An investigation of radial basis function approximation methods with application in dynamic investment model

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Abstract

The present study is an attempt to investigate some features of Radial Basis Functions (RBFs) approximation methods related to variational problems. Thereby authors applied some properties of RBFs to develop a direct method which reduces constrained variational problem to a static optimization problem. To assess the applicability and effectiveness of the method, some examples are examined. Dynamic investment problem with free endpoint in unbounded domain is solved, accordingly the effectiveness of the proposed method is verified. To improve the accuracy and stability of the method we have used various shape parameter strategies with equally spaced and scattered centers. Finally, two new shape parameter strategies are proposed and then it is shown that the proposed strategies increase the accuracy and stability of the method.

Keywords: Gaussian RBF; multiquadric RBF; constrained variational problems; dynamic investment model; shape parameter

1. Introduction

In the previous decades, some computational aspects of global RBF and local RBF interpolation have been considered by engineers, scientists and mathematicians in different fields such as neural networks, solving ODEs, PDEs and integral equations (IEs). For instance, Franke and Schaback (1998) used RBFs for solving PDEs, Golbabai and Seifollahi (2006) used RBFs to solve the second kind of IEs and Golbabai and Saeedi (2014) applied RBF method to solve national saving model.

Indeed, for smooth problems, RBFs create exponential convergence rate (Cheney, 2000). (Schaback, 1995 and 2005; Madych, 1992; Madych and Nelson, 1992) have collaborated in finding error bounds of some RBFs. The implementation of RBF based approaches to solve functional equation is straightforward and the accuracy and efficiency of these methods have made them popular.

Though the calculus of variations and its variant's studies began over 300 years ago, scientists are still interested in it. Its various applications are abundant in fields including geometry, differential equations and in diverse areas such as mechanics, economics and, renewable resources. It is evident that to analytically find optimal solutions for many problems is impossible. Thus numerical techniques are an indispensable tool for solving many variational problems.

Special attention has been given to applications of Walsh functions (Chen and Hsiao, 1975), Block-pulse functions (Hwang and Shih, 1985) , Legendre polynomials (Chang and Wang, 1983), Chebyshev polynomials (Horng and Chou, 1985), Triangular orthogonal functions (Babolian et. al, 2007) and Walsh-Wavelets (Glabisz, 2004) for solving a variational problem.

In order to attain a more comprehensive and deeper insight into RBF features, the use of RBF approximation method for solving constrained variational problems with fixed and free boundaries is examined in the current research.

The present paper tries to develop a new method that facilitates reduction of variational problem into a system of algebraic equations via expanding the candidate functions as a linear combination of RBFs with unknown coefficients.

The paper is organized as follows: In section 2, some fundamental properties of few RBFs are described. Besides some existing shape parameter strategies are described including variable shape parameter (VSP) and constant shape parameter, which have previously been utilized in interpolation and solving some PDEs, as well as two new VSPs, introduced in this work to perform the efficiency of the method. Then in sec. 3 the variational problem and its variants are described.

Section 4 deals with the RBF method for constrained variational problem and well-definedness of the reduced static optimization problem. In section 5, the previously described method is applied to some test problems as well as dynamic investment problem and it is verified that the RBF direct method provides a more accurate and more stable solution for the problems.

2. RBF approximation

Given a set of centers $X_1,...,X_N$ in R^d, the RBF interpolation takes the form

$$y(X) = \sum_{j=1}^{N} \alpha_j \phi(\|X - X_j\|_2, c) \tag{1}$$

where $r = \|X\|_2 = \sqrt{x_1^2 + ... + x_d^2}$.
Common choices of RBFs are Gaussian (GA), $\phi(r) = \exp(-cr^2)$ Multiquadric (MQ), $\phi(r) = \sqrt{c^2 + r^2}$ and Inverse Multiquadric (IMQ), $\phi(r) = \dfrac{1}{\sqrt{c^2 + r^2}}$. The coefficients α are chosen
by enforcing the interpolation condition

$$y(X_i) = f(X_i) \tag{2}$$

at a set of grid points that typically coincide with the centers. Applying the interpolation conditions at N centers leads to a $N \times N$ linear system:

$$W\alpha = f \tag{3}$$

to be solved for expansion coefficients α where the entries of the interpolation matrix W are given by $W_{ij} = \phi_j(X_i) = \phi(\|X_i - X_j\|_2, c), i, j = 1,...,N$, $\alpha = [\alpha_1,...,\alpha_N]^T$, $y = [y_1,...,y_N]^T$.

