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On the detection and estimation of the simple harmonizable processes

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Abstract

Simple harmonizable processes (SHP) introduced by Soltani and Parvardeh (2006) are a large class of nonstationary processes which includes stationary and periodically correlated (PC) processes. Detection and estimation of SHP structure are important problems when dealing with nonstationary data. In this paper, we study the spectral properties of simple processes and propose a method to detect and estimate SHP structure. As an example, we discuss the detection, estimation and prediction of periodically correlated processes. The performance of this method is investigated through extensive Monte Carlo simulations. This method is compatible with other method.

Keywords: Periodically correlated; simple random measure; simple processes; simulation

1. Introduction

Research activities on nonstationary processes are growing. A large class of nonstationary processes is simple processes which were introduced and studied by Soltani and Parvardeh (2006). Simple harmonizable processes form a large class of harmonizable processes that includes stationary and harmonizable periodically correlated (cyclostationary) processes. Assume $\{B_1, \dots, B_m\}$ are a partition for $[0,2\pi)$, and $T_j: B_1 \to B_j$, $j = 1, \dots, m$ are one-to-one and measurable mappings, $T_1(x) = x$. Also assume $\mathbf{\Psi} = (\Psi_1, \dots, \Psi_m)$ is an independently scattered multivariate random measure supported by B_1.

Then the simple random measure Φ is defined by

$$\Phi(dx) = \mathbf{\Psi} o T_j^{-1}(dx)\ , x \in B_j\ , j = 1, \dots, m.$$

A discrete time harmonizable simple process (DTHSP) is defined by

$$X(t) = \int_0^{2\pi} e^{itx} \Phi(dx) \quad , t \in \mathbb{Z},$$

where $\mathbb{Z}$ is the set of integers and Φ is a simple random measure on $[0,2\pi)$.

We assume the spectral density $\mathbf{f}(x) = E\big(\mathbf{\Psi}(x)\mathbf{\Psi}^*(x)\big)$ is full rank with respect to lebesgue measure. (*Shows the conjugate transpose).

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Let $B_j = \left[\frac{2\pi(j-1)}{T}, \frac{2\pi j}{T}\right)$, $T_j(x) = \alpha_j + \beta_j x$, where $\alpha_j = \frac{2\pi(j-1)}{T}$, and $\beta_j = 1$, $j = 1, \dots, m$. The resulted simple process is a periodically correlated process with period m (PC-m in short), and conversely each PC-m has the above representation (Soltani and Parvardeh (2006)).

The theory of PC processes was introduced by Gladyshev (1961). These processes are applied in various areas of researches, such as climatology, hydrology, engineering, signal processing and economics (Gardner et al. (2006); Hurd and Miamee (2007)).

We summarize the content of this paper as follows: In Section 2, notations and preliminaries are provided. We study simple random measures, simple processes and their spectral properties. In Section 3, we propose a procedure to detect and estimate SHP structure. Simulation study is provided in Section 4.

2. Preliminaries

Soltani and Parvardeh (2006) showed a simple random measure has the following spectral representation:

$$\Phi(A) = \int_0^{2\pi} h(x,A)\Lambda(dx) \quad , A \subset [0,2\pi],$$

where Λ is an independently scattered random measure on $[0,2\pi)$, $E|\Lambda(dx)|^2 = \frac{dx}{2\pi}$, and $h(x, A)$ is given by

$$\sum_{k=1}^{m}\sum_{j=1}^{m} a_{jk}(T_k^{-1}(x)) I_{(A\cap B_j)}(T_j T_k^{-1}(x)) I_{B_k}(x),$$

where $\mathbf{S}(x) = [a_{jk}(x)]$, $x \in B_1$, is square root of $\mathbf{f}(x)$, i. e.,
$\mathbf{f}(x) = \mathbf{S}(x)\mathbf{S}^*(x)$.

