(s)} \int_0^{\beta_i(t)} g_i(t,s,v,w,x_1(v,w),\ldots,x_n(v,w))dvdw \quad (12)$$

are compact and continuous operators for $i = 1,2,\ldots,n$.

**Proof:** Let us fix arbitrarily $1 \leq i \leq n$. First notice that the continuity of $G_i((x_j)_{j=1}^n)(t,s)$ for any fixed $(x_j)_{j=1}^n \in BC(R_+ \times R_+)^n$ is obvious. Moreover, by (12), $G_i$ is well defined on $BC(R_+ \times R_+)^n$. Now we show that $G_i$ is a continuous operator on $BC(R_+ \times R_+)^n$. To verify this, take $((x_j)_{j=1}^n) \in BC(R_+ \times R_+)^n$ and $\varepsilon > 0$ arbitrarily. Moreover take $((y_j)_{j=1}^n) \in BC(R_+ \times R_+)^n$ with $\|x_j - y_j\| < \varepsilon$. Then we have

$$|G_i((x_j)_{j=1}^n)(t,s) - G_i((y_j)_{j=1}^n)(t,s)| \leq m_i(t,s) | \int_0^{\zeta_i(s)} \int_0^{\beta_i(t)} [g_i(t,s,v,w,x_i(v,w),\ldots,x_n(v,w)) - g_i(t,s,v,w,y_1(v,w),\ldots,y_n(v,w))]dvdw |$$

So, using condition (iv) we can find a $T > 0$ such that for $\|(t,s)\| > T$

$$|G_i((x_j)_{j=1}^n)(t,s) - G_i((y_j)_{j=1}^n)(t,s)| \leq \varepsilon$$

and if $t,s \in [0,T]$, then

$$|G_i((x_j)_{j=1}^n)(t,s) - G_i((y_j)_{j=1}^n)(t,s)| \leq m_T \beta_T \zeta_T \vartheta_{i,T}(\varepsilon)),$$

where

$$\beta_T = \sup\{\beta_i(t) : t \in [0,T], 1 \leq i \leq n\},$$
$$\zeta_T = \sup\{\zeta_i(t) : t \in [0,T], 1 \leq i \leq n\},$$

$m_T = \sup\{m_i(t,s) : t,s \in [0,T],$
$1 \le i \le n\},$
$\vartheta_{i,T}(\varepsilon) = \sup\{|g_i(t,s,v,w,x_1,\ldots,x_n)$
$-g_i(t,s,v,w,y_1,\ldots,y_n)|:$
$t,s \in [0,T], v \in [0,\beta_T], w \in [0,\zeta_T],$
$x_i, y_i \in [-b,b], |x_i - y_i| \le \varepsilon\}$

with $b = \max_i \|x_i\| + \varepsilon$. By using the continuity of $g_i$ on the compact set $[0,T]\times[0,T]\times[0,\beta_T]\times[0,\zeta_T]\times[-b,b]^n$, we have $\vartheta_{i,T}(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, $G_i$ is a continuous function on $BC(R_+ \times R_+)^n$. To finish the proof we only need to verify that $G_i$ is compact. Let $X_1,\ldots,X_n$ be nonempty and bounded subsetsof $BC(R_+\times R_+)$, and assume that $T>0$ and $\varepsilon > 0$ are arbitrary constants. Let $t_1,t_2,s_1,s_2 \in [0,T]$, with $|t_2 - t_1| \le \varepsilon$, $|s_2 - s_1| \le \varepsilon$ and $x_i \in X_i$. we have