It is obvious that the interpolation problem (2) will be well posed if and only if the matrix W is nonsingular.

In the RBF literature for example Fasshuar (2007) there has been enough discussion around the positive definite and conditionally positive definite functions which leads to positive definite matrix W that guarantees well posedness of interpolation problem (2).

For positive definite radial basis functions an extension of Bochner's theorem guarantees the invertibility of the matrix W, for any set of distinct grid points (also non-uniformly spaced and in R^d, $d > 1$. Characterization of positive definite functions in terms of Fourier transforms was established by Bochner (Fasshuar, 2007).

Theorem 2.1. A (complex valued) function $\Phi \in C(R^d)$ is positive definite on R^d if and only if it is the Fourier transform of finite non-negative Borel measure μ on R^d, i.e.

$$\Phi(x) = \mu(x) = \frac{1}{\sqrt{(2\pi)^d}} \int_{R^d} e^{-ix.y} d\mu(y), \tag{4}$$

$x \in R^d$.

Following theorem enables us to obtain a criterion for checking conditional positive definiteness of radial functions.

Theorem 2.2. (Cheney, 2000) Let $\phi \in C[0, \infty] \cap C\infty(0, \infty)$. Then the function $\Phi = \phi(\|.\|^2)$ is conditionally positive definite radial functions on R^d for all d if and only if $(-1)^m \phi^{(m)}$ is completely monotone on $(0, \infty)$.

Also, for GA-RBFs using (1) it is easy to obtain the following properties that facilitate computing via RBFs:

$$y^N(x) = \alpha^T \Phi(x) \tag{5}$$

where

$$\alpha^T = [\alpha_1,...,\alpha_N], \Phi(x) = [\phi_1(x),...\phi_N(x)].$$

It is easy to see

$$\frac{d}{dx} y^N(x) = \alpha^T A \Phi(x), \tag{6}$$

and A = diag$\{-2c_1(x - x_1), -2c_2(x - x_2), ..., -2c_N(x - x_N)\}$.
$(y^N(x))^2$ can be written as a quadratic from

$$(y^N(x))^2 = \alpha^T B \alpha \tag{7}$$

where $B_{ij} = \phi_j(x_i)$, B is a symmetric matrix.
Also,

$$\left(\frac{d}{dx} y^N(x)\right)^2 = \alpha^T S \alpha \tag{8}$$

$S = A\Phi\Phi^T A^T$,

$S_{ij} = 4c^2(x - x_i)(x - x_j)\phi_i(x)\phi_j(x)$, S is a symmetric matrix.

One can use relations (5), (6) to obtain

$$y^N(x)\frac{d}{dx}y^N(x) = \alpha^T D\alpha, \quad (9)$$

where $D = \Phi\Phi^T A^T$ and $D_{ij} = -2c(x-x_j)\phi_i(x)\phi_j(x)$.
It is possible to provide a similar discussion with MQ-RBF and IMQ-RBF.

2.1. Choosing the shape parameter

Many RBFs are defined by a constant called the shape parameter. The choice of shape parameter has a significant effect on the accuracy and stability of an RBF method. It has been shown that interpolants by RBF in R^d with finite smoothness of even order converge to a polynomial spline interpolant as the scalar parameter of the RBFs goes to zero, i.e., the radial basis functions becomes increasingly flat.

There are some methods for choosing shape parameter. The most typical is calculating the errors with different shape parameters and choosing the best one (trial and error procedure). This strategy can be used if we know the function f, so the job of finding interpolant for f may be to some extent pointless. If f is not known, then to decide what "best" means becomes somewhat difficult.

Fasshuar (2007) suggests a criterion based on "trade-off principle", i.e. the fact that the error and condition number cannot both be kept small. So he suggests to define "best" as "to be the smallest c for which MATLAB does not report a close to singular warning". *linsolve(W,b)* in MATLAB solves a linear system Wx=b using LU factorization with partial pivoting when W is square and QR factorization with column pivoting otherwise. Indeed, it returns a warning if W is square and ill-conditioned or if it is rank efficient and non-square.