The $h(x, A)$ defines a kernel that for $x \in B_k$, $h(x,.)$ is supported by the atoms $T_j T_k^{-1}(x), j = 1, \dots, m$, with corresponding masses $a_{jk}(T_k^{-1}(x))$. We refer to $h(.,.)$ as the spectral kernel of the simple random measure Φ. Assume the Cholesky decomposition for the density $\mathbf{f}$, is given by

$$a_{jk}(x) = \begin{cases} a_{j-k}\left(T_j(x)\right) & j > k \\ 0 & j \le k \end{cases},$$

then the kernel $h(x, A)$ will be given by

$$\sum_{k=1}^{m}\sum_{j=k}^{m} a_{j-k}\left(T_j T_k^{-1}(x)\right) I_{(A\cap B_j)}(T_j T_k^{-1}(x)) I_{B_k}(x),$$

and for $x \in B_k$, $h(x,.)$ is supported by the atoms $T_j T_k^{-1}(x), j = k, \dots, m$, with corresponding masses $a_{j-k}(T_k^{-1}(x))$.

Therefore a DTHSP has the following spectral representation,

$$X(t) = \int_D \int_D e^{ity} h(x, dy) \Lambda(dx)$$

$$= \int_D \sum_{k=1}^{m}\sum_{j=1}^{m} e^{itT_jT_k^{-1}(x)} a_{jk}\left(T_k^{-1}(x)\right) I_{B_k}(x)\, \Lambda(dx)$$

$$= \sum_{k=1}^{m} \int_{B_k} \sum_{j=1}^{m} e^{itT_jT_k^{-1}(x)} a_{jk}\left(T_k^{-1}(x)\right) \Lambda(dx).$$

In the next section, we apply a procedure to estimate the functions $T_j(x)$, $j = 2, \dots, m$.

3. Estimating Procedure

Assume $X(0), \dots, X(N-1)$ are a path from $X(t)$. Let

$$d_X(\lambda) = N^{-1/2} \sum_{t=0}^{N-1} X_t e^{it\lambda}, \lambda \in [0,2\pi)$$

be the discrete finite Fourier transform (FFT) of the finite sequence $X(0)$, ..., $X(N-1)$.

We apply the following procedure to estimate the functions $T_j(x)$, $j = 2, \dots, m$.

Since X_t is harmonizable, $d_X(\lambda)$ estimates $\Phi(d\lambda)$. Also, from SHP property (Soltani and Parvardeh (2006)),
$E\left(\Phi(d\lambda)\overline{\Phi(d\lambda')}\right) = 0$, for $\lambda' \ne T_j(\lambda), \lambda \in B_1, j = 2, \dots, m$.
Consider the joint spectral coherency as

$$C(\lambda, \lambda') = \frac{|g(\lambda, \lambda')|}{\sqrt{|g(\lambda, \lambda) g(\lambda', \lambda')|}}, \lambda, \lambda' \in [0,2\pi),$$

where $g(\lambda, \lambda') = E\left(\Phi(d\lambda)\overline{\Phi(d\lambda')}\right)$.

The joint spectral coherency can be estimated by

$$\hat{C}(\lambda, \lambda') = \hat{\rho} = Corr(|d_X(\lambda)|, |d_X(\lambda')|),$$

where $\hat{\rho}$ is the sample correlation.

For PC processes (Hurd (1989)), $C(\lambda, \lambda') = 0$, for $\lambda' \ne T_j(\lambda)$, $\lambda \in B_1, j = 2, \dots, m$.

In application we will use the bootstrap estimation method to produce more samples for the finite Fourier transforms. For $\lambda^* \in B_j$, the estimation for $T_j(\lambda)$ is $\lambda \in B_1$ so that $\hat{C}(\lambda, \lambda')$ attains its maximum on $B_1 \times B_j$ at (λ, λ').

Outline of T_j's estimation:

(i) The block bootstrap methodology is used to produce a sample of size n for $d_X(\lambda)$, using $X(0), \dots, X(N-1)$ for given $\lambda \in [0,2\pi)$.

(ii) $\hat{C}(\lambda, \lambda')$ is calculated for $\lambda \in B_1$ and $\lambda' \in B_j, j = 2, \dots, m$, using $\{d_1(\lambda), \dots, d_n(\lambda)\}$ and $\{d_1(\lambda'), \dots, d_n(\lambda')\}$.