$|G_i((x_j)_{j=1}^n)(t_2,s_2) - G_i((x_j)_{j=1}^n)(t_1,s_1)| \le$
$\le |m_i(t_2,s_2)\int_0^{\zeta_i(s_2)}\int_0^{\beta_i(t_2)}$
$g_i(t_2,s_2,v,w,x_1(v,w),\ldots,$
$x_n(v,w))dvdw$
$-m_i(t_2,s_2)\int_0^{\zeta_i(s_2)}\int_0^{\beta_i(t_2)}$
$g_i(t_1,s_1,v,w,x_1(v,w),\ldots,x_n(v,w))$
$dvdw\,|$
$+|m_i(t_2,s_2)\int_0^{\zeta_i(s_2)}\int_0^{\beta_i(t_2)}$
$g_i(t_1,s_1,v,w,x_1(v,w),\ldots,x_n(v,w)))dvdw$
$-m_i(t_2,s_2)\int_0^{\zeta_i(s_2)}\int_0^{\beta_i(t_1)}$
$g_i(t_1,s_1,v,w,x_1(v,w),\ldots,x_n(v,w))dvdw\,|$
$+|m_i(t_2,s_2)\int_0^{\zeta_i(s_2)}\int_0^{\beta_i(t_1)}$
$g_i(t_1,s_1,v,w,x_1(v,w),\ldots,x_n(v,w))dvdw$
$-m_i(t_1,s_1)\int_0^{\zeta_i(s_1)}\int_0^{\beta_i(t_1)}$
$g_i(t_1,s_1,v,w,x_1(v,w),\ldots,x_n(v,w))dvdw\,|$
$\le m_T\,|\int_0^{\zeta_i(s_2)}\int_0^{\beta_i(t_2)}$
$[g_i(t_2,s_2,v,w,x_1(v,w),\ldots,x_n(v,w))$
$-g_i(t_1,s_1,v,w,x_1(v,w),\ldots,x_n(v,w))]dvdw\,|$
$+m_T\,|\int_0^{\zeta_i(s_2)}\int_{\beta_i(t_1)}^{\beta_i(t_2)} g_i(t_1,s_1,v,w,x_1(v,w),\ldots,$
$x_n(v,w))dvdw\,|$
$+m_T\,|\int_{\zeta_i(s_1)}^{\zeta_i(s_2)}\int_0^{\beta_i(t_1)}$
$g_i(t_1,s_1,v,w,x_1(v,w),\ldots,x_n(v,w))dvdw\,|$
$\le m_T(\zeta_T\beta_T\omega_r^T(g_i,\varepsilon)$
$+\zeta_T U_r^T\omega^T(\beta_i,\varepsilon) + \beta_T U_r^T\omega^T(\zeta_i,\varepsilon)),$ (13)

where $r = \sup\{\|x_i\| : x_i \in X_i, 1 \le i \le n\}$,
$\omega_r^T(g_i,\varepsilon) = \sup\{|g_i(t_1,s_1,v,w,x_1,\ldots,x_n) -$
$g_1(t_2,s_2,v,w,x_1,\ldots,x_n)|:$
$t_1,t_2,s_1,s_2 \in [0,T], |t_2 - t_1| \le \varepsilon,$
$|s_2 - s_1| \le \varepsilon, v, \in [0,\beta_T], w \in [0,\zeta_T], x_i \in [-r,r]\},$
$\omega^T(\beta_i,\varepsilon) = \sup\{|\beta_i(a) - \beta_i(b)| : a,b \in [0,T],$
$|a-b| \le \varepsilon\},$
$\omega^T(\zeta_i,\varepsilon) = \sup\{|\zeta_i(a) - \zeta_i(b)| : a,b \in [0,T],$
$|a-b| \le \varepsilon\},$
$U_r^T = \sup\{|g_1(t,s,v,w,x_1,\ldots,x_n)| : t,s \in [0,T],$
$v \in [0,\beta_T], w \in [0,\zeta_T], x_i \in [-r,r]\}.$