For any matrix W the quantity

$$cond(W) = \|W\|_2\|W^{-1}\|_2 = \frac{\sigma_{max}}{\sigma_{min}}, \quad (10)$$

is called l_2-condition number where σ_{max}, σ_{min} are respectively the largest and smallest singular values of W that in positive definite matrices, it is equal to ratio of largest and smallest eigenvalues.
Also, it is possible to use *p*-norm condition number ($p = 1,2,...,\infty$) that produces results comparable to l_2--condition number due to equivalency of *p*-norms property.

For distinct center points if a constant shape parameter is used, the system matrix (3) for MQ-RBF, GA-RBF and IMQ-RBF is known to be nonsingular (Sarra and Sturgill, 2009) whereas for the variable shape parameter case it notable theoretical progress has not been enough to establish invertibility of the system.

Using a variable shape parameter rather than constant shape parameter is strategy that facilitates producing different entries in the RBF matrices which may result in a more well-conditioned system.
The formula

$$c_j = (c_{min}^2\left(\frac{c_{max}^2}{c_{min}^2}\right)^{\frac{j-1}{N-1}})^{\frac{1}{2}} , \quad j = 1,...,N$$

gives an exponentially varying shape parameter that was suggested to produce a distinct value shape parameter. Linearly varying shape parameter is another strategy:

$$c_j = c_{min} + (\frac{c_{max}-c_{min}}{N-1})j , \quad j = 0,...,N-1 \quad (11)$$

also the random shape strategy

$$c_j = c_{min} + (c_{max}-c_{min})rand(1,N). \quad (12)$$

The command *rand(1,N)* is the MATLAB function that returns N uniformly distributed pseudo random numbers on the unit interval (Sarra and Sturgill, 2009; Golbabai and Rabiei, 2012) proposed sinusoidal shape parameter (SSP) that produces N shape parameters in the interval [c_{min}, c_{max}]

$$c_j = c_{min} + (c_{max}-c_{min})\sin\left(\frac{(j-1)\pi}{2(N-1)}\right), \quad (13)$$

$$j = 1,...,N.$$

In this paper we propose Cubic Root (CR) shape parameter strategy as:

$$c_j = (c_{min}^3 c_{max}^2 \frac{j-1}{N-1})^{1/3} , \quad j = 1,...,N, \quad (14)$$

and a Square Root (SR) shape parameter strategy:

$$c_j = (c_{min}^3 c_{max}^2 \frac{j-1}{N-1})^{1/2} , \quad j = 1,...,N. \quad (15)$$

that are increasing and convex functions, aid generating different entries in the RBF matrices together with a decrease in the condition number of matrices, however, because of cubic root property, shape parameters produced by CR have smaller

range than SR so different effect of them on the solutions is expected. In our experiments the results are compared over a range of average shape parameters which will be denoted by

$$c_{avg} = \frac{c_{min} + c_{max}}{2}.$$

3. Constrained variational problem

In this work we concentrate on the problem of finding the extremum of the functional

$$I(u) = \int_{x_a}^{x_b} L(x, u, \dot{u}) dx, \qquad (16)$$

where $u \in Lip[x_a, x_b]$ is the class of Lipschitz functions mapping $[x_a, x_b]$ to R^N, then each component of y is an absolutely continuous function.

$$u(x_a) = A_0, \quad u(x_b) = B_0, \qquad (17)$$

which two points A_0 and B_0 in R^N are given.
There is great variety of variational problems:
1. Boundary conditions: In certain variational problems the boundary conditions of the competing functions or its derivative are not fully described. In these situations transversality conditions may be helpful.
2. Isoperimetric problem: requiring competing functions to comply with restrictions of the type

$$\int_{\Omega} G(x, u(x), \dot{u}(x)) dx = D, \qquad (18)$$

where G: $\Omega \times R \times R^N \to R^d$, $D \in R^d$ are known; some of the constraints could come in the form of inequalities.
3. Pointwise constraints: establishing that feasible functions must respect the condition

$$E(x, u(x), \dot{u}(x)) = 0, \qquad (19)$$

for all $x \in \Omega$ where E is a known function as above, and we could also have some inequalities. This case will be treated with the same strategy as the previous case.

Theorem 3.1. (Tonelli's theorem) (Clarke, 2013) If the Lagrangian L(x, u, v) satisfies the following assumptions:
1. Coercivity of rank r > 1 for certain constants α > 0 and β, we have
L(t, x, v) ≥ α‖v‖^r+ β, ∀(t, x, v) ∈ [a, b] × R^N× R^N
2. Convexity in v: L_{vv} is positive semidefinite ($L_{vv} \geq 0$) everywhere, there exists a solution of the basic problem (15) relative to the class of absolutely continuous (A.C.) functions.

