IJST (2015) 39A3 (Special issue): 399-406
Iranian Journal of Science & Technology

http://ijsts.shirazu.ac.ir

An algorithmic approach for solution of nonlinear Fredholm-Hammerstein integral equations

H. Saberi Najafi[1], S. A. Edalatpanah[1, 2]* and H. Aminikhah[1]

[1]*Department of Mathematics, Faculty of Sciences, University of Guilan, University Campus 2, Rasht, Iran*
[2]*Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran*
[2]*Young Researchers Club, Lahijan Branch, Islamic Azad University, Lahijan, Iran*
E-mail: saedalatpanah@gmail.com

Abstract

In this paper, we present a new computational technique for solving nonlinear Fredholm integral equations of the second kind. This proposed method is based on Galerkin method and is computationally very attractive. Moreover, for reducing the operations in comparing similar works, we design our algorithm based on transformations of orthogonal polynomials in approximation coefficients calculating scheme. Finally, for showing the reliability and efficiency of this method, we use some numerical examples.

Keywords: Nonlinear Fredholm integral equation; Hammerstein equations; Galerkin method; Legendre polynomials; Trial space; Taylor expansion

1. Introduction

Various types of integral equations often occur in a wide variety of areas including mechanics, the theory of optimal control, economics models, vehicular traffic, chemical engineering processes, physical and biological sciences; see (Corduneanu, 1973; Burton, 1983; Agarwal et al., 1999; Deimling, 2010; Ladopoulos, 2013) and the references therein.

For instance, integral equations of the Fredholm-Hammerstein type have been one of the most important domains of applications of the ideas and methods of nonlinear functional analysis and in particular of the theory of nonlinear operators of monotone type (Cardinali and Papageorgiou, 1999).

Furthermore, this kind of integral equations appear in nonlinear physical phenomena such as electro-magnetic fluid dynamics, reformulation of boundary value problems with a nonlinear boundary condition (Atkinson, 1997). This equation is as follows:

$$u(t)-\int_a^b K(t,y)[u(y)]^m\,dy=f(t). \qquad (1)$$

where, $f(t)\in L^2[a,b]$, $K(t,y)\in L^2[a,b]\times[a,b]$, are known functions, $u(t)$ is the unknown function to be determined, $t\in I=[a,b]$ and $m>1$ is a positive integer (Hammerstein, 1930). Throughout this paper, since any finite interval $[a,b]$ can be transformed to interval $[-1,1]$ by linear maps, without loss of generality, we assume that $a=-1, b=1$.

Several numerical methods for approximating the solution of the above Fredholm- Hammerstein integral equations are known. For example, Tricomi in [1985, Sec. 4.6] introduced the classical method of successive approximations. Kumar and Sloan (1987), introduced a new collocation-type method for the numerical solution of Fredholm-Hammerstein integral equations. Brunner (1992) applied this method to nonlinear Volterra integral and integro-differential equations and discussed its connection with the iterated collocation method. Guoqiang (1993) obtained the asymptotic error expansion of this method for nonlinear Volterra–Hammerstein integral equations at mesh points. Elnagar and Kazemi (1996) investigated the Chebyshev spectral method to an equivalent equation of nonlinear Volterra-Hammerstein integral equations and discussed some convergence results. Hernandez et al. (2000) applied a one-parametric family of secant-type iterations for Eq. (1) and established a semilocal convergence result for these iterations by means of a technique based on a new system of recurrence relations. Kaneko et al. (2003) developed the Petrov–Galerkin method and the iterated Petrov–Galerkin method for Eq. (1) and established a framework for fast algorithms to obtain approximate solutions based on Alpert's

