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On the characteristic of projectively invariant Pseudo-distance on Finsler spaces

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Abstract

A projective parameter of a geodesic as solution of certain ODE is defined to be a parameter which is invariant under projective change of metric. Using projective parameter and Poincaré metric, an intrinsic projectively invariant pseudo-distance can be constructed. In the present work, solutions of the above ODE are characterized with respect to the sign of parallel Ricci tensor on a Finsler space. Moreover, the Ricci tensor is used to define a Finsler structure and it is shown that, the pseudo-distance is trivial on complete Finsler spaces of positive semi-definite Ricci tensor and it is a distance on a Finsler space of parallel negative definite Ricci tensor.

***Keywords**:* Finsler metric; Schwarzian derivative; Ricci tensor; projective parameter; pseudo-distance

1. Introduction

If any geodesic of two Finsler spaces $(M,F)$and $(M,\bar{F})$ coincide as a set of points, then Fand d_F are said to be projectively related. It is well-known that two Finsler spaces are projectively related, if and only if there is a 1-homogeneous scalar field $P(x,y)$ called the projective factor satisfying $\bar{G}^i(x,y) = G^i(x,y) + P(x,y)y^i$, where G^i and $\bar{G}^i$are the corresponding spray vector fields. In general the parameter "t" of a geodesic $\gamma := x^i(t)$ on (M,F) does not remain invariant under projective change of metrics. A parameter which remains invariant under projective change of metrics is called projective parameter. The projective parameter is defined first for geodesics of general affine connection, (Thomas, 1925; Eisenhart, 1927; Berwald, 1937). In (Bidabad & Sepasi, 2015), this parameter is carefully spelled out for geodesics of Finsler spaces by the present authors as solutions of the following differential equation.

$$\{p,s\} := \frac{\frac{d^3p}{ds^3}}{\frac{dp}{ds}} - \frac{3}{2}\left[\frac{\frac{d^2p}{ds^2}}{\frac{dp}{ds}}\right]^2 = \frac{2}{n-1}\mathrm{Ric}_{jk}\left(x(s),\frac{dx}{ds}\right)\frac{dx^j}{ds}\frac{dx^k}{ds}, \tag{1}$$

where, $\{p,s\}$ is known in the literature as Schwarzian derivative and "s" is the arc-length parameter of γ. The Schwarzian derivative is defined to be an operator which is invariant under all linear fractional transformations $t \to \frac{at+b}{cf+d}$ where, $ad - bc \neq 0$. That is

$$\left\{\frac{af+b}{cf+d}, t\right\} = \{f,t\}. \tag{2}$$

By means of the equation (2), the projective parameter is unique up to all linear fractional transformations. In (Bidabad & Sepasi, 2015), the projective parameter and the Poincaré metric are used to define a projectively invariant pseudo-distance denoted by $d_M$and it is shown that in a complete Einstein-Finsler space with negative constant Ricci scalar, the projectively invariant pseudo-distance is a constant multiple of the Finslerian distance. Recall that two Finsler structures F and $\bar{F}$ are said to be homothetic if there is a constant λ such that $F = \lambda\bar{F}$; as a corollary, the following results are obtained;

Theorem A. Let (M,F) and $(M,\bar{F})$ be two complete Einstein Finsler spaces with $\mathrm{Ric}_{ij} = -c^2 g_{ij}$, and $\overline{\mathrm{Ric}}_{ij} = -\bar{c}^2\bar{g}_{ij}$, respectively, if F and $\bar{F}$ are projectively related then they are homothetic.

Theorem B. Let $(M,F)$and $(M,\bar{F})$ be two complete Finsler spaces of constant negative flag curvature, if

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F and $\bar{F}$ are projectively related then they are homothetic.

The last result is also obtained by Shen (2001) using another technique of proof.

In the present work, solutions of the differential equation (1) are characterized with respect to the parallel Ricci tensor in any of the Berwald, Chern or Cartan connections as follows.