Coercivity condition could be replaced by some certain weakened conditions while existence of a solution to problem (15) is establishable. Under the hypothesis of Tonelli's theorem, if in addition the Lagrangian is autonomous, then any solution y∗, of the problem (15) is Lipschitz on $[x_a, x_b]$. The problem or its Lagrangian is said to be autonomous when L has no dependence on the t variable.

4. RBF direct method

Using relations (5)-(9) the dynamic optimization problem (16)-(19) will be reduced to the following constrained static optimization problem

$$\min J(\alpha) = \int_{x_a}^{x_b} L(x, \alpha^T \Phi(x), \alpha^T \frac{d\Phi}{dx}) dx \qquad (20)$$

subject to

$$\alpha^T \Phi(x_a) = A_0, \qquad (21)$$

$$\alpha^T \Phi(x_b) = B_0, \qquad (22)$$

$$\int_{x_a}^{x_b} G(x, \alpha^T \Phi(x), \alpha^T \frac{d\Phi}{dx}) dx = D, \qquad (23)$$

$$E(x_s, \alpha^T \Phi(x_s), \alpha^T \frac{d\Phi}{dx}\bigg|_{x=x_s}) = 0, \qquad (24)$$

$$s = 1,...,M.$$

where $x_1,...,x_M$ are collocation points in the domain selected to enforce new condition. Comparing the various integration rules (Newton-cotes formulas, extrapolating methods, Gaussian integration), computational efforts being equal, Gaussian integration yields the most accurate results (Stoer, 2002). So using an approximate Guassian quadrature with m nodes on the problem (20)-(24) yield:
min

$$J(\alpha) = \frac{x_b - x_a}{2} \sum_{k=1}^{m} w_k L(x_k, \alpha^T \Phi(x_k), \alpha^T \frac{d\Phi(x_k)}{dx}), \qquad (25)$$

subject to

$$\alpha^T \Phi(x_a) = A_0, \qquad (26)$$

$$\alpha^T \Phi(x_b) = B_0, \qquad (27)$$

$$\frac{x_b - x_a}{2} \sum_{k=1}^{m} w_k G(x_k, \alpha^T \Phi(x_k), \alpha^T \frac{d\Phi(x_k)}{dx}) = D, \qquad (28)$$

$$E(x_s, \alpha^T \Phi(x_s), \alpha^T \frac{d\Phi}{dx}\Big|_{x=x_s}) = 0, \ s=1,...,M. \quad (29)$$

where $x_k = \dfrac{x_b + x_a}{2} + \dfrac{x_b - x_a}{2} z_k$, z_ks and w_ks are nodes and coefficients of Gaussian quadrature and RBF centers x_i, $i=1, ...N$ are nodes which are selected in a specific way.

To solve the optimization problem (25)-(29) our choice is to use Lagrange multipliers method, so we establish Lagrangian

$$J^*(\alpha) = J(\alpha) + \lambda_1(\alpha^T \Phi(x_a) - A_0) + \lambda_2(\alpha^T \Phi(x_b) - B_0)$$

$$+ \Gamma \ (\frac{x_b - x_a}{2}\sum_{k=1}^{m} w_k G(x_k, \alpha^T \Phi(x_k), \alpha^T \frac{d\Phi(x_k)}{dx}) - D),$$

$$+ \sum_{s=1}^{M} \mu_s \ E(x_s, \alpha^T \Phi(x_s), \alpha^T \frac{d\Phi}{dx}\Big|_{x=x_s}) \quad (30)$$

then we should calculate coefficients $\alpha^T = [\alpha_1,...,\alpha_N], \lambda_1, \lambda_2, \Gamma, \mu_1,...,\mu_M$ by solving the following algebraic system

$$\frac{\partial J^*}{\partial \alpha} = 0, \quad (31)$$

$$\frac{\partial J^*}{\partial \lambda_i} = 0, \qquad i = 1, 2. \quad (32)$$

$$\frac{\partial J^*}{\partial \Gamma} = 0, \quad (33)$$

$$\frac{\partial J^*}{\partial \mu_s} = 0, \qquad s = 1,.. M \quad (34)$$

Equation (31), (33), (34) are linear if corresponding operators $L(x,u,\dot{u})$, $G(x,u,\dot{u})$, $E(x,u,\dot{u})$ be quadratic. If L, G, E are instead non-quadratic, then the system (31)-(34) will be a nonlinear system which needs to be solved using some appropriate iterative techniques to evaluate the RBF approximation.