*Corresponding author
Received: 5 July 2014 / Accepted: 22 December 2014

Wavelets. Contea and Prete (2006) proposed discrete collocation methods for Volterra integral equations of Hammerstein type, where the Laplace transform of the kernel rather than the convolution kernel itself is known a priori. In recent years, numerous numerical methods have been proposed to find Hammerstein integral equations. These methods include: Nyström type methods (Sommariva, 2005; Cheng and Huang, 2011), projection methods (Cai and Xu, 2008; Jiang and Xu, 2010), methods of special functions (Alipanah and Dehghan, 2007; Maleknejad et al., 2010; Babolian and Mordad, 2011), wavelets (Jang et al., 2005; Mahmoudi, 2005; Maleknejad et al., 2006; Xiao et al., 2006; Lepik, 2006; Maleknejad et al., 2007; Ordokhani and Razzaghi, 2008; Babolian and Shahsavaran; 2009; Saeedi et al., 2011; Aziz, 2013; Saberi Najafi et al., 2014), homotopy techniques and the Adomian decomposition method (Abbasbandy, 2006; Dong et al., 2013), Toeplitz matrix method (Abdou et al., 2009), polynomial interpolation procedures (Maleknejad and Lotfi, 2005; Borzabadi et al., 2006; Bica et al., 2012) and multigrid methods (Atkinson and Han, 2007). The main problem for solution of Eq. (1) is $[u(y)]^m$. Some methods used quadrature formula methods and spline approximations (for example, see (Maleknejad and Lotfi, 2005)). Some methods considered $[u(y)]^m$ as an independent variable (for example, see Eq. (7) in (Babolian and Shahsavaran; 2009)). Some other methods applied equivalent equation as follows:

$$[u(y)]^m = \psi(y,u(y)) = \psi(y,\sum_{i=1}^{n} c_i \int_a^b K(y,t) b_i(t)\,dt + f(y)).$$

For more details, see (Elnagar and Kazemi, 1996; Kaneko et al., 2003; Maleknejad et al., 2006). These require a huge number of arithmetic operators, high computational costs and a large storage capacity. Some methods also used operational vector. For example, in (Mahmoudi, 2005), Eq. (1) is studied using Legendre wavelets basis as $\Psi(y) = [\psi_1(y), \psi_2(y), ..., \psi_{2^{k-1}M}(y)]^T$. Then this author approximated $[u(y)]^m$ as follows:

$$[u(y)]^m = [U^T \Psi(y)]^m = U_m^{*T} \Psi(y),$$

where U_m^{*T} is a column vector, whose elements are nonlinear combinations of the elements of the vector *U*. U_m^{*T} is called the operational vector of the *mth* power of the function $u(y)$. The method proposed only works under the condition that the operational vector Y^* is a vector function of Y, moreover, this function must be explicitly known (see Eq. (4.2) in (Mahmoudi, 2005)). Generally, it is difficult to meet this requirement in practice.

In this paper, we present an algorithmic approach which is new and different from all the existing methods. Our method is based on Galerkin methods and transformations of Legendre polynomials. This method is very simple to apply and offers several advantages in reducing computational costs. Furthermore, it is worthy to note that this method can be utilized as an accurate algorithm to solve linear and nonlinear integro-differential equations and functional integral equations arising in physics and other fields of applied mathematics.

2. Solving the Nonlinear Fredholm-Hammerstein Integral Equations

First of all, we give some basic notations and preliminary results which are essential tools for describing our main results.

2.1. Properties of Legendre Polynomials

Legendre polynomials are important in approximation theory and numerical analysis and in some quadrature rules based on these polynomials such as Gauss-Legendre rule that appears in the theory of numerical integration (Chihara, 2011).
Consider the well-known Legendre polynomials of order $n, P_n(x),$ which are derived from the following recursive formula (for $x \in [-1,1]$):

$$\begin{aligned} &P_0(x) = 1, \\ &P_1(x) = x, \\ &P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x). \end{aligned} \tag{2}$$

Moreover, these polynomials are orthogonal with respect to the weight function $\omega(x) = 1$ and satisfy:

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{mn}, \tag{3}$$

where, δ_{mn} is the Kronecker delta.

Furthermore, for writing x^n in terms of Legendre polynomials we have the following formula:

$$x^n = \sum_{l=n,n-2,\ldots} \gamma_l P_l(x), \tag{4}$$

where,

$$\gamma_l = \frac{(2l+1)n!}{2^{(n-l)/2}(0.5(n-l))!(l+n+1)!!}. \quad (5)$$

We note that $n!!$ is double factorial and defined as:

$$n!! = \begin{cases} n(n-2)(n-4)(n-6)...(4)(2) & If \quad n \ is \ even, \\ n(n-2)(n-4)(n-6)...(3)(1) & If \quad n \ is \ odd, \\ 1 & If \quad n = 0, -1. \end{cases}$$

The first few powers in terms of Legendre polynomials are:

$$1 = P_0(x),$$
$$x = P_1(x),$$
$$x^2 = \frac{1}{3}[P_0(x) + 2P_2(x)],$$
$$x^3 = \frac{1}{5}[3P_1(x) + 2P_3(x)],$$
$$x^4 = \frac{1}{35}[7P_0(x) + 20P_2(x) + 8P_4(x)],$$
$$\vdots$$

For more details, see (Abramowitz and Stegun, 1972).