Theorem 1. Let (M, F) be a Finsler space of parallel Ricci tensor. Then the Ricci tensor is constant along geodesics, and solutions of (1) are classified as follows.

i) If $\{p, s\} = c^2$, with $c > 0$, then

$$p = \frac{\alpha\cos(cs) + \beta\sin(cs)}{\gamma\cos(cs) + \delta\sin(cs)}. \tag{3}$$

ii) If $\{p, s\} = -c^2$ with $c > 0$, then

$$p = \frac{\alpha e^{cs} + \beta e^{-cs}}{\gamma e^{cs} + \delta e^{-cs}}. \tag{4}$$

iii) If $\{p, s\} = 0$, then

$$p = \frac{\alpha + \beta s}{\gamma + \delta s}. \tag{5}$$

Next a new approach to the study of pseudo-distances is established and the following results are obtained.

Theorem 2. Let (M, F) be a connected complete Finsler space of positive semi-definite Ricci tensor. Then the intrinsic projectively invariant pseudo-distance is trivial, that is, $d_M = 0$.

Theorem 3. Let (M, F) be a connected (complete) Finsler space of negative-definite parallel Ricci tensor with respect to the Berwald or Chern connection. Then the intrinsic projectively invariant pseudo-distance d_M is a (complete) distance.

These results are generalizations of Riemannian works (Kobayashi, 1978) and (Kobayashi & Sasaki, 1978) and establish a new approach to the study of projective geometry in Finsler spaces.

2. Preliminaries

A (globally defined) Finsler structure on a differential manifold M is a function on the tangent bundle $F: TM \to [0, \infty)$ with the following properties, i) Regularity: F is C^∞ on the entire slit tangent bundle TM_0, ii) Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$, iii) Strong convexity: The Hessian matrix $(g_{ij}) := ([1/2F^2]_{y^i y^j})$ is positive-definite on $TM_0 = TM\backslash 0$. The pair (M, F) is known as a Finsler space.

Every Finsler structure F induces a spray $G = y^i \frac{\partial}{\partial x^i} - G^i(x, y)\frac{\partial}{\partial y^i}$ on TM, where

$$G^i(x, y) := \frac{1}{2} g^{il}\left\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\right\}.$$

G is a globally defined vector field on TM. The projection of a flow line of G on M is called a geodesic. The differential equation of a geodesic in the local coordinates is given by $\frac{d^2x^i}{ds^2} + G^i\left(x(s), \frac{dx}{ds}\right) = 0$, where the parameter$(t) = \int_{t_0}^{t} F(\gamma, \frac{d\gamma}{dr})dr$is arc length parameter. Everywhere in this work, the differential manifold M is supposed to be connected.

For a non-null, $y \in T_xM$, the Riemann curvature $Ry: T_xM \to T_xM$ is defined by $R_y(u) = R^i_k u^k \frac{\partial}{\partial x^i}$, where $R^i_k(y) := \frac{\partial G^i}{\partial x^k} - \frac{1}{2}\frac{\partial^2 G^i}{\partial y^k \partial x^j} y^j + G^j \frac{\partial^2 G^i}{\partial y^k \partial y^j} - \frac{1}{2}\frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}$. The Ricci scalar is defined by $Ric := R^i_i$, see for instance (Akbar-Zadeh, 1988) or (Bao, Chern, & Shen, 2000). In the present work, we use the definition of Ricci tensor introduced by Akbar-Zadeh as, $Ric_{ik} := 1/2(F^2 Ric)_{y^i y^k}$. Moreover, by homogeneity we have $Ric_{ik} l^i l^k = Ric$.

Let us consider the spray $G^i := \gamma^i_{jk} y^j y^k$, where $\gamma^i_{jk} := \frac{1}{2} g^{is}(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j})$, $N^i{}_j := \frac{1}{2}\frac{\partial G^i}{\partial y^j}$, $l^i := \frac{y^i}{F}$ and, $\hat{l} := \frac{\delta}{\delta x^i} = l^i\left(\frac{\partial}{\partial x^i} - N^k{}_i \frac{\partial}{\partial y^k}\right)$, (Bao, Chern and Shen 2000).

