

---

## Generalized double sinh-Gordon equation: Symmetry reductions, exact solutions and conservation laws

G. Magalakwe<sup>1</sup>, B. Muatjetjeja<sup>2</sup> and C. M. Khalique<sup>2\*</sup>

<sup>1</sup>*School of Computer, Statistical and Mathematical Sciences, North-West University, Potchefstroom Campus, Private Bag X 2520, Republic of South Africa*

<sup>2</sup>*International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X 2046, Mmabatho 2735, Republic of South Africa*

*E-mails: Gabriel.Magalakwe@nwu.ac.za, Ben.Muatjetjeja@nwu.ac.za & Masood.Khalique@nwu.ac.za*

---

### Abstract

This paper aims to study a generalized double sinh-Gordon equation, which appears in several physical phenomena such as integrable quantum field theory, kink dynamics and fluid dynamics. Lie symmetry analysis together with the simplest equation method is used to obtain exact solutions for this equation. Moreover, we derive conservation laws for the equation by using two different approaches, namely, the direct method and the new conservation theorem due to Ibragimov.

**Keywords:** Generalized double sinh-Gordon equation; Lie symmetry analysis; simplest equation method; conservation laws

---

### 1. Introduction

Physical phenomena in physics and other fields are often described by nonlinear partial differential equations (NLPDEs). Finding exact solutions of NLPDEs is one of the most important task, since they provide a better understanding of the physical phenomena. During the past several decades researchers have developed numerous methods to find exact solutions of NLPDEs. Some of the methods found in the literature are the Hirota bilinear method (Hirota, 2004; Ma et al., 2012; Zhang and Khalique, 2014), the dynamical system method (Li, 2013; Zhang et al., 2013; Zhang and Chen, 2009), the  $F$ -expansion method (Wang and Li, 2005), the homogeneous balance method (Wang et al., 1996), the  $(G'/G)$ -expansion method (Wang et al., 2008), the Weierstrass elliptic function expansion method (Chen and Yan, 2006), the exponential function method (He and Wu, 2006), the tanh function method (Wazwaz, 2004), the extended tanh function method (Wazwaz, 2007), the simplest equation method (Kudryashov, 2005; Vitanov, 2010) and the Lie group analysis method (Bluman and Kumei, 1989; Olver, 1993; Ibragimov, 1994–1996).

In this paper we study the generalized double sinh-Gordon equation

$$u_{tt} - ku_{xx} + 2\alpha \sinh(nu) + \beta \sinh(2nu) = 0, \quad n \geq 1, \quad (1)$$

where  $k$ ,  $\alpha$  and  $\beta$  are non-zero real constants, which appear in several physical phenomena such as integrable quantum field theory, kink dynamics and fluid dynamics. It should be noted that when  $n=k=1$ ,  $\alpha=1/2$  and  $\beta=0$ , (1) reduces to the sinh-Gordon equation (Wazwaz, 2005). Furthermore, if  $k=a$ ,  $\alpha=b/2$  and  $\beta=0$ , (1) becomes the generalized sinh-Gordon equation (Wazwaz, 2006). Various methods have been used to study (1). In (Wazwaz, 2005) the tanh method and variable separable ODE method were employed to find the exact solutions of (1). The authors of Tang and Huang (2007) studied the existence of periodic wave, solitary wave, kink and anti-kink wave and unbounded wave solutions of (1) by using the method of bifurcation theory of dynamical systems. The solitary and periodic wave solutions of (1) were obtained in (Kheiri and Jabrari, 2010) by employing  $(G'/G)$ -expansion method. In addition, it was shown that the solutions obtained in (Kheiri and Jabrari, 2010) are more general than those obtained in (Wazwaz, 2005). Recently, in (Magalakwe and Khalique, 2013) new exact solutions of (1) were found by employing exponential function method.

The purpose of this paper is twofold. Firstly, we use the Lie group analysis along with the simplest

---

\*Corresponding author

equation method to obtain exact solutions of the generalized double sinh-Gordon equation (1). Secondly, we derive conservation laws for the equation by using two different techniques, namely, the direct method and the new conservation theorem due to Ibragimov.

The Lie symmetry method is based on symmetry and invariance principles and is a systematic method for solving differential equations analytically. It was first developed by Sophus Lie (1842-1899) and since then has become an essential mathematical tool for anyone investigating mathematical models of physical, engineering and natural problems. The Lie group analysis methods are presented in many books, see for example (Bluman and Kumei, 1989; Olver, 1993; Ibragimov, 1994–1996).