2.2. Galerkin Method

The basic idea in the weighted residual method is to assume that the unknown function $u(t)$ in Eq. (1) can be approximated by the sum of $N+1$ *trail functions* $b_i(t)$, *i.e,*

$$u(t) \approx u_{app}(t) = \sum_{i=0}^{N} c_i b_i(t). \quad (7)$$

In this way the function u_{app} is a linear combination of $b_i(t)$ and it's expansion coefficients c_i have to be determined uniquely. Substituting the approximate solution given by Eq. (7) into Eq. (1), the result is the *residual function* defined by:

$$E(t, c_i) = E(t, u_{app}(t)) = L[u_{app}(t)] - f(t), \quad (8)$$

where,

$$L[u_{app}(t)] = u_{app}(t) - \int_a^b K(t, y)[u_{app}(y)]^m \, dy. \quad (9)$$

Since the residual function is identically equal to zero for the exact solution, the challenge is to choose the coefficients c_i so that the residual function is minimized. In integral form this can be achieved with the condition:

$$\int_a^b \omega_i(t) E(t, c_i) \, dt = 0, \quad (10)$$

where $\omega_i(t)$ are weight function. Galerkin approach makes the residual $E(t, c_i)$ orthogonal to $N+1$ given independent function on the domain $[a, b]$. In this approach the weighting function is chosen to be identical to the trail functions.

2.3. Main Results

Consider the approximate solution given by Eq. (7) with $b_i(t) = t^i$, *i.e*

$$u_{app}(t) = \sum_{i=0}^{N} c_i t^i. \quad (11)$$

Let,

$$\psi(y, u(y)) = [u_{app}(y)]^m = [\sum_{i=0}^{N} c_i y^i]^m, \quad (12)$$

by Taylor expansion of $\psi(y, t)$ at y=0, we can write:

$$\begin{aligned} &\psi(y, u(y)) = \psi(y = 0, u(y)) + \\ &(\frac{\partial}{\partial y}\psi(y = 0, u(y)))y + (\frac{1}{2!}\frac{\partial^2}{\partial y^2}\psi(y = \\ &0, u(y)))y^2 + \cdots + (\frac{1}{N!}\frac{\partial^N}{\partial y^N}\psi(y = 0, u(y)))y^N \\ &+ \cdots + (\frac{1}{(D)!}\frac{\partial^D}{\partial y^D}\psi(y = 0, u(y)))y^D, \end{aligned} \quad (13)$$

where, $D = mN$.

Then, we have:

$$\begin{aligned} &\psi(y, u(y)) = ([\sum_{i=0}^{N} c_i y^i]^m)_{y=0} + \\ &(\frac{\partial}{\partial y}[\sum_{i=0}^{N} c_i y^i]^m)_{y=0} y + (\frac{1}{2!}\frac{\partial^2}{\partial y^2}[\sum_{i=0}^{N} c_i y^i]^m)_{y=0} y^2 \\ &+ \cdots + (\frac{1}{N!}\frac{\partial^N}{\partial y^N}[\sum_{i=0}^{N} c_i y^i]^m)_{y=0} y^N \\ &+ \cdots + (\frac{1}{D!}\frac{\partial^D}{\partial y^D}[\sum_{i=0}^{N} c_i y^i]^m)_{y=0} y^D. \end{aligned} \quad (14)$$

Moreover, since

$$(\frac{\partial^s}{\partial y^s}[\sum_{i=s+1}^{N} c_i y^i]^m)_{y=0} = 0, \quad (15)$$