3. Projective parameter on Ricci parallel Finsler spaces

Let the Ricci tensor of (M, F) be parallel with respect to any of the Cartan, Berwald or Chern connections. We recall the Abel's identity in ordinary differential equations.

Consider the second-order linear ordinary differential equation:

$$\ddot{y} + P(x)\dot{y} + Q(x)y = 0. \tag{6}$$

Consider the two linearly independent solutions, $y_1(x)$ and $y_2(x)$. Then, the Wronskian of y_1 and y_2, $w(y_1, y_2) := \dot{y_1}y_2 - y_1\dot{y_2}$ satisfies $\dot{w} + Pw = 0$, therefore,

$$w = w_0 e^{-\int P(x)dx}. \tag{5}$$

Proposition 1. If y_1 and $y_2$are linearly independent solutions of the ordinary differential equation

$$\ddot{y} + Q(s)y(s) = 0, \tag{8}$$

where $Q(s) = \frac{1}{n-1} Ric_{jk}\left(x(s), \frac{dx}{ds}\right)\frac{dx^j}{ds}\frac{dx^k}{ds}$,
then the general solution of (1) is given by

$$u(t) = \frac{\alpha y_1 + \beta y_2}{\gamma y_1 + \delta y_2}, \tag{9}$$

where, $\alpha\delta - \beta\gamma \neq 0$.

Proof: According to (2), it suffices to show that y_1/y_2 is a solution of (1). The term $P(x)$ in (7) is zero, hence the Wronskian $w(y_1, y_2)$ is constant. We may assume that
$w(y_1, y_2) = 1$. Then, $\dot{u} = 1/y_2^2$, $\frac{\ddot{u}}{\dot{u}} = -2\frac{\dot{y_2}}{y_2}$ and
$\dot{\left(\frac{\ddot{u}}{\dot{u}}\right)} = \frac{-2\ddot{y_2}y_2 + 2(\dot{y_2})^2}{y_2^2} = -2\frac{\ddot{y_2}}{y_2}\ 2\left(\frac{\dot{y_2}}{y_2}\right)^2$, thus

$$\frac{\dddot{u}\dot{u} - (\ddot{u})^2}{(\dot{u})^2} = -2\,\frac{(-Q(s))y_2(s)}{y_2(s)} + \frac{1}{2}\left(\frac{\ddot{u}}{\dot{u}}\right)^2,$$

$$\frac{\dddot{u}}{\dot{u}} - \left(\frac{\ddot{u}}{\dot{u}}\right)^2 = 2\,Q(s) + \frac{1}{2}\left(\frac{\ddot{u}}{\dot{u}}\right)^2,$$

$$\frac{\dddot{u}}{\dot{u}} - \frac{3}{2}\left(\frac{\ddot{u}}{\dot{u}}\right)^2 = 2\,Q(s).$$

This completes proof of the proposition.

Proof of Thoerem 1. Assume that the Ricci tensor is parallel with respect to the Cartan connection. Let us denote the horizontal and vertical Cartan covariant derivatives by $\nabla^c{}_{\frac{\delta}{\delta x^k}}$ and $\nabla^c{}_{\frac{\partial}{\partial y^k}}$ respectively. Hence

$$\nabla^c{}_{\frac{\delta}{\delta x^k}} Ric_{ij} = \frac{\delta Ric_{ij}}{\delta x^k} - Ric_{ir}\Gamma^r_{jk} - Ric_{jr}\Gamma^r_{ik} = 0, \tag{10}$$

$$\nabla^c{}_{\frac{\partial}{\partial y^k}} Ric_{ij} = \frac{\partial Ric_{ij}}{\partial y^k} - Ric_{ir}\frac{A^r_{jk}}{F} - Ric_{jr}\frac{A^r_{ik}}{F} = 0, \tag{11}$$