It is well known that conservation laws play a vital role in the solution process of differential equations (DEs). There is no doubt that the existence of a large number of conservation laws of a system of partial differential equations (PDEs) is an indication of its integrability (Bluman and Kumei, 1989). Comparison of different approaches to conservation laws for some partial differential equations in fluid mechanics is given in (Naz et al., 2008).

The organization of the paper is as follows. In Section 2, we obtain symmetry reductions of the generalized double sinh-Gordon equation (1) using Lie group analysis based on the optimal systems of one-dimensional subalgebras of (1). Exact solutions are also obtained using the simplest equation method. In Section 3 we first recall some definitions and theorems on conservation laws. We then construct conservation laws for (1) using two approaches; the direct method and the new conservation theorem due to Ibragimov. Finally concluding remarks are presented in Section 4.

**2. Symmetry reductions and exact solutions of (1)**

We assume that the vector field of the form  $X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}$  will generate the symmetry group of (1). Applying the second prolongation  $X^{[2]}$  to (1) we obtain an overdetermined system of eight linear partial differential equations, namely

$$\begin{aligned} \xi_u = 0, \tau_u = 0, \eta_{uu} = 0, \xi_t - k\tau_x = 0, \\ \tau_t - \xi_x = 0, \tau_{tt} - k\tau_{xx} - 2\eta_{tu} = 0, \xi_{tt} - k\xi_{xx} + 2k\eta_{xu} = 0, \\ -2\beta n\eta + 4\alpha\tau_t \sinh(nu) - 2\alpha\eta_u \sinh(nu) \\ + 2\alpha n\eta \cosh(nu) + 4\beta n\eta \cosh^2(nu) \\ + 4\beta\tau_t \cosh(nu) \sinh(nu) - 2\beta\eta_u \cosh(nu) \sinh(nu) \\ + \eta_{tt} - k\eta_{xx} = 0. \end{aligned}$$

Solving the above equations one obtains the following three Lie point symmetries:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = kt \frac{\partial}{\partial x} + x \frac{\partial}{\partial t}.$$

*2.1. One-dimensional optimal system of subalgebras*

In this subsection we first obtain the optimal system of one-dimensional subalgebras of (1). Thereafter the optimal system will be used to obtain the optimal system of group-invariant solutions of (1). For this purpose we invoke the method given in (Olver, 1993). Recall that the commutator of  $X_i$  and  $X_j$ , denoted by  $[X_i, X_j]$ , is given by

$$[X_i, X_j] = X_i X_j - X_j X_i$$

and the adjoint transformations are given by

$$\begin{aligned} Ad(\exp(\varepsilon X_i))X_j = X_j - \varepsilon[X_i, X_j] \\ + \frac{1}{2}\varepsilon^2[X_i, [X_i, X_j]] - \dots \end{aligned}$$

The commutator table of the Lie point symmetries of (1) and the adjoint representations of the symmetry group of (1) on its Lie algebra are presented in Table 1 and Table 2, respectively.

**Table 1.** Commutator table of the Lie algebra of system (1)

$[X_i, X_j]$	$X_1$	$X_2$	$X_3$
$X_1$	0	0	$X_2$
$X_2$	0	0	$kX_1$
$X_3$	$-X_2$	$-kX_1$	0

**Table 2.** Adjoint table of the Lie algebra of system (1)

Ad	$X_1$	$X_2$	$X_3$
$X_1$	$X_1$	$X_2$	$X_3 - \varepsilon X_2$
$X_2$	$X_1$	$X_2$	$X_3 - k\varepsilon X_1$
$X_3$	$(\frac{1}{2}k\varepsilon^2 + 1)X_1 + \varepsilon X_2$	$\varepsilon k X_1 + (\frac{1}{2}k\varepsilon^2 + 1)X_2$	$X_3$

Thus, from Tables 1 and 2 and following the method given in (Olver, 1993) one can conclude that an optimal system of one-dimensional subalgebras of (1) is given by  $\{cX_1 + X_2, X_2, X_3\}$ , where  $c$  is a non-zero constant.

## 2.2. Symmetry reductions of (1)

Here the optimal system of one-dimensional subalgebras constructed above will be used to obtain symmetry reductions. Thereafter, we will obtain the exact solutions of (1).