A general formulation for the minimizing functions subject to constraints on the variables is:

$$\min \quad f(x) \quad (35)$$

subject to

$$p_i(x) = 0, \quad i \in H, \quad (36)$$

$$p_i(x) \geq 0, \quad i \in I, \quad (37)$$

where f and the functions c_i are all smooth, real valued functions on a subset of R^n, and I and H are two finite sets of indices. We define the feasible set Ω:

$$\Omega = \{x | p_i(x) = 0, i \in H ; p_i(x) \geq 0, i \in I\}.$$

Definition 2.1. Given the point x*and the active set $A(x^*) = H \cup \{i \in I | p_i(x) = 0\}$ we say that the linear independence constraint qualification (LICQ) holds if the set of active constraint gradients $\{\nabla p_i(x*), i \in A(x^*)\}$ is linearly independent.

Theorem 3.2. (Hoy et. Al., 2001) In a constrained optimization problem

$$\min \quad f(x)$$

subject to

$$g_1(x) = 0, \ ..., g_m(x) = 0,$$

where $x \in R^n$, if the objective function f is quasi-convex and the constraint functions $g_1, ..., g_m$ are all quasi-concave, then any local optimal solution to the problem is also globally optimal.

Theorem 3.3. (Nocedal and Wright, 1999) Suppose that x* is a local solution of (35) and the LICQ holds at x*. Then there is a Lagrange multiplier vector λ*, with components λ*$_i$, i ∈ H ∪ I such that the following conditions are satisfied at (x*, λ*)

$$\nabla_x L(x^*, \lambda^*) = 0, \quad (38)$$

$$c_i(x) = 0, \qquad for \ all \ i \in H \quad (39)$$

$$c_i(x) \geq 0, \qquad for \ all \ i \in I \quad (40)$$

$$\lambda^* \geq 0, \qquad for \ all \ i \in I \quad (41)$$

$$\lambda_i^* c_i(x^*) = 0, \ for \ all \ i \in H \bigcup I \quad (42)$$

These conditions are often known as the Karush-Kuhn-Tucker conditions, or KKT conditions for short.

Theorem 3.4. (Nocedal and Wright, 1999) Suppose that for some feasible point x*∈ R^n there is a Lagrange multiplier vector λ* such that the KKT conditions are satisfied. Suppose also that

$$w^T \nabla_{xx}(x^*, \lambda^*)w > 0, \ for \ all \ w \in F(\lambda^*), w \neq 0. \quad (43)$$

Then x* is a strict local solution for (35). Define w ∈ F(λ*) if and only if

$$\nabla c_i(x^*)T_w = 0, \ for all \ i \in H, \quad (44)$$

$$\nabla c_i(x^*)T_w = 0, \ for\ all\ i \in A(x^*) \cap I \ with \ \lambda_i^* > 0, \qquad (45)$$

$$\nabla c_i(x^*)T_w \geq 0, \ for\ all\ i \in A(x^*) \cap I \ with \lambda_i^* = 0 \qquad (46)$$

Theorem 3.5. Let the Lagrangian L(x, u, v) fulfills the assumptions coercivity of rank $r > 1$ and convexity in (y, v). Then the operator (16) satisfies the following properties:

1. I(u) is a convex operator,

2. If we assume $u(x) = \sum_{i=1}^{N} \alpha_i \phi_i(x) = \alpha^T \Phi(x)$

where $\phi_i's$ are positive definite RBFs described previously, then constrained optimization problem

$$\min J(\alpha) = \int_{x_a}^{x_b} L(x, \alpha^T \Phi(x), \alpha^T \frac{d\Phi}{dx}) dx \qquad (47)$$

subject to

$$\alpha^T \Phi(x_a) = A_0, \quad \alpha^T \Phi(x_b) = B_0, \qquad (48)$$

admits a solution, so the problem (47)-(48) is well-defined.