where, $\Gamma^i_{jk} = \frac{1}{2}g^{is}\left(\frac{\delta g_{sj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} + \frac{\delta g_{ks}}{\delta x^j}\right)$ and $A^i_{jk} := g^{ih}A_{hjk} = g^{ih}\frac{F}{4}\frac{\partial g_{hj}}{\partial y^k}$are the coefficients of Cartan tensor. Consider the geodesic, $\gamma := x^i(s)$, where"s" is the arc-length
parameter. Contracting (10) by $\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}(Ric_{ir}\Gamma^r_{jk})$ gives

$$\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}\left(\frac{\partial Ric_{ij}}{\partial x^k} - N^l{}_k\frac{\partial Ric_{ij}}{\partial y^l}\right) - \frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}\left(Ric_{ir}\Gamma^r_{jk}\right) - \frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}(Ric_{jr}\,\Gamma^r_{ik}) = 0.$$

By means of (11) and, $y^jA^i_{jk} = 0$, we have

$$\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dRic_{ij}}{ds} - \frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}N^l{}_k\left(Ric_{ir}\frac{A^r_{jl}}{F} + Ric_{jr}\frac{A^r_{il}}{F}\right) - 2\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}Ric_{jr}\Gamma^r_{ik} = 0.$$

Therefore

$$\frac{dRic_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}}{ds} - 2Ric_{ij}\frac{d^2x^i}{ds}\frac{dx^j}{ds} - 0 + 2Ric_{ij}\frac{d^2x^i}{ds}\frac{dx^j}{ds} = 0.$$

and we have

$$Ric_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = \text{constant}. \tag{12}$$

Following the method just used, we can prove that if the Ricci tensor is parallel with respect to the Berwald or Chern connection, then along the geodesic γ parameterized by arc-length, we have $Ric_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = \text{constant}$.

Considering the above assertion and Lemma 1, the equation (1) reduces to a second order ODE with constant coefficients. Therefore, sign of Ricci tensor explicitly determines a projective parameter "p" as an elementary function of "s" given by (3), (4)and (5). This completes the proof of Theorem.

4. Positive semi-definite Ricci tensor

Let $I = (-1, 1)$ be an open interval with the Poincaré metric $ds_I^2 = \frac{4du^2}{(1-u^2)^2}$. The Poincaré distance between two points a and b in I is given by

$$\rho(a, b) = \left|\ln\frac{(1-a)(1+b)}{(1-b)(1+a)}\right|, \tag{13}$$

(Okada, 1983). A geodesic $f : I \to M$ on the Finsler space (M, F) is said to be a projective map, if the natural parameter u on I is a projective parameter. We now come to the main step for defining the pseudo-distance d_M on (M, F). We proceed in analogy with the treatment of Kobayashi in Riemannian geometry, (Kobayashi, 1978). Although he has confirmed that the construction of intrinsic pseudo-distance is valid for any manifold with an affine connection, or more generally a projective connection (Kobayashi, 1977), we restrict our consideration to the pseudo-distances induced by the Finsler structure F on a connected manifold M. Given any two points x and yin (M, F), we consider a chain αof geodesic segments joining these points. That is

- a chain of points $x = x_0, x_1, \dots, x_k = y$ on M;
- pairs of points $a_1, b_1, \dots, a_k, b_k$ in I;

- projective maps $f_1, \dots, f_k$, $f_i : I \to M$, such that
$f_i(a_i) = x_{i-1}$, $f_i(b_i) = x_i$, where, $i = 1, \dots, k$.

By virtue of the Poincaré distance $\rho(.,.)$on I we define the length $L(\alpha)$ of the chain α by $L(\alpha) := \sum_i \rho(a_i, b_i)$, and we put

$$d_M(x, y) := \inf L(\alpha), \tag{14}$$

where the infimum is taken over all chains $\alpha$of geodesic segments from x to y.