### Case 1. $cX_1 + X_2$

The symmetry generator  $cX_1 + X_2$  gives rise to the group-invariant solution

$$u = W(z), \quad (2)$$

where  $z = x - ct$  is an invariant of the symmetry  $cX_1 + X_2$  and  $W$  is an arbitrary function of  $z$ . The insertion of (2) into (1) yields the ODE

$$(c^2 - k)W''(z) + 2\alpha \sinh(nW(z)) + \beta \sinh(2nW(z)) = 0. \quad (3)$$

Using the transformation

$$W(z) = \frac{1}{n} \ln(H(z)) \quad (4)$$

on (3) we obtain the nonlinear second-order ordinary differential equation

$$2(c^2 - k)H(z)H''(z) - 2(c^2 - k)H'(z)^2 + 2\alpha H(z)^3 - 2\alpha nH(z) + \beta nH(z)^4 - \beta n = 0. \quad (5)$$

The integration of the above equation and reverting back to original variables, yields

$$\begin{aligned} & \pm \int \left[ \frac{1}{k - c^2} (2\alpha n \exp(nu) + 2\alpha n \exp(3nu)) + \frac{1}{2} \beta n \right. \\ & \left. + \frac{1}{2} \beta n \exp(4nu) + c_1 \exp(2nu) \right]^{-\frac{1}{2}} n \exp(nu) du \\ & = x - ct + c_2, \end{aligned}$$

where  $c_1$  and  $c_2$  are constants of integration.

### Case 2. $X_2$

The symmetry operator  $X_2$  results in the group-invariant solution of the form

$$u = W(z), \quad (6)$$

where  $z = x$  is an invariant of  $X_2$  and  $W$  is an arbitrary function satisfying the ODE

$$-kW''(z) + 2\alpha \sinh(nW(z)) + \beta \sinh(2nW(z)) = 0. \quad (7)$$

Again using the transformation (4), equation (7) becomes

$$\begin{aligned} & -2kH(z)H''(z) + 2kH'(z)^2 + 2\alpha nH(z)^3 - 2\alpha nH(z) \\ & + \beta nH(z)^4 - \beta n = 0, \end{aligned} \quad (8)$$

whose solution is

$$\begin{aligned} & \pm \int \left[ \frac{1}{k} (2\alpha n \exp(nu) + 2\alpha n \exp(3nu)) + \frac{1}{2} \beta n \right. \\ & \left. + \frac{1}{2} \beta n \exp(4nu) + c_1 \exp(2nu) \right]^{-\frac{1}{2}} n \exp(nu) du \\ & = x + c_2, \end{aligned}$$

where  $c_1$  and  $c_2$  are constants of integration and we obtain a steady state solution of (1).

### Case 3. $X_3$

The symmetry  $X_3$  gives rise to the group-invariant solution

$$u = W(z), \quad (9)$$

where  $z = x^2 - kt^2$  is an invariant of  $X_3$  and  $W$  satisfies the ODE

$$\begin{aligned} & 4kzW'''(z) - 2kW'(z) + 2\alpha \sinh(nW(z)) \\ & + \beta \sinh(2nW(z)) = 0. \end{aligned}$$

## 2.3. Exact solutions of (1) using simplest equation method

In this subsection we invoke the simplest equation method (Kudryashov, 2005; Vitanov, 2010) to solve the highly nonlinear ODE (5). This will then give us the exact solution for the generalized double sinh-Gordon equation (1). The simplest equations that we will use are the Bernoulli and Riccati equations. For details, see for example (Adem and Khalique, 2013).

### 2.3.1. Solutions of (1) using the Bernoulli equation as the simplest equation

The balancing procedure (Vitanov, 2010) yields  $M = 1$  so the solutions of (5) take the form

$$H(z) = A_0 + A_1 G, \quad (10)$$

where  $G$  satisfies the Bernoulli equation (Adem and Khalique, 2013). Inserting (10) into (5) and using the Bernoulli equation (Adem and Khalique, 2013) and thereafter, equating the coefficients of powers of  $G^i$  to zero, we obtain an algebraic system of five equations in terms of  $A_0, A_1$ , namely

$$\begin{aligned} \beta n A_0^4 - \beta n + 2cn A_0^3 - 2cn A_0 &= 0, \\ 4\beta n A_0^3 A_1 - 2a^2 A_0 A_1 k - 2cn A_1 + 6cn A_0^2 A_1 \\ + 2a^2 A_0 A_1 c^2 &= 0, \\ cn A_0 A_1^2 - a A_0 A_1 b k + \beta n A_0^2 A_1^2 + A_0 A_1 a b c^2 &= 0, \\ \beta n A_1^2 + 2A_1^2 b^2 c^2 - 2A_1^2 b^2 k &= 0, \\ 4\beta A_0 A_1^3 n + 4A_0 A_1 b^2 c^2 - 4A_0 A_1 b^2 k + 2cn A_1^3 \\ + 2A_1^2 a b c^2 - 2k A_1^2 a b &= 0. \end{aligned}$$