Since $x_i$ was arbitrary element of $X_i$, $i = 1,\ldots,n$ in (13), we obtain

$\omega^T(G_i(X_1\times\ldots\times X_n),\varepsilon) \le$
$m_T(\zeta_T\beta_T\omega_r^T(g_i,\varepsilon) +$
$\zeta_T U_r^T\omega^T(\beta_i,\varepsilon) + \beta_T U_r^T\omega^T(\zeta_i,\varepsilon),)$

and by the uniform continuity of $g_i$, $\beta_i$ and $\zeta_i$ on the compact sets $[0,T]\times[0,T]\times[0,\beta_T]\times[0,\zeta_T]\times[-r,r]^n$, $[0,T]$ and $[0,T]$ respectively, we have $\omega_r^T(g_i,\varepsilon) \to 0$, $\omega^T(\beta_i,\varepsilon) \to 0$ and $\omega^T(\zeta_i,\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore we obtain

$$\omega_0^T(G_i(X_1\times\ldots\times X_n)) = 0$$

and, finally

$$\omega_0(G_i(X_1\times\ldots\times X_n)) = 0. \quad (14)$$

On the other hand, for all $x_i, y_i \in X_i$ $(i = 1,\ldots,n)$ and $t,s \in R_+$ we get

$$|G_i(x_1,\dots,x_n)(t,s)-G_i(y_1,\dots,y_n)(t,s)|\le$$
$$\le m_i(t,s)|\int_0^{\zeta_i(s)}\int_0^{\beta_i(t)}$$
$$[g_i(t,s,v,w,x_1(v,w),\dots,x_n(v,w))-$$
$$g_i(t,s,v,w,y_1(v,w),\dots,y_n(v,w)))]dvdw\ |$$

where

$$\theta_i(t,s)=\sup\{m_i(t,s)|\int_0^{\zeta_i(s)}\int_0^{\beta_i(t)}$$
$$[g_i(t,s,v,w,x_1(v,w),\dots,x_n(v,w))$$
$$-g_i(t,s,v,w,y_1(v,w),\dots,y_n(v,w))]$$
$$dvdw\ |: x_1,y_1,\dots,x_n,y_n\in BC(R_+\times R_+)\}.$$

Thus

$$diamG_i(X_1\times\dots\times X_n)(t,s)\le\theta_i(t,s). \tag{15}$$

Taking $t,s\to\infty$ in the inequality (15), then using (iv) we arrive at

$$\limsup_{\|(t,s)\|\to\infty} diamG_i(X_1\times\dots\times X_n)(t,s)=0. \tag{16}$$

Further, combining (14) and (16) we get

$$\limsup_{t,s\to\infty} diamG_i(X_1\times\dots\times X_n)(t,s)+\omega_0(G_i(X_1\times\dots\times X_n))=0 \tag{17}$$

or, equivalently

$$\mu(G_i(X_1\times\dots\times X_n))=0.$$

Therefore, $G_i$ is compact and the proof is complete.

**Theorem 5.** Under the assumptions (i)-(v), Eq. (1) has at least one solution in $BC(R_+\times R_+)^n$.

**Proof:** We define the operators $F_i,T_i: BC(R_+\times R_+)^n\to BC(R_+\times R_+)$ by

$$F_i(x_1,\dots,x_n)(t,s)=x_i(t,s)$$

and

$$T_i(x_1,\dots,x_n)(t,s)=f_i(t,s,x_1(t,s),\dots,x_n(t,s),$$
$$\int_0^{\zeta_i(s)}\int_0^{\beta_i(t)}g_i(t,s,v,w,x_1(v,w),\dots,x_n(v,w))dvdw).$$

Using conditions (i)-(iv), for arbitrary fixed $t,s\in R_+$, we have

$$|T_i(x_1,\dots,x_n)(t,s)|\le$$
$$\le|f_i(t,s,x_1(t,s),\dots,x_n(t,s),$$
$$\int_0^{\zeta_i(s)}\int_0^{\beta_i(t)}g_i(t,s,v,w,x_1(v,w),\dots,x_n(v,w))dvdw)$$
$$-f_i(t,s,0,\dots,0)|$$
$$+|f_i(t,s,0,\dots,0)|$$
$$\le\phi(\max_i|x_i(t,s)|)+\Phi_i(m_1(t,s)|\int_0^{\zeta_i(s)}\int_0^{\beta_i(t)}$$
$$g_i(t,s,v,w,x_1(v,w),\dots,x_n(v,w))dvdw\ |)$$
$$+|f_i(t,s,0,\dots,0)|$$
$$\le\varphi(\max_i|x_i(t,s)|)+M+\Phi_i(D).$$