Proposition 2. Let (M, F) be a Finsler space. Then for any points x, y, and z in M, d_M satisfies
i. $d_M(x, y) = d_M(y, x)$.
ii. $d_M(x, z) \le d_M(x, y) + d_M(y, z)$.
iii. If $x = y$ then $d_M(x, y) = 0$, but the inverse is not always true.

Traditionally, $d_M(x, y)$ is called the pseudo-distance of any two points x and y on M. From the property (2) of Schwarzian derivative, and the fact that the projective parameter is invariant under fractional transformation, the pseudo-distance d_M is projectively invariant.

Proof of Theorem 2. In order to prove Theorem 2 we need the following Lemmas.

Lemma 3. Let (M, F) be a complete Finsler space. Consider two points $x_0$and x_1 on M. If there exists a geodesic $x(u)$ with projective parameter u, $-1 < u < +1$, such that$x_0 = x(u_0)$ and $x_1 = x(u_1)$ for some u_0 and u_1 in $\mathbb{R}$ then

$$d_M(x_0, x_1) = 0.$$

Proof: Linear equation of the segment passing through the points $(u_0, -1/2)$and $(u_1, 1/2)$ is given by $\hat{u} = \frac{u}{u_1 - u_0} - \frac{1}{2}\frac{(u_1 + u_0)}{u_1 - u_0}$. Here, $\hat{u}$ is a linear transformation of u and is also a projective parameter. We have $-\frac{1}{2} < \hat{u} < \frac{1}{2}$ whenever $u_0 < u < u_1$.

Next, we consider the chain α of projective maps, a_n and b_n where

$$f_n = x(n\hat{u}), \quad a_n = -\frac{1}{2n}, \quad b_n = \frac{1}{2n}.$$

Note that $f_n\left(-\frac{1}{2n}\right) = x\left(n\left(-\frac{1}{2}n\right)\right) = x\left(-\frac{1}{2}\right) = x(u_0)$ and $\rho\left(-\frac{1}{2n}, \frac{1}{2n}\right) = \left|\ln\frac{\left(1+\frac{1}{2n}\right)\left(1+\frac{1}{2n}\right)}{\left(1-\frac{1}{2n}\right)\left(1-\frac{1}{2n}\right)}\right|$.

Considering n sufficiently large, we have $d_M(x_0, x_1) = \inf L(\alpha) = 0$. This completes the proof of Lemma 3.

Lemma 4. Let (M, F) be a complete Finsler space and $x(s)$ a geodesic with arc-length parameter $-\infty < s < \infty$. Assume that there exists a (finite or infinite) sequence of open intervals $I_i = (a_i, b_i), i = 0, \pm 1, \pm 2, \dots$ such that;
i) $a_{i+1} \le b_i, \lim_{i \to -\infty} a_i = -\infty, \lim_{i \to +\infty} b_i = +\infty$ and $\bigcup_i \bar{I}_i = (-\infty, +\infty)$;
ii) in each interval $I_i = (a_i, b_i)$, a projective parameter "u" moves from $-\infty$ to $+\infty$ whenever t moves from a_i to b_i. Then, for any pair of points x_0 and $x_1$on this geodesic, we have

$$d_M(x_0, x_1) = 0.$$

Proof: By means of Lemma1, the distance between any two points in the same interval I_i is zero. Two consecutive open intervals I_i and I_{i+1} have either a point as a boundary point or an interval in common. In each case, given $\epsilon > 0$, there exist the points S_i and S_{i+1} in I_i and I_{i+1} respectively such that $d_M(x(s_i), x(s_{i+1})) < \epsilon$. This completes the proof of Lemma 4.

The following Lemmas permit us to construct the open intervals I_i in Lemma 3. The proofs are given in (Kobayashi & Sasaki, 1978).