Proof:

$$I(\lambda u + (1-\lambda)v) =$$

$$\int_{x_a}^{x_b} L(x, \lambda u + (1-\lambda)v, \lambda \dot{u} + (1-\lambda)\dot{v}) dx$$

$$\leq \lambda \int_{x_a}^{x_b} L(x, u, \dot{u}) dx + (1-\lambda) \int_{x_a}^{x_b} L(x, v, \dot{v}) dx$$

$$\leq \lambda I(u) + (1-\lambda)I(v)$$

so I is a convex operator.

As discussed in (Sec.2) interpolation of function u(x) by positive definite RBFs is a well defined problem and ϕ_is, i = 1, ..., N are independent functions. The convexity of I(u) yields

$$J(\lambda \alpha + (1-\lambda)\beta) =$$

$$\int_{x_a}^{x_b} L(x, \lambda \alpha^T \Phi + (1-\lambda)\beta^T \Phi, \lambda \alpha^T \frac{d\Phi}{dx} +$$

$$(1-\lambda)\beta^T \frac{d\Phi}{dx}) dx$$

$$\leq \lambda \int_{x_a}^{x_b} L(x, \alpha^T \Phi, \alpha^T \frac{d\Phi}{dx}) dx +$$

$$(1-\lambda)\int_{x_a}^{x_b} L(x, \beta^T \Phi, \beta^T \frac{d\Phi}{dx}) dx$$

$$= \lambda J(\alpha) + (1-\lambda)J(\beta)$$

so J(α) is a multivariate convex function in R^N.

Since the equations (48) are linear they are quasi-concave functions and according to theorem 3.2 the problem (47)-(48) is a well-defined problem and has unique solution. Δ

Remark: If we assume the quasi-concavity of operators G and E, using theorems 3.2 – 3.4 the theorem 3.5 is establishable for the problem (25)-(29) (Note that quasi-convexity of G yields quasi-concavity of −G).

5. Numerical Experiments

Example 1. Dynamic Investment model (Hoy et. al., 2001)
Suppose that a firm's output depends on the amount of capital it employs. Let Q =q(K) where Q is the firm's output level, q is the production function and K is the amount of capital employed.

Authors suppose that the firm should purchase its own capital. Once purchased, the capital lasts for a long time. Let I(t) be the amount of capital purchased (investment) at time t and c[I(t)] be a function that gives the cost of purchasing (investing) the amount I(t) of capital at time t; then profit at time t is

$$\pi[K(t), I(t)] = pq[K(t)] - c(I(t)), \qquad (49)$$

If the firm's objective is to choose K and I to maximize discounted sum of profits over an interval of time running from the present time (t = 0) to a given time horizon, T, this is given by the functional

$$\max J(I(t)) = \int_0^T e^{-\rho t} \pi[K(t), I(t)] dt, \qquad (50)$$

where ρ is the firm's discount rate and $e^{-\rho t}$ is the continuous-time discounting factor.

Assume that capital depreciates at the rate δ. The amount (stock) of capital owned by the firm at time t is K(t) and changes according to the differential equation

$$K'(t) = I(t) - \delta K(t), \qquad (51)$$

which says that, at each point in time, the firm's capital stock increases by the amount of investment and decreases by the amount of depreciation.

The problem facing the firm at each point in time is to decide how much capital to purchase. This is a truly dynamic problem because the present investment affects current profit, and it is a current expense, additionally it affects future profits, for it affects the amount of capital available for future production.

Assume in the investment model the firm's

production function is given by $\pi[K(t),I(t)] = K - aK^2 - I^2, a > 0$ and the price of the firm's output is a constant 1\$, the cost of investment is equal to I^2\$, then the firm's profit at a point in time is $\pi = K(t) - aK(t)^2 - I(t)^2$ So we are faced with the following constrained variational problem

$$\max \ J[I(t)] = \int_0^{+\infty} e^{-\rho t}(K(t) - aK(t)^2 - I(t)^2)dt \qquad (52)$$

subject to

$$K'(t) = I(t) - \delta K(t), \qquad (53)$$

$$K(0) = K_0. \qquad (54)$$

If we assume $K(0) = 1$, using RBF direct method with constant shape parameter c= 2.5 and N= 100 equally spaced centers, the solution will be obtained as in Fig. 1. It illustrates the better accuracy of MQ case in comparison with GA case. The authors used the change of variable τ=f(t) where f is a differnetiable, strictly increasing the function of t which maps the infinite interval [0,+ ∞) onto finite interval [−1,1). An example is $f(t) = \dfrac{1+t}{1-t}$, other possibilities could be found in (Garg et. al., 2011). Using the described procedure in Sec. 4 the problem (52)-(54) will be reduced to an algebraic linear system which was solved by MATLAB software.