We obtain the following expansion of the above relation:

$$\psi(y,u(y)) = ([c_0]^m) + (\frac{\partial}{\partial y}[c_0 + c_1 y]^m)_{y=0} y +$$
$$(\frac{1}{2!}\frac{\partial^2}{\partial y^2}[\sum_{i=0}^{2} c_i y^i]^m)_{y=0} y^2 + (\frac{1}{3!}\frac{\partial^3}{\partial y^3}[\sum_{i=0}^{3} c_i y^i]^m)_{y=0} y^3$$
$$+\cdots+(\frac{1}{N!}\frac{\partial^N}{\partial y^N}[\sum_{i=0}^{N} c_i y^i]^m)_{y=0} y^N +$$
$$(\frac{1}{(N+1)!}\frac{\partial^{N+1}}{\partial y^{N+1}}[\sum_{i=0}^{N} c_i y^i]^m)_{y=0} y^{N+1} +$$
$$\cdots+(\frac{1}{D!}\frac{\partial^D}{\partial y^D}[\sum_{i=0}^{N} c_i y^i]^m)_{y=0} y^D.$$

Therefore, we can write:

$$[u_{app}(y)]^m = \sum_{i=0}^{D} T_i y^i, \tag{16}$$

where,

$$T_s = \begin{cases} (\frac{1}{s!}\frac{\partial^s}{\partial y^s}[\sum_{i=0}^{s} c_i y^i]^m)_{y=0}, & for \quad s = 0,1,\ldots,N, \\ (\frac{1}{s!}\frac{\partial^s}{\partial y^s}[\sum_{i=0}^{N} c_i y^i]^m)_{y=0}, & for \quad s = N+1,\ldots,D. \end{cases} \tag{17}$$

Now by transformations of orthogonal polynomials based on formula (4), we will obtain an efficient method to solve Eq. (1). This method is as follows:

Algorithm1.

Step 1. Choose $u_{app}(y) = \sum_{i=0}^{N} c_i y^i$ and $[u_{app}(y)]^m = \sum_{i=0}^{D} T_i y^i$

Step 2. Use formula (4) and set $u_{app}(t) = \sum_{i=0}^{N} \alpha_i P_i(t)$ and $[u_{app}(y)]^m = \sum_{i=0}^{D} \beta_i P_i(y)$.

Step 3. Apply Galerkin method and solve the following nonlinear equations:

$$\frac{2}{2j+1}\alpha_j - \sum_{i=0}^{D} \beta_i K_{i,j} = f_j, \quad j = 0,\ldots,N, \quad i = 0,\ldots,D, \tag{18}$$

where,

$$f_j = \int_{-1}^{1} f(t) P_j(t) dt, \tag{19}$$

$$K_{i,j} = \int_{-1}^{1}\int_{-1}^{1} K(t,y) P_i(y) P_j(t) dy dt. \tag{20}$$

3. Illustrative examples

In this section, we give some numerical experiments to illustrate the results obtained in previous sections.

Example 3.1. Consider the following nonlinear Fredholm-Hammerstein integral equation

$$u(t) = e^t - \frac{t(e^4+3)}{4e^2} + \int_{-1}^{1} ty[u(y)]^2 dy.$$

where, $f = e^t - \frac{t(e^4+3)}{4e^2}$ and the exact solution is $u(t) = e^t$.

To solve the above problem using our method, we do the following steps:

Let us consider $N = 2$.Then we construct trail space as follows:

$$u_{app}(t) = \sum_{i=0}^{2} c_i t^i = c_0 + c_1 t + c_2 t^2,$$

and by formula (4) we have:

$$u_{app}(t) = \sum_{i=0}^{2} \alpha_i P_i(t),$$

where,

$$\alpha_0 = c_0 + \frac{1}{3}c_2, \alpha_1 = c_1, \alpha_2 = \frac{2}{3}c_2.$$

Similarly,

$$[u_{app}(y)]^2 = \sum_{i=0}^{4} T_i y^i,$$

where,

$$T_0 = c_0^2, T_1 = 2c_0c_1, T_2 = c_1^2 + 2c_0c_2, T_3 = 2c_1c_2, T_4 = c_1^2.$$

and by formula (4) we have:

$$[u_{app}(y)]^2 = \sum_{i=0}^{4} \beta_i P_i(y),$$

where,

$$\beta_0 = T_0 + \frac{1}{3}T_2 + \frac{7}{35}T_4, \beta_1 = T_1 + \frac{3}{5}T_3,$$
$$\beta_2 = \frac{2}{3}T_2 + \frac{20}{35}T_4, \beta_3 = \frac{2}{5}T_3, \beta_4 = \frac{8}{35}T_4.$$