Lemma 5. In the ODE (8), if $Q(s) = 0$ for all $s \in \mathbb{R}$ then every solution $y(s)$ has at least one zero unless $Q(s) = 0$ and $y(s)$ is constant $c \neq 0$.

In the sequel the Sturm's separation theorem which claims; given a homogeneous second order linear differential equation and two continuous linear independent is needed.

Solutions $v(x)$ and $u(x)$ with x_0 and x_1 successive roots of $v(x)$, so $u(x)$ has exactly one root in the open interval (x_0, x_1).

Lemma 6. Let $y_1(s)$and $y_2(s)$ be two linearly independent solutions of (8). If a and b are two consecutive zeros of y_2 then $u = y_1(s)/y_2(s)$ or $u = -y_1(s)/y_2(s)$ is a projective parameter on the interval (a, b) which moves from $-\infty$ to $+\infty$ as s moves from a to b.

The differential equation (8) is said to be oscillatory at $s = \pm\infty$ if the zeros

$$\dots < a_{-2} < a_{-1} < a_0 < a_1 < a_2 < \cdots,$$

of the solution $y(s)$ have the property $\lim_{h \to -\infty} a_h = \infty$ and $\lim_{h \to +\infty} b_h = +\infty$. Then the sequence of intervals $I_i = (a_i, b_i)$ satisfies the condition of Lemma 3. This fact proves Theorem 2 in this case.

Next, we consider the case that (8) is nonoscillatory at $s = +\infty$. That is, $y_2(s)$ does not vanish for sufficiently large s. According to the Sturm's theorem, this condition is independent of choice of a particular solution $y_2(s)$.

Lemma 7. If the differential equation (8) is non-oscillatory at $s = +\infty$, then there is a solution $y_2(s)$ which is uniquely determined up to a constant factor satisfying

$$\lim_{s\to+\infty} \frac{y_2(s)}{y_1(s)} = 0, \tag{15}$$

for any solution $y_1(s)$ linearly independent of $y_2(s)$.

The solution $y_2(s)$ in Lemma 6 is called a principal solution. Here, we consider a weaker version of comparison Theorem of Sturm as follows.

Lemma 8. Consider two differential equations
(i) $\ddot{y}(s) + Q_1(s)y(s) = 0,$
(ii) $\ddot{y}(s) + Q_2(s)y(s) = 0,$
with $Q_1(s) > Q_2(s)$. Let $y_1(s)$ and $y_2(s)$ be solutions of (i) and (ii) respectively such that

$$\frac{\dot{y}_1(a)}{y_1(a)} \leq \frac{\dot{y}_2(a)}{y_2(a)}. \tag{16}$$

If $y_1(s)$ and $y_2(s)$ have no zero in the interval $a < s < +\infty$, then for $s > a$

$$\frac{\dot{y}_1(s)}{y_1(s)} \leq \frac{\dot{y}_2(s)}{y_2(s)}. \tag{17}$$

If $y_2(a) = 0$, then the term $\frac{\dot{y}_2(a)}{y_2(a)}$ is considered to be ∞.

One can refer to (Du and Kwnog, 1990) and (Kobayashi and Sasaki, 1978) for more details about this subject.

Lemma 9. Assume that the differential equation (8) is nonoscillatory at $s = +\infty$ and that $Q(s) \geq 0$. Let $y(s)$ be a principal solution as in Lemma 6. If a is the largest zero of $y_2(s)$ and if $y_1(s)$ is a solution linearly independent of $y_2(s)$, then $y_1(s)$ vanishes at some $s > a$.