Thus,

$$\|T(x_1,\dots,x_n)\|\le\varphi(\max_i\|x_i\|)+M+\Phi_i(D) \tag{18}$$

and $T(x_1,\dots,x_n)\in BC(R_+\times R_+)$ for any $(x_1,\dots,x_n)\in BC(R_+\times R_+)^n$. Due to Inequality (18) and using (v), the function $T_i$ maps $\overline{B}_{r_0}\times\overline{B}_{r_0}\times\dots\times\overline{B}_{r_0}$ into $B_{r_0}$. Also, applying (10) and definitions of $G_i$, $F_i$ and $T_i$, it is easy to verify that

$$|T_i(x_1,\dots,x_n)(t,s)-T_i(y_1,\dots,y_n)(t,s)|\le$$
$$\phi(|F_i(x_1,\dots,x_n)(t,s)-F_i(y_1,\dots,y_n)(t,s)|)$$
$$+\Phi_i(|G_i(x_1,\dots,x_n)(t,s)-$$
$$G_i(y_1,\dots,y_n)(t,s)|).$$

Thus, $T_i$ satisfies (6), $i=1,\dots,n$, now an application of Corollary 3.3 completes the proof.
The following examples illustrate the applicability of our results.

**Example 6.** Consider the following system of functional integral equations

$$\begin{cases} x_1(t,s)=\dfrac{ts}{(ts+1)(|x_1(t,s)|+1)}+ \\ \dfrac{1}{e^{ts}}\arctan(\int_0^t\int_0^s \\ \dfrac{v^3\cos(x_1(v,w))+e^w\sin(x_2^4(v,w))}{(2+\sin(x_1(v,w)))} \\ dvdw) \\ x_2(t,s)=\sin(ts)+\dfrac{|x_2(t,s)|}{|x_2(t,s)|+1}+ \\ \int_0^{\sqrt{t}}\int_0^{\sqrt{s}} \\ \dfrac{\sqrt{1+\sin^2(ux_1(u,v))}+ts(uv)^{11}(1+x_2^4(u,v))}{(1+t^7s^7)(1+x_2^4(u,v))} \\ dudv \end{cases} \tag{19}$$

Eq. (19) is a special case of Eq. (1)
where

$$\beta_1(t)=\zeta_1(t)=t,\ \beta_2(t)=\zeta_2(t)=\sqrt{t},$$

$$f_1(t,s,x_1,x_2,z)=\frac{ts}{(ts+1)(|x_1|+1)}+\frac{1}{e^{ts}}\arctan(z),$$

$$f_2(t,s,x_1,x_2,z)=\sin(ts)+\frac{|x_2|}{|x_2|+1}+z,$$

$$g_1(t,s,v,w,x_1,x_2)=\frac{v^3 cos(x_1)+e^w sin(x_2{}^4)}{(2+sin(x_1))},$$

$$g_2(t,s,v,w,x_1,x_2)=\frac{\sqrt{1+\sin^2(vx_1)}+ts(vw)^{11}(1+x_2^4)}{(1+t^7s^7)(1+x_2^4)}.$$

From the definitions of $\beta_i$, $\zeta_i$, $f_1$ and $f_2$, hypothesis (i) and (iv) of Theorem 5 are obviously satisfied. Also we have

$$|f_1(t,s,x_1,x_2,z_1)-f_1(t,s,y_1,y_2,z_2)|\leq$$
$$\leq\left|\frac{ts}{(ts+1)(|x_1|+1)}+\frac{1}{e^{ts}}\arctan(z_1)-\frac{ts}{(ts+1)(|y_1|+1)}-\frac{1}{e^{ts}}\arctan(z_2)\right|$$
$$\leq\frac{|x_1-y_1|}{(|x_1|+1)(|y_1|+1)}+\frac{1}{e^{ts}}|z_1-z_2|$$
$$\leq\frac{|x_1-y_1|}{|x_1-y_1|+1}+\frac{1}{e^{ts}}|z_1-z_2|$$

and similarly

$$|f_2(t,s,x_1,x_2,z_1)-f_2(t,s,y_1,y_2,z_2)|\leq\frac{|x_2-y_2|}{|x_2-y_2|+1}+|z_1-z_2|.$$