Solving the above system of algebraic equations, with the aid of Maple, one possible solution is

$$A_1 = \frac{b(A_0^2 - 1)}{a A_0}, \alpha = \frac{\beta(A_0^2 + 1)}{2A_0}, \beta = -\frac{2b^2(c^2 - k)}{n A_1^2}.$$

Thus, reverting back to the original variables, a solution of (1) is (Adem and Khalique, 2013)

$$\begin{aligned} u(t, x) = \frac{1}{n} \ln(A_0) \\ + A_1 a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b \cosh[a(z+C)] - b \sinh[a(z+C)]} \right\}, \end{aligned} \tag{11}$$

where  $z = x - ct$  and  $C$  is an arbitrary constant of integration.

2.3.2. Solutions of (1) using the Riccati equation as the simplest equation

In this case the balancing procedure yields  $M = 1$ . So the solutions of (5) take the form

$$H(z) = A_0 + A_1 G, \tag{12}$$

where  $G$  satisfies the Riccati equation (Adem and Khalique, 2013). Substituting (12) into (5) and making use of the Riccati equation (Adem and Khalique, 2013), we obtain an algebraic system of equations in terms of  $A_0, A_1$  by equating the coefficients powers of  $G^i$  to zero. The resulting algebraic equations are

$$\begin{aligned} -2A_0 A_1 b k v + 2cn A_0^3 - \beta n + 2A_1^2 k v^2 - 2A_1^2 c^2 v^2 \\ - 2cn A_0 + \beta n A_0^4 + 2A_0 A_1 b c^2 v = 0, \\ -2A_1^2 a^2 k + 2a^2 A_1^2 c^2 + \beta n A_1^4 = 0, \\ -4a A_0 A_1 k v + 4\beta n A_0^3 A_1 + 2A_1^2 b k v + 4a A_0 A_1 c^2 v \\ - 2A_1^2 b c^2 v - 2A_0 A_1 b^2 k + 6cn A_0^2 A_1 - 2cn A_1 \\ + 2A_0 A_1 b^2 c^2 = 0, \\ -abk + \beta n A_0 A_1 + abc^2 + cn A_1 = 0, \\ 2a^2 A_0 A_1 c^2 + a A_1^2 b c^2 - a A_1^2 b k - 2a^2 A_0 A_1 k + cn A_1^3 \\ + 2\beta n A_0 A_1^3 = 0. \end{aligned}$$

Solving the above equations, we get

$$\begin{aligned} A_0 &= \frac{A_1 b + \sqrt{A_1^2 b^2 + 4a^2 - 4A_1^2 a v}}{2a}, \\ \alpha &= \frac{-\beta(A_0^2 a + a - A_1^2 v)}{2a A_0}, \\ \beta &= \frac{-2a(c^2 - k)}{n A_1^2}, \end{aligned}$$

and consequently, the solutions of (1) are (Adem and Khalique, 2013)

$$u(t, x) = \frac{1}{n} \ln(A_0 + A_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left[\frac{1}{2}\theta(z+C)\right] \right\}) \tag{13}$$

and

$$\begin{aligned} u(t, x) = \frac{1}{n} \ln(A_0 + A_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{1}{2}\theta z\right) \right. \\ \left. + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C \cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta} \sinh\left(\frac{\theta z}{2}\right)} \right\}), \end{aligned} \tag{14}$$

where  $z = x - ct$  and  $C$  is an arbitrary constant of integration.

3. Conservation laws of (1)

In this section conservation laws will be derived for (1). However, first we briefly present some notations, definitions and theorems that will be utilized later. For details the reader is referred to (Ibragimov, 2007).