Figure 1 displays the optimal path of investment for the case in which $K_0 < K$. Along the optimal path, investment declines, with increasing time, investment converges to a constant amount so that in the long run the firm's investment is just a replacement of depreciation.

It is shown in Fig. 2 that based on Root Mean Square criterion:

$$RMS = \sqrt{\frac{1}{M}\sum_{i=0}^{M}(f(x_i) - P_f(x_i))^2}. \qquad (55)$$

The CR-VSP and SR-VSP produce a considerably more stable solution over most of the average shape parameter range in comparison to the other methods and the CR-VSP is considerably more accurate than other shape parameter strategies. The results are presented in Fig. 2. Results of numerical experiments over a range of the distance s =$c_{max} - c_{min}$ have reported that taking s > 1 upgraded the accuracy and reduced fluctuation of the graph in comparison to small s.

Example 2. Constrained variational problem
Consider the functional

$$J(y) = \int_0^1 \dot{y}^2 + x^2 dx, \qquad (56)$$

whose minimum should be found with respect to the integral condition $\int_0^1 y^2(x)dx = 2,$ and boundary conditions y(0)=0, y(1)=0, (Yousefi and Dehghan, 2010).

Numerical result of GA and MQ case with constant shape parameter was obtained by Maple software that shows RMS error around 1e−4 for N= 5, 9, 25, and for various center points choosing methods such as equally spaced, Gauss-Lobatto and Chebyshev roots. To improve the accuracy of the method authors used root square varying shape parameter with *Digits* environment variable in Maple that controls the number of digits that Maple uses when making calculations with software floating-point numbers. In table (1) it is demonstrated that with N = 25 equally spaced centers, increasing floating-point numbers enhances accuracy of the method as high as six decimal digits in comparison with when default floating-point number is used. It asserts that the main part of error of the RBF method which is reported in these examples is due to machine epsilon.

Example 3. An optimal control problem with free endpoint
Consider the differential equation $x'(t) = 3x(t) + u(t) \ \ a.e. \ \ on \ [0,1]$ with the initial condition $x(0) = x_0$ where $x_0 \in$ R is arbitrarily chosen. The objective functional J is as follows:

$$J(u) = x(1)^2 + \int_0^1 u(t)^2 dt. \qquad (57)$$

$$x(t) = \frac{x_0}{e^{-6}+5} e^{-3t}(e^{6(t-1)}+5) \quad \forall t \in [0, 1] \qquad (58)$$

and the control

$$u(t) = -\frac{30x_0}{e^{-6}+5} e^{-3t} \qquad \forall t \in [0, 1]. \qquad (59)$$