(iii) We fix $\lambda^* \in B_j$ and find $\lambda \in B_1$ such that it maximizes $\hat{C}(\lambda, \lambda')$.

(iv) Step (iii) is repeated to find $\lambda_1, \dots, \lambda_J \in B_1$ corresponding to $\lambda_1^*, \dots, \lambda_J^* \in B_j$.

(v) We let $\lambda_k^* = \hat{T}_j(\lambda_k), k = 1, \dots, J$;that estimates $T_j, j = 2, \dots, m$.

4. Simulation study

In this section, we report simulation results for the proposed procedure. In order to investigate the performance of the procedure, we generate $N = 100, 200, 500$ *and* 1000 observations from first order periodic autoregressive (PAR(1)) process X_t,

$$X(t) = \phi(t)X(t-1) + \varepsilon(t),$$

where $\phi(t) = 0.6 + 0.4\cos(2\pi t/m)$, m is the period and $\varepsilon(t)$ is standard normal white noise. The Monte Carlo simulations are based on 10000 repetitions for different values of $m = 2, 3, 5$. Also, Moving Blocks Bootstrap (MBB) method is used to generate more samples (B=10). Table 1 shows the results for the estimation procedure. The columns show real value (T_j), mean estimated value $(\hat{T}_j)$, mean absolute error (MAE) and mean square error

(MSE) of $\hat{T}_j$ in second Fourier frequency $(2\pi/N)$.

As can be seen in Table 1, the estimated values are very close to real values. Also the values of MAE and MSE are very close to zero. These results show that the proposed method performs well in estimation procedure, especially as N grows.

Table 1. Estimation results based on proposed method

m		N	Real Value	Estimated Value	MAE	MSE
		100	3.20	3.19	0.00214	4.10079e-05
2	T_2	**200**	3.17	3.17	0.00162	2.64198e-06
		500	3.15	3.15	0.00092	8.59883e-07
		1000	3.14	3.14	0.00078	3.65745e-07
	T_2	**100**	2.15	2.16	0.00234	6.16785e-05
		200	2.12	2.13	0.00207	9.64198e-06
3		**500**	2.10	2.12	0.00198	8.98675e-07
		1000	2.10	2.08	0.00106	5.65435e-07
		100	4.25	4.27	0.00546	7.22318e-05
	T_3	**200**	4.22	4.20	0.00453	1.76567e-05
		500	4.20	4.21	0.00298	6.98468e-06
		1000	4.19	4.18	0.00187	3.23753e-06
		100	1.31	1.27	0.00765	6.94543e-05
	T_2	**200**	1.28	1.29	0.00546	9.97875e-06
		500	1.26	1.27	0.00324	4.32675e-06
		1000	1.26	1.26	0.00187	1.31234e-06
5		**100**	2.57	2.54	0.00654	7.56435e-05
	T_3	**200**	2.54	2.52	0.00534	3.09870e-05
		500	2.52	2.51	0.00341	9.07776e-06
		1000	2.51	2.52	0.00198	6.15466e-06
	T_4	**100**	3.83	3.79	0.00876	7.89876e-05
		200	3.80	3.82	0.00657	5.45786e-05
		500	3.78	3.80	0.00543	1.13465e-05
		1000	3.77	3.79	0.00291	9.98234e-06
		100	5.08	5.07	0.01045	9.54324e-05
	T_5	**200**	5.05	5.06	0.00891	7.76549e-05
		500	5.03	5.04	0.00677	2.98512e-05
		1000	5.03	5.03	0.00293	1.05609e-05

To compare of our method with the other method, we apply the proposed method to predict PC processes. Soltani and Parvardeh (2006) showed the best predictor of $\hat{X}(t+\tau), \tau > 0$ is given by

$$\hat{X}(t+\tau) = \sum_{k=1}^{m} \hat{X}_k(t+\tau),$$

where

$$\hat{X}_k(t) = \sum_{l=-\infty}^{+\infty} (\hat{g}_{t,k})(l) Z_{k,l},$$

$$\hat{g}_{t,k}(x) = \sum_{j=1}^{m} e^{itT_j(x)} a_{jk}(x),$$

and $\{Z_{k,l}\}, k = 1, \dots, m$ are orthogonal white noise series.