3.1. Preliminaries

We consider a  $k$  th-order system of PDEs

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(k)}) = 0, \quad \alpha = 1, \dots, m, \tag{15}$$

of  $n$  independent variables  $x = (x^1, x^2, \dots, x^n)$  and  $m$  dependent variables  $u = (u^1, u^2, \dots, u^m)$ . Let  $u_{(1)}, u_{(2)}, \dots, u_{(k)}$  denote the collections of all first, second, ...,  $k$  th-order partial derivatives. This means that,  $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \dots$ , respectively, where the total derivative operator with respect to  $x^i$  is given by

$$D_i = \partial / \partial x^i + u_i^\alpha \partial / \partial u^\alpha + u_{ij}^\alpha \partial / \partial u_j^\alpha + \dots, \quad i = 1, \dots, n. \tag{16}$$

Now consider the system of adjoint equations to the system of  $k$  th-order differential equations (15),

which is defined by (Ibragimov, 2007)

$$E_\alpha^*(x, u, v, \dots, u_{(k)}, v_{(k)}) = 0, \quad \alpha = 1, \dots, m, \quad (17)$$

where

$$E_\alpha^*(x, u, v, \dots, u_{(k)}, v_{(k)}) = \frac{\delta(v^\beta E_\beta)}{\delta u^\alpha}, \quad (18)$$

$$\alpha = 1, \dots, m, v = v(x)$$

and  $v = (v^1, v^2, \dots, v^m)$  are new dependent variables.

The system (15) is said to be *self-adjoint* if the substitution of  $v = u$  into the system of adjoint equations (17) yields the same system (15).

If the system (15) admits the symmetry operator

$$X = \xi^i \partial / \partial x^i + \eta^\alpha \partial / \partial u^\alpha \quad (19)$$

then the system of adjoint equations (17) admits the operator

$$Y = \xi^i \partial / \partial x^i + \eta^\alpha \partial / \partial u^\alpha + \eta_*^\alpha \partial / \partial v^\alpha, \quad (20)$$

$$\eta_*^\alpha = -[\lambda_\beta^\alpha v^\beta + v^\alpha D_i(\xi^i)],$$

where the operator (20) is an extension of (19) to the variable  $v^\alpha$  and  $\lambda_\beta^\alpha$  are obtainable from

$$X(E_\alpha) = \lambda_\alpha^\beta E_\beta. \quad (21)$$

**Theorem 3.1.** (Ibragimov, 2007) Every Lie point, Lie-Bäcklund and nonlocal symmetry (19) admitted by the system (15) gives rise to a conservation law for the system consisting of the equation (15) and the adjoint equation (17), where the components  $T^i$  of the conserved vector  $T = (T^1, \dots, T^n)$  are determined by

$$T^i = \xi^i L + W^\alpha \frac{\delta L}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \delta L / \delta u_{i_1 i_2 \dots i_s}^\alpha, \quad (22)$$

$$i = 1, \dots, n,$$

with Lagrangian given by

$$L = v^\alpha E_\alpha(x, u, \dots, u_{(k)}). \quad (23)$$

The differentiation of  $L$  in (22) up to second-order derivative, yields

$$T^i = \xi^i L + W^\alpha \left[ \frac{\partial L}{\partial u_i^\alpha} - D_j \left( \frac{\partial L}{\partial u_{ij}^\alpha} \right) \right] + D_j(W^\alpha) \left( \frac{\partial L}{\partial u_{ij}^\alpha} \right). \quad (24)$$

### 3.2. Construction of conservation laws of (1)

In this section conservation laws will be constructed for (1) by two different methods, namely the direct method and the new conservation theorem.

We recall that the equation (1) admits the following three Lie point symmetry generators:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x \frac{\partial}{\partial t} + kt \frac{\partial}{\partial x}.$$

#### 3.2.1. Application of the direct method

It is well-known that there exists a fundamental relationship between the point symmetries admitted by a given equation and the conservation laws of that equation. Following (Khalique and Mahomed, 2009), we see that the conservation law

$$D_t T^1 + D_x T^2 = 0, \quad (25)$$

which must be evaluated on the partial differential equation, can be considered together with the following requirements:

$$X^{[n]}(T^1) + T^1 D_x(\xi) - T^2 D_x(\tau) = 0, \quad (26)$$

$$X^{[n]}(T^2) + T^2 D_t(\tau) - T^1 D_t(\xi) = 0 \quad (27)$$

in which  $X^{[n]}$  is the  $n$ th prolongation of a point symmetry of the original equation. The order of the extension is equal to the order of the highest derivative in  $T^1$  and  $T^2$ . Consequently, for the given  $X$ , (25)-(27) can be solved to obtain the conserved vectors or tuple  $T = (T^1, T^2)$ .