Thus, by taking $m_1(t,s)=\frac{1}{e^{ts}}$, $m_2(t,s)=1$, $\varphi(t)=\frac{t}{t+1}$ and $\Phi_1(t)=\Phi_2(t)=t$, the functions $f_1$ and $f_2$ satisfy assumption (ii) of Theorem 5. Also, $g_1$ and $g_2$ are continuous on $R_+\times R_+\times R_+\times R_+\times R\times R$ and since

$$\left|\frac{1}{e^{ts}}\int_0^t\int_0^s\frac{v^3\cos(x_1(v,w))+e^w\sin(x_2^4(v,w))}{(2+\sin(x_1(v,w)))}dvdw\right|\leq$$
$$\left|\int_0^t\int_0^s\frac{v^3+e^w}{e^{ts}}dvdw\right|$$
$$\leq\frac{\frac{s^4t}{4}+e^ts-s}{e^{ts}}$$

and

$$\left|\int_0^{\sqrt{t}}\int_0^{\sqrt{s}}\frac{\sqrt{1+\sin^2(ux_1(u,v))}+ts(uv)^{11}(1+x_2^4(u,v))}{(1+t^7s^7)(1+x_2^4(u,v))}dudv\right|\leq$$
$$\left|\int_0^{\sqrt{t}}\int_0^{\sqrt{s}}\frac{\sqrt{2}}{1+t^7s^7}+\frac{ts(uv)^{11}}{1+t^7s^7}dudv\right|$$
$$\leq\frac{\sqrt{2ts}+(ts)^{\frac{13}{2}}}{1+t^7s^7}$$

for all $t,s\in R_+$, so we obtain

$$D\leq\sup\left\{\frac{\frac{s^4t}{4}+e^ts-s}{e^{ts}}+\frac{\sqrt{2ts}+(ts)^{\frac{13}{2}}}{1+t^7s^7}:s,t\in R_+\right\}<\infty.$$

Moreover,

$$\lim_{\|(t,s)\|\to\infty}\frac{1}{e^{ts}}\int_0^t\int_0^s\frac{v^3\cos(x_1(v,w))+e^w\sin(x_2^4(v,w))}{(2+\sin(x_1(v,w)))}dvdw=0,$$

$$\lim_{\|(t,s)\|\to\infty}\frac{1}{e^{ts}}\int_0^t\int_0^s\left[\frac{v^3\cos(x_1(v,w))+e^w\sin(x_2^4(v,w))}{(2+\sin(x_1(v,w)))}-\frac{v^3\cos(y_1(v,w))+e^w\sin(y_2^4(v,w))}{(2+\sin(y_1(v,w)))}\right]dvdw=0$$

and

$$\lim_{\|(t,s)\|\to\infty}\int_0^{\sqrt{t}}\int_0^{\sqrt{s}}\left[\frac{\sqrt{1+\sin^2(ux_1(u,v))}+ts(uv)^{11}(1+x_2^4(u,v))}{(1+t^7s^7)(1+x_2^4(u,v))}-\frac{\sqrt{1+\sin^2(uy_1(u,v))}+ts(uv)^{11}(1+y_2^4(u,v))}{(1+t^7s^7)(1+y_2^4(u,v))}\right]dudv=0,$$

uniformly with respect to $x_1,x_2,y_1,y_2\in BC(R_+)$, which show that assumption (iv) is satisfied. Furthermore, we have

$$M=\sup\{|f_i(t,s,0,0,0)|:t,s\in R_+,i=1,2\}$$
$$=\sup\{\frac{ts}{ts+1},\sin(ts),t,s\in R_+\}=1.$$

So, taking $r_0 \geq 2 + D$ then we see thatassumptions (iii) and (v) of Theorem 5 are satisfied. Hence by that theorem the system of integral equations (19) has at least one solution in the space $BC(R_+ \times R_+)^2$.

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