So we have,

$$\sum_{i=0}^{2} \alpha_i P_i(t) - \int_{-1}^{1} ty[\sum_{i=0}^{4} \beta_i P_i(y)] dy = f.$$

Now, we multiply both sides of the above relation

with $\{P_j(t)\}_{j=0}^{2}$, which, from orthogonality of Legendre polynomials we obtain the following nonlinear equations:

$$\begin{bmatrix} 2\alpha_0 \\ 2/3\alpha_1 \\ 2/5\alpha_2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 4/9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} 872/371 \\ -422/749 \\ 609/4255 \\ 167/8296 \\ 40/18063 \end{bmatrix}.$$

Therefore, we get:

$$c_0 = \frac{1613}{1619}, c_1 = \frac{2109}{1891}, c_2 = \frac{1827}{3404}.$$

Table 1 and Fig. 1 show the absolute values of error for N = 2, 4 and 6.

Example 3.2. As the second example consider the following integral equation:

$$u(t) = \sin(\frac{\pi}{2}t) - \frac{11t}{3} + \int_{-1}^{1} 4\pi t y \sin(\pi y)[u(y)]^4 \, dy.$$

with exact solution $u(t) = \sin(\frac{\pi}{2}t)$. Table 2 and Fig. 2 are the numerical results for Example 3.2.

Example 3.3. As the third example consider the following integral equation:

$$u(t) = e^{t-1}(e - e^2 + 1) + \int_{-1}^{1} e^{x-2y}[u(y)]^3 \, dy,$$

with exact solution $u(t) = e^t$. Table 3 and Fig. 3 illustrate the numerical results for Example 3.3.

Example 3.4. As the last example consider the following integral equation:

$$u(t) = 1 - t(5.6 + 25t - t^3 - t^5) + \int_{-1}^{1} (t - 2y)[u(y)]^3 \, dy.$$

with exact solution $u(t) = 1 - \frac{t^2}{10}(t-1)^2$. Table 4 and Fig. 4 are the numerical results for Example 3.4.

Remark: A notable difference of our method from the conventional Galerkin method such as models (Elnagar and Kazemi, 1996; Kaneko et al., 2003; Maleknejad and Lotfi, 2005; Maleknejad et al., 2006; Babolian and Shahsavaran; 2009) with high computational costs and large storage capacity or model (Mahmoudi, 2005) with impractical conditions in general case, lies in the fact that the expansion coefficients can be computed algorithmically without evaluating the independent variable (Kazemi, 1996; Kaneko et al., 2003; Maleknejad and Lotfi, 2005; Babolian and Shahsavaran; 2009) or any requirement to redundant operational vector and vector function (Mahmoudi, 2005). Thus, our method may be reckoned as a cheap and accurate solver for integral equations based on orthogonal polynomials.

Table 1. Absolute errors for Example 3.1

t_i	$N = 2$	$N = 4$	$N = 6$
-1.0	4.985309447948438e-002	1.004328462252735e-003	7.142154386019506e-006
-0.8	1.759576087826509e-003	4.045100380887234e-004	1.812817149282608e-006
-0.6	2.846767360847835e-002	1.843740711369168e-004	2.349736868190178e-006
-0.4	3.426378701625354e-002	2.673281180076703e-004	1.824023998509006e-008
-0.2	2.402447544707376e-002	3.433596111147574e-004	2.209651693174664e-006
0.0	3.705981679884784e-003	3.094137594139568e-005	1.381050315218957e-007
0.2	1.941672292683716e-002	3.272926702755363e-004	2.249326875869429e-006
0.4	3.645796829031300e-002	3.252658055907798e-004	2.519121500288435e-007
0.6	3.656477246333578e-002	1.499283565222775e-004	2.564063457555221e-006
0.8	6.481273361322426e-003	4.824575942268261e-004	1.828749566445964e-006
1.0	6.998327552762440e-002	1.204253212619744e-003	8.158223572252155e-006