The condition (25) on the equation (1) gives

$$\frac{\partial T^1}{\partial t} + u_t \frac{\partial T^1}{\partial u} + (ku_{xx} - 2\alpha \sinh(nu) - \beta \sinh(2nu)) \frac{\partial T^1}{\partial u_i}$$

$$+ u_{tx} \frac{\partial T^1}{\partial u_x} + \frac{\partial T^2}{\partial x} + u_x \frac{\partial T^2}{\partial u} + u_{tx} \frac{\partial T^2}{\partial u_t} + u_{xx} \frac{\partial T^2}{\partial u_x} = 0.$$

Since  $T^1$  and  $T^2$  are independent of the second derivatives of  $u$ , it implies that the coefficients of  $u_{tt}, u_{tx}$  and  $u_{xx}$  must be zero. Hence,

$$\frac{\partial T^1}{\partial u_x} + \frac{\partial T^2}{\partial u_t} = 0, \quad (28)$$

$$k \frac{\partial T^1}{\partial u_t} + \frac{\partial T^2}{\partial u_x} = 0, \quad (29)$$

$$\frac{\partial T^1}{\partial t} + u_t \frac{\partial T^1}{\partial u} - (2\alpha \sinh(nu) + \beta \sinh(2nu)) \frac{\partial T^1}{\partial u_t} \quad (30)$$

$$+ \frac{\partial T^2}{\partial x} + u_x \frac{\partial T^2}{\partial u} = 0.$$

We now construct the conservation laws for (1) using the three admitted Lie point symmetries.

We start with the translation symmetry  $X_1 = \partial/\partial t$ , which is already in its extended form. The symmetry conditions (26)-(27) yield

$$\frac{\partial T^1}{\partial t} = 0, \quad \frac{\partial T^2}{\partial t} = 0, \tag{31}$$

respectively. Therefore from (28)-(30) and (31) the components of the conserved vector of (1) associated with the symmetry  $X_1$  are given by

$$T^1 = \frac{c_4 u_t^2}{k} + c_4 u_x^2 + c_5 u_x + \frac{2c_4}{k} \left[ \frac{2\alpha \cosh(nu)}{n} + \frac{\beta \cosh(2nu)}{2n} \right] + j(x) + c_6,$$

$$T^2 = -2c_4 u_t u_x - c_5 u_t + c_7,$$

where  $c_4, c_5, c_6$  and  $c_7$  are arbitrary constants and  $j(x)$  is an arbitrary function of  $x$ .

Continuing in the same manner using  $X_2$  and  $X_3$  we obtain the components of the conserved vector for equation (1) as

$$T^1 = \frac{c_4 u_t^2}{k} + c_4 u_x^2 + c_5 u_x + \frac{2c_4}{k} \left[ \frac{2\alpha \cosh(nu)}{n} + \frac{\beta \cosh(2nu)}{2n} \right] + c_8$$

$$T^2 = -2c_4 u_t u_x - c_5 u_t + p(t),$$

and

$$T^1 = c_5 u_x + c_6 x,$$

$$T^2 = -c_5 u_t + c_6 k t,$$

respectively, where  $c_4, c_5, c_6$  and  $c_8$  are constants and  $p(t)$  is an arbitrary function of  $t$ . However, we note that the symmetry  $X_3$  gives a trivial conserved vector.

### 3.2.2. Application of the new conservation theorem

In this subsection we use the new conservation theorem given in (Ibragimov, 2007) and construct conservation laws for (1). For applications of this theorem, see for example (Tracina et al., 2014; Gandarias and Khaliq, 2014; da Silva and Freire, 2014). The adjoint equation of (1), by invoking (18), is

$$E^*(t, x, u, v, \dots, u_{xx}, v_{xx}) = \frac{\delta}{\delta u} [v(u_{tt} - k u_{xx} + 2\alpha \sinh(nu) + \beta \sinh(2nu))] = 0, \tag{32}$$

where  $v = v(t, x)$  is a new dependent variable. Thus from (32) we have

$$v_{tt} - k v_{xx} + 2nv[\alpha \cosh(nu) + \beta \cosh(2nu)] = 0. \tag{33}$$

It is clear from the adjoint equation (33) that equation (1) is not self-adjoint. By recalling (23), we obtain the Lagrangian for the system of equations (1) and (33) as