We are now in a position to complete the proof of the theorem 2 where the differential equation (8) is nonoscillatory at $s = +\infty$, or $s = -\infty$.
If (8) is non-oscillatory at $s = +\infty$ but oscillatory at $s = -\infty$, we take a principal solution $y_2(s)$ and another solution $y_1(s)$ linearly independent of $y_2(s)$. Let $\ldots < a_{-2} < a_{-1} < a_0 < a_1 < a_2 < \cdots$, be the zeros of $y_2(s)$. Then the sequence of intervals, $I_i = (a_i, b_i)$, for $i = \cdots, -2, -1, 0, 1, 2, \ldots, k$ with $a_{k+1} = +\infty$, equipped with a projective parameter $u = \frac{y_1}{y_2}$ or $u = -\frac{y_1}{y_2}$ satisfy the requirements of Lemma 3. We note that Lemma 8 implies that u is a projective parameter in the last interval $I_k = (a_k, +\infty)$. If (8) is nonoscillatory at $s = -\infty$ but oscillatory at $s = +\infty$, we replace Lemma 6 and Lemma 8 by the analogous Lemmas for $s = -\infty$. Assume that (8) is nonoscillatory at $\pm\infty$. Let $y_2(s)$ be a principal solution for $s = +\infty$ and not for $s = -\infty$. Let $y_1(s)$ be a principal solution for $s = -\infty$. then $y_1(s)$ and $y_2(s)$ are linearly independent. We obtain a sequence of intervals I_i, $i = 0, 1, \ldots, k$ with a projective parameter $u = \frac{y_1}{y_2}, u = -\frac{y_1}{y_2}, u = \frac{y_2}{y_1}$ or $u = -\frac{y_2}{y_1}$ satisfying the requirements of Lemma
3. In this case, there are some overlaps among these intervals.

If $y_2(s)$ is a principal solution for both $s = +\infty$ and $s = -\infty$ then we consider $y_1(s)$ as a solution linearly independent of $y_2(s)$. We obtain a sequence of intervals I_i, $i = 0, 1, \ldots, k$, with a projective parameter $u =$ or $u = -\frac{y_1}{y_2}$ satisfying the requirements of Lemma 3. In this case, there are no overlaps of intervals. This completes the proof of Theorem2.

5. Parallel negative-definite Ricci tensor

We recall the following theorem which will be used in the sequel.

Theorem A. Let (M, F) be a connected (complete) Finsler space for which the Ricci tensor satisfies, $Ric_{ij} \leq -c^2 g_{ij}$, as matrices, for a positive constant c. Then d_M is a (complete) distance. (Bidabad & Sepasi, 2015).

Proof of Theorem 3. Let us consider the Finsler structure defined by means of the Ricci tensor as $\hat{F}(x, y) = \sqrt{-Ric_{ij}(x, y)y^i y^j}$.
One can easily check that $\hat{F}$ satisfies all properties of a Finsler structure on M. More precisely;
i) by definition $\hat{F}$ is C^∞ on the entire slit tangent bundle TM_0;
ii) The Ricci tensor $Ric_{ij}(x, y)$ is 0-homogeneous, hence $\hat{F}(x, \lambda y) = \lambda \hat{F}(x, y)$ for all $\lambda > 0$;
iii) again, according to the 0-homogeneity of $Ric_{ij}(x, y)$, through straightforward calculation we get, $\hat{g}_{ij} = \left[\frac{1}{2}\hat{F}^2\right]_{y^i y^j} = -Ric_{ij}(x, y)$.

The Ricci tensor is supposed to be negative-definite thus the Hessian matrix $(\hat{g}_{ij})$ is positive-definite. Next, we show that the spray coefficients of $\hat{F}$ and F are equal, that is, $\hat{G}^i = G^i$, hence we have