We replace T_j by $\hat{T}_j$. Also $a_{jk}(x)$ can be estimated as follows:

Soltani and Azimmohseni (2007) defined the periodogram of the finite PC sequence $X(0), \dots, X(N-1)$, as follows:

$$\boldsymbol{I}_X^T(\lambda) = \boldsymbol{d}_X^T(\lambda) \boldsymbol{d}_X^{T^*}(\lambda),$$

where $\boldsymbol{d}_X^T(\lambda)$ is given by

$$\left(d_X(T_1(\lambda)), d_X(T_2(\lambda)) \dots, d_X(T_m(\lambda))\right)', \lambda \in B_1 = \left[0, \frac{2\pi}{m}\right).$$

They showed that $\hat{\mathbf{f}}(\lambda) = \frac{\mathbf{I}_X^T(\lambda)}{2\pi}$, is an asymptotically unbiased estimator for $\mathbf{f}(\lambda)$, $\lambda \in B_1$. Therefore we can estimate $\mathbf{S}(x)$ as the square root of $\hat{\mathbf{f}}(x)$.

We use these results to compute $\hat{X}(N)$ based on $\{X(1), \dots, X(N-1)\}$ and compare our method with the method that was introduced by Hurd (2005; 2007).

The results are summarized in Table 2. The first column shows the value of $X(N)$. Other columns present predicted value $(\hat{X}(N))$, empirical 95% lower bound (LB), empirical 95% upper bound (UB), MAE and MSE, respectively. The results show the values of MAE and MSE for the proposed method is smaller than the other method, especially for large values of N. Also, the empirical confidence interval based on the proposed method has smaller length and it seems that the proposed method performs better than the other method.

Table 2. Prediction results based on proposed and Hurd methods

M	N	Method	$X(N)$	$\hat{X}(N)$	LB	UB	MAE	MSE
2	100	**Proposed**	0.60	0.60	0.52	0.68	0.031	0.0015
		Hurd	0.60	0.60	0.52	0.68	0.031	0.0015
	200	**Proposed**	0.60	0.60	0.53	0.67	0.028	0.0012
		Hurd	0.60	0.60	0.52	0.67	0.032	0.0016
	500	**Proposed**	0.65	0.65	0.59	0.71	0.023	0.0008
		Hurd	0.65	0.65	0.57	0.73	0.031	0.0015
	1000	**Proposed**	0.62	0.62	0.57	0.67	0.020	0.0006
		Hurd	0.62	0.62	0.54	0.70	0.034	0.0015
3	100	**Proposed**	1.63	1.63	1.55	1.71	0.031	0.0015
		Hurd	1.63	1.63	1.55	1.71	0.031	0.0016
	200	**Proposed**	1.60	1.60	1.53	1.67	0.027	0.0011
		Hurd	1.60	1.60	1.52	1.68	0.031	0.0015
	500	**Proposed**	1.68	1.68	1.62	1.73	0.023	0.0008
		Hurd	1.68	1.68	1.60	1.75	0.031	0.0015
	1000	**Proposed**	1.67	1.67	1.62	1.72	0.019	0.0006
		Hurd	1.67	1.67	1.60	1.75	0.031	0.0015
5	100	**Proposed**	2.15	2.15	2.07	2.23	0.032	0.0016
		Hurd	2.15	2.15	2.07	2.23	0.032	0.0016
	200	**Proposed**	2.15	2.15	2.08	2.22	0.027	0.0012
		Hurd	2.15	2.15	2.07	2.23	0.031	0.0016
	500	**Proposed**	2.13	2.13	2.07	2.18	0.024	0.0009
		Hurd	2.13	2.13	2.05	2.20	0.032	0.0016
	1000	**Proposed**	2.05	2.05	2.00	2.10	0.019	0.0006
		Hurd	2.05	2.05	1.97	2.13	0.031	0.0015

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