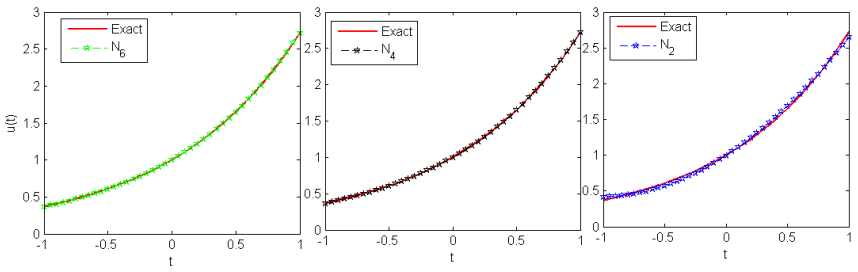

Fig. 1. Comparison plot of exact and approximation solution of Example 3.1, for N=2, 4 and 6

Table 2. Absolute errors for Example 3.2

t_i	*N = 2*	*N = 4*	*N = 6*
-1.0	2.226326862717143e-001	1.156312547339200e-002	1.585432956718869e-004
-0.8	2.704963272221794e-002	1.614745215159363e-003	4.097514427281457e-005
-0.6	7.543738261191890e-002	5.918714028074756e-005	5.111269450774891e-005
-0.4	9.873217778378740e-002	3.501780277965771e-003	2.952098078878862e-006
-0.2	6.449045712060453e-002	3.381778210274855e-003	4.848712811389433e-005
0.0	0	0	0
0.2	6.449045712060461e-002	3.381778210274855e-003	4.848712811389433e-005
0.4	9.873217778378735e-002	3.501780277965771e-003	2.952098078878862e-006
0.6	7.543738261191879e-002	5.918714028063654e-005	5.111269450785994e-005
0.8	2.704963272221794e-002	1.614745215159363e-003	4.097514427281457e-005
1.0	2.226326862717143e-001	1.156312547339200e-002	1.585432956718869e-004

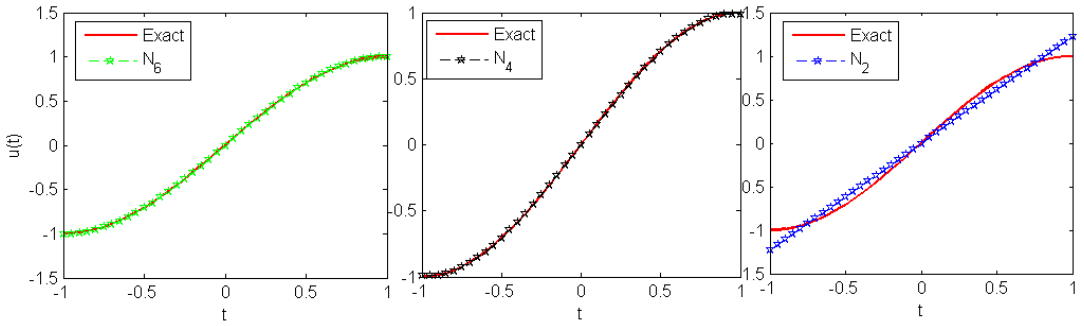

Fig. 2. Comparison plot of exact and approximation solution of Example 3.2, for N=2, 4 and 6

Table 3. Absolute errors for Example 3.3

t_i	*N = 2*	*N = 4*	*N = 6*
-1.0	6.114637826600777e-002	1.007226482430568e-003	7.142541026405880e-006
-0.8	7.182258258249030e-003	4.022114717202596e-004	1.812637060893962e-006
-0.6	2.191242883679023e-002	1.826774958223476e-004	2.349833650883149e-006
-0.4	3.013027195289242e-002	2.684198300844098e-004	1.829614970638716e-008
-0.2	2.234783022597919e-002	3.438430367994627e-004	2.209626592808434e-006
0.0	4.521346434996243e-003	3.081232507717857e-005	1.381096741415178e-007
0.2	1.607420806158011e-002	3.280393708791074e-004	2.249292343936560e-006
0.4	3.055316318097057e-002	3.266365281899830e-004	2.518480561874981e-007
0.6	2.806253697596839e-002	1.479258245749016e-004	2.564151238448886e-006
0.8	4.653532638009050e-003	4.798138344495051e-004	1.828667472558720e-006
1.0	8.378579217285997e-002	1.207549465869207e-003	8.158206790120914e-006