$$L = v[u_{tt} - k u_{xx} + 2\alpha \sinh(nu) + \beta \sinh(2nu)]. \tag{34}$$

(i) We first consider the Lie point symmetry generator  $X_1 = \partial/\partial t$ . It can easily be seen from (20) that the operator  $Y_1$  is the same as  $X_1$  and that the Lie characteristic function  $W = -u_t$ . Thus, by using (24), the components  $T^i, i = 1, 2$ , of the conserved vector  $T = (T^1, T^2)$  are given by

$$T^1 = v(-k u_{xx} + 2\alpha \sinh(nu) + \beta \sinh(2nu)) + u_t v_t,$$

$$T^2 = -k u_t v_x + k v u_{tx}.$$

**Remark:** The conserved vector  $T$  contains the arbitrary solution  $v$  of the adjoint equation (33) and hence gives an infinite number of conservation laws. This remark also applies to the two cases given below.

(ii) For the symmetry  $X_2 = \partial/\partial x$ , we have  $W = -u_x$ . Thus, by using (24), the symmetry generator  $X_2$  gives rise to the following components of the conserved vector:

$$T^1 = v_t u_x - v u_{tx},$$

$$T^2 = v(u_{tt} + 2\alpha \sinh(nu) + \beta \sinh(2nu)) - k v_x u_x.$$

(iii) The symmetry  $X_3 = x\partial/\partial t + kt\partial/\partial x$  has the Lie characteristic function  $W = -x u_t - k t u_x$ . Thus, invoking (24), we obtain the conserved vector  $T$ , given by

$$T^1 = xv(-k u_{xx} + 2\alpha \sinh(nu) + \beta \sinh(2nu)) + xv_t u_t + k t v_t u_x - k v u_x - k t v u_{tx},$$

$$T^2 = k t v(u_{tt} + 2\alpha \sinh(nu) + \beta \sinh(2nu)) - k x v_x u_x - k^2 t v_x u_x + k v u_t + k x v u_{tx}.$$

## 4. Concluding remarks

We have studied the generalized double sinh-Gordon equation (1) using the Lie symmetry analysis. Symmetry reductions based on the optimal

systems of one-dimensional subalgebras of (1) and exact solutions with the help of simplest equation method were obtained. These exact solutions obtained here are different from the ones obtained in (Wazwaz, 2005; Wazwaz, 2006; Wazwaz, 2005; Tang and Huang, 2007; Kheiri and Jabrari, 2010; Magalakwe and Khalique, 2013). Also, the correctness of the solutions obtained here has been verified by substituting them back into (1). Finally, conservation laws for (1) were derived by employing two different methods; the direct method and the new conservation theorem. The usefulness of conservation laws was discussed in the introduction.

### Acknowledgements

G. M. would like to thank SANHARP, NRF and North-West University, Mafikeng Campus, South Africa for their financial support.