$$\widehat{G}^i = \frac{1}{2}(-\mathrm{Ric})^{ih}\left(\frac{\partial^2\widehat{F}^2}{\partial y^h\,\partial x^j}y^j - \frac{\partial\widehat{F}^2}{\partial x^h}\right)$$
$$= \frac{1}{2}(-\mathrm{Ric})^{ih}\left(\frac{\partial^2\left(-\mathrm{Ric}_{lr}y^l y^r\right)}{\partial y^h\,\partial x^j}y^j - \frac{\partial\left(-\mathrm{Ric}_{lr}y^l y^r\right)}{\partial x^h}\right)$$
$$= \frac{1}{2}(-\mathrm{Ric})^{ih}\left(-2\frac{\partial\left(\mathrm{Ric}_{hl}y^l\right)}{\partial x^j}y^j + \frac{\partial(\mathrm{Ric}_{lr})}{\partial x^h}y^l y^r\right)$$
$$= \ \mathrm{Ric}^{ih}\frac{\partial\left(\mathrm{Ric}_{hl}y^l\right)}{\partial x^j}y^j - \frac{1}{2}\,\mathrm{Ric}^{ih}\frac{\partial(\mathrm{Ric}_{lr})}{\partial x^h}y^l y^r. \qquad (18)$$

Let Ricci tensor be parallel with respect to the Berwald connection ∇^b. Similar arguments hold well for Chern connection. We have

$$\nabla^b{}_{\frac{\delta}{\delta x^j}}\mathrm{Ric}_{hl} = \frac{\delta \mathrm{Ric}_{hl}}{\delta x^j} - \mathrm{Ric}_{hr}G^r{}_{lj} - \mathrm{Ric}_{lr}G^r{}_{hj} = 0,$$
$$G^r{}_{lj} = \frac{1}{2}\frac{\partial^2 G^r}{\partial y^l\,\partial y^j}. \qquad (19)$$

$$\nabla^b{}_{\frac{\partial}{\partial y^k}}\mathrm{Ric}_{ij} = \frac{\partial \mathrm{Ric}_{ij}}{\partial y^k} = 0. \qquad (20)$$

Contracting (19) with $\mathrm{Ric}^{ih}y^j y^l$ leads to

$$\mathrm{Ric}^{ih}y^j y^l\frac{\partial \mathrm{Ric}_{hl}}{\partial x^j} - \mathrm{Ric}^{ih}\mathrm{Ric}_{ha}G^a - \frac{1}{2}\mathrm{Ric}^{ih}\mathrm{Ric}_{la}\frac{\partial G^a}{\partial y^h}y^l = 0.$$
$$\mathrm{Ric}^{ih}y^j y^l\frac{\partial \mathrm{Ric}_{hl}}{\partial x^j} - G^i - \frac{1}{2}\mathrm{Ric}^{ih}\mathrm{Ric}_{la}\frac{\partial G^a}{\partial y^h}y^l = 0. \quad (21)$$

On the other hand

$$-\frac{1}{2}\mathrm{Ric}^{ih}y^r y^l\frac{\partial \mathrm{Ric}_{lr}}{\partial x^h} + \frac{1}{2}\,\mathrm{Ric}^{ih}y^r y^l\mathrm{Ric}_{la}G^a{}_{rh} + \frac{1}{2}\mathrm{Ric}^{ih}y^r y^l\mathrm{Ric}_{ra}G^a{}_{lh} = 0.$$
$$-\frac{1}{2}\mathrm{Ric}^{ih}y^r y^l\frac{\partial \mathrm{Ric}_{lr}}{\partial x^h} + \frac{1}{2}\,\mathrm{Ric}^{ih}y^r\mathrm{Ric}_{ra}\frac{\partial G^a}{\partial y^h} = 0. \quad (22)$$

Considering, (18), (21) and (22) we have $\widehat{G}^i = G^i$. As a consequence, we have $\widehat{\mathrm{Ric}}_{ij} = \mathrm{Ric}_{ij}$. On the other hand, we just asserted that $\hat{g}_{ij}(x,y) = -\mathrm{Ric}_{ij}(x,y)$. Thus, we have $\hat{g}_{ij}(x,y) = -\widehat{\mathrm{Ric}}_{ij}(x,y)$. According to TheoremA, $\hat{d}_M$is a (complete) distance. The two spaces (M,F) and $(M,\widehat{F})$ are affine and we have $d_M = \hat{d}_M$. Hence, d_M is a (complete)
distance.

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