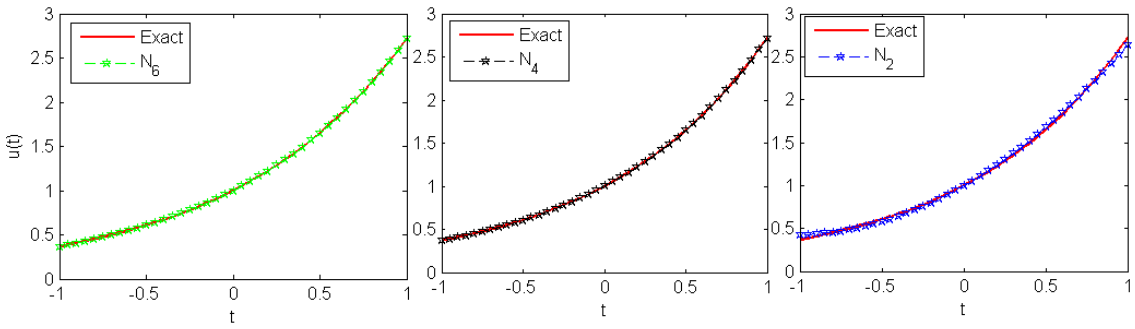

Fig. 3. Comparison plot of exact and approximation solution of Example 3.3, for N=2, 4 and 6

Table 4. Absolute errors for Example 3.4

t_i	$N = 2$	$N = 4$	$N = 6$
-1.0	1.344011965602714e-001	7.337915747843371e-003	0
-0.8	9.323421997845260e-002	2.351515844836061e-004	4.440892098500626e-016
-0.6	1.255635793743202e-001	1.504943471576703e-004	0
-0.4	6.204288162733052e-002	2.290566424352880e-004	1.110223024625157e-016
-0.2	1.476787326251638e-002	3.437378955982728e-004	1.110223024625157e-016
0.0	5.072468529522056e-002	1.999424033731212e-004	0
0.2	2.701155447078174e-002	4.423101718640243e-004	0
0.4	3.755551921079992e-002	2.093421871821155e-004	0
0.6	8.883253574952432e-002	1.209226642779004e-004	0
0.8	4.425949514539118e-002	2.745804949899955e-004	2.220446049250313e-016
1.0	1.956196026015979e-001	6.845054366512504e-003	4.440892098500626e-016

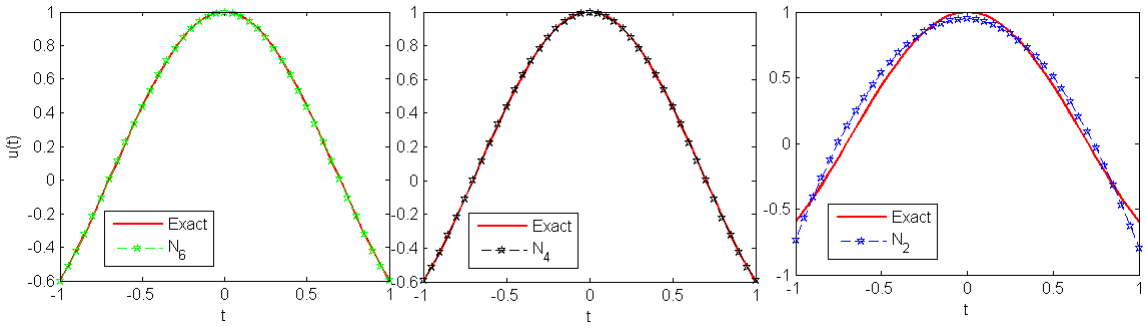

Fig. 4. Comparison plot of exact and approximation solution of Example 3.4, for N=2, 4 and 6

5. Conclusions

In this paper, we have proposed a new computational method for solution of Fredholm-Hammerstein integral equation. This method offers several advantages in reducing computational costs. On the other hand, this method is very simple to apply and to make an algorithm. Thus, this method may be reckoned as a simple and accurate solver for integral equations and it is worthy to note that this method can be utilized as an accurate algorithm to solve linear and nonlinear integro-differential equations and functional integral equations arising in physics and other fields of applied mathematics.

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