### References

- Adem, K. R., & Khalique, C. M. (2013). Exact Solutions and Conservation Laws of a (2+1)-Dimensional Nonlinear KP-BBM Equation. *Abstract and Applied Analysis*, Article ID 791863.
- Bluman, G. W., & Kumei, S. (1989). *Symmetries and Differential Equations*. New York: Springer-Verlag.
- Chen, Y., & Yan, Z. (2006). The Weierstrass elliptic function expansion method and its applications in nonlinear wave equations. *Chaos Solitons and Fractals*, 29, 948–964.
- Da Silva, P. L., & Freire, I. L. (2014). Strict self-adjointness and certain shallow water models. arXiv:1312.3992v2 [math-ph] 25 Mar 2014.
- Gandarias, M. L., & Khalique, C. M. (2014). Nonlinearly Self-Adjoint, Conservation Laws and Solutions for a Forced BBM Equation. *Abstract and Applied Analysis*, Article ID 630282.
- He, J. H., & Wu, X. H. (2006). Exp-function method for nonlinear wave equations. *Chaos Solitons and Fractals*, 30, 700–708.
- Hirota, R. (2004). *The Direct Method in Soliton Theory*. Cambridge: Cambridge University Press.
- Ibragimov, N. H. (Ed.), (1994–1996). *CRC Handbook of Lie Group Analysis of Differential Equations*. Vols.1–3, Boca Raton, FL: CRC Press.
- Ibragimov, N. H. (2007). A new conservation theorem. *Journal of Mathematical Analysis and Applications*, 333, 311–328.
- Khalique, C. M., & Mahomed, F. M. (2009). Soil water redistribution and extraction flow models: Conservation laws. *Nonlinear Analysis: Real World Applications*, 10, 2021–2025.
- Kheiri, H., & Jabrari, A. (2010). Exact solutions for the double sinh-Gordon and generalized form of the double sinh-Gordon equations by using  $G'/G$ -expansion method. *Turkish Journal of Physics*, 34, 73–82.
- Kudryashov, N. A. (2005). Simplest equation method to look for exact solutions of nonlinear differential equations. *Chaos Solitons and Fractals*, 24, 1217–1231.
- Li, J. (2013). *Singular Traveling Wave Equations: Bifurcations and Exact Solutions*. Beijing, China: Science Press.
- Ma, W.-X., Zhang, Y., Tang, Y., & Tu, J. (2012). Hirota bilinear equations with linear subspaces of solutions. *Applied Mathematics and Computation*, 218, 7174–7183.
- Magalakwe, G., & Khalique, C. M. (2013). New exact solutions for a generalized double sinh-Gordon equation. *Abstract and Applied Analysis*, Article ID 268902.
- Naz, R., Mahomed, F. M., & Mason, D. P. (2008). Comparison of different approaches to conservation laws for some partial differential equations in fluid mechanics. *Applied Mathematics and Computation*, 205, 212–230.
- Olver, P. J. (1993). *Applications of Lie Groups to Differential Equations, Graduate Texts in Mathematics*. Berlin: Springer-Verlag. 2nd edition.
- Tang, S., & Huang, W. (2007). Bifurcation of travelling wave solutions for the generalized double sinh-Gordon equation. *Applied Mathematics and Computation*, 189, 1774–1781.
- Tracinà, R., Bruzón, M. S., Gandarias, M. L., & Torrisi, M. (2014). Nonlinear self-adjointness, conservation laws, exact solutions of a system of dispersive evolution equations. *Communications in Nonlinear Science Numerical Simulation*, 19, 3036–3043.
- Vitanov, N. K. (2010). Application of simplest equations of Bernoulli and Riccati kind for obtaining exact traveling-wave solutions for a class of PDEs with polynomial nonlinearity. *Abstract and Applied Analysis*, 15, 2050–2060.
- Wang, M., & Li, X. (2005). Applications of  $F$ -expansion to periodic wave solutions for a new Hamiltonian amplitude equation. *Chaos Solitons and Fractals*, 24, 1257–1268.
- Wang, M., Xiangzheng, L. X., & Jinliang, Z. J. (2008). The  $(G'/G)$ -expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Physics Letter A*, 372, 417–423.
- Wang, M., Zhou, Y., & Li, Z. (1996). Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics. *Physics Letter A*, 216, 67–65.
- Wazwaz, A. M. (2006). The tanh function method for traveling wave solutions of nonlinear equations. *Applied Mathematics and Computation*, 154, 713–723.
- Wazwaz, A. M. (2005). Exact solutions to the double sinh-Gordon equation by the tanh method and a variable separated ODE method. *Computers and Mathematics with Application*, 50, 1685–1696.
- Wazwaz, A. M. (2005). The tanh method: exact solutions of the sine-Gordon and the sinh-Gordon equations. *Applied Mathematics and Computation*, 167, 1196–1210.
- Wazwaz, A. M. (2006). Exact solutions for the generalized sine-Gordon and generalized sinh-Gordon equations. *Chaos Solitons and Fractals*, 28, 127–135.

- Wazwaz, A. M. (2007). The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations. *Applied Mathematics and Computation*, 184, 1002–1014.
- Zhang, L., & Chen, L. Q. (2009). Envelope compacton and solitary pattern solutions of generalized nonlinear Schrödinger equation, *Nonlinear Analysis: Theory, Methods & Applications*, 70, 492–496.
- Zhang, L., Chen, L. Q., & Huo, X. (2013). The effects of horizontal singular straight line in a generalized nonlinear Klein–Gordon model equation. *Nonlinear Dynamics*, 72, 489–901.
- Zhang, L., & Khalique, C. M. (2014). Traveling Wave Solutions and Infinite-Dimensional Linear Spaces of Multiwave Solutions to Jimbo-Miwa Equation. *Abstract and Applied Analysis*, Article ID 963852.