Figure 3 demonstrates the superiority of GA-RBF to MQ-RBF in absolute error criterion and condition number criterion. In Fig. 4 the results are shown from five shape parameter strategies which demonstrates the better accuracy of CR and SR to other shape strategies (especially in GA-RBF) meanwhile better condition number. In Fig. 5 it is demonstrated that absolute error has stable behavior when number of centers (N) increases in GA-RBF and MQ-RBF cases.
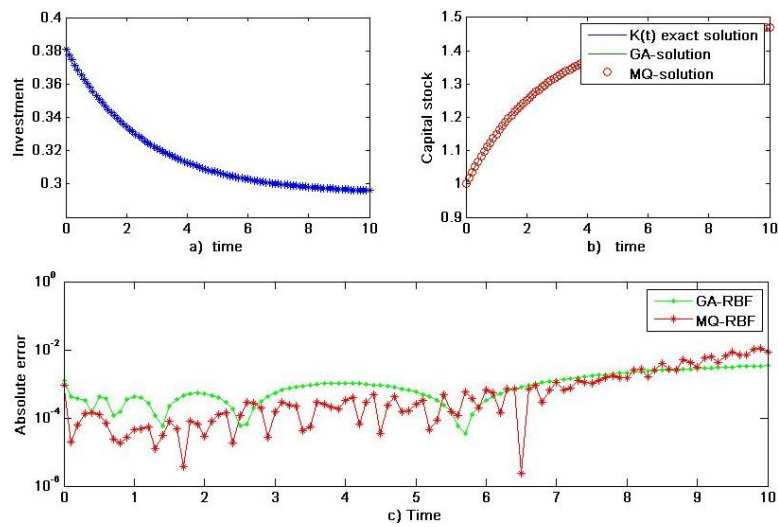
Fig. 1. (Example 1) a) Up left: Optimal Investment path over time. b) Up right: Capital Stock over time. c) Below: Comparing accuracy of MQ and GA RBFs
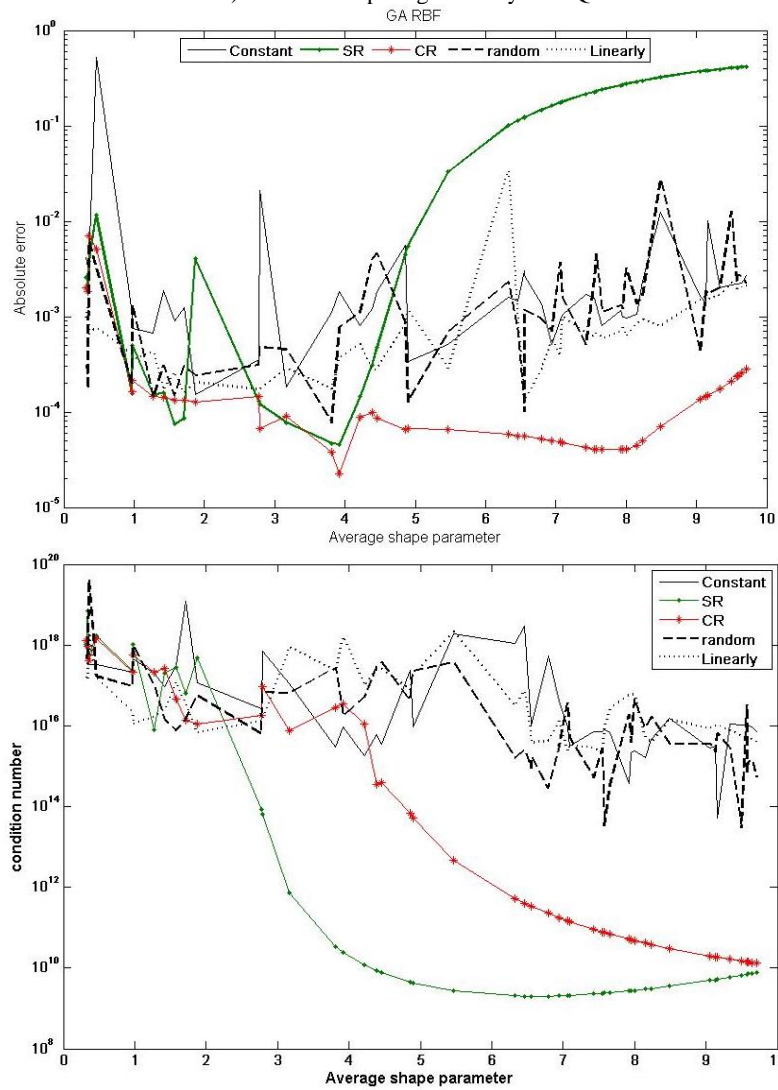

Fig. 2. (Example 1) a) Up: Comparison of absolute error of five shape parameter strategies. b) Down: Comparison of condition number of five shape parameter strategies

Table 1. Results for Example 2

| X	Exact Solution	Results of (Yousefi and Dehghan, 2010)	Results of the method
0	0.0	0.0	0.0
0.1	0.618033988750	5.0×10^{-11}	1.12×10^{-12}
0.2	1.175570504585	4.15×10^{-10}	1.15×10^{-13}
0.3	1.618033988750	2.50×10^{-10}	3.50×10^{-12}
0.4	1.902113032590	4.10×10^{-10}	5.00×10^{-12}
0.5	2.0	3.50×10^{-11}	2.50×10^{-13}
0.6	1.902113032590	4.10×10^{-10}	3.00×10^{-12}
0.7	1.618033988750	2.50×10^{-10}	5.00×10^{-12}
0.8	1.175570504585	4.15×10^{-10}	1.35×10^{-12}
0.9	0.618033988750	5.00×10^{-11}	6.50×10^{-12}
1.0	0.0	0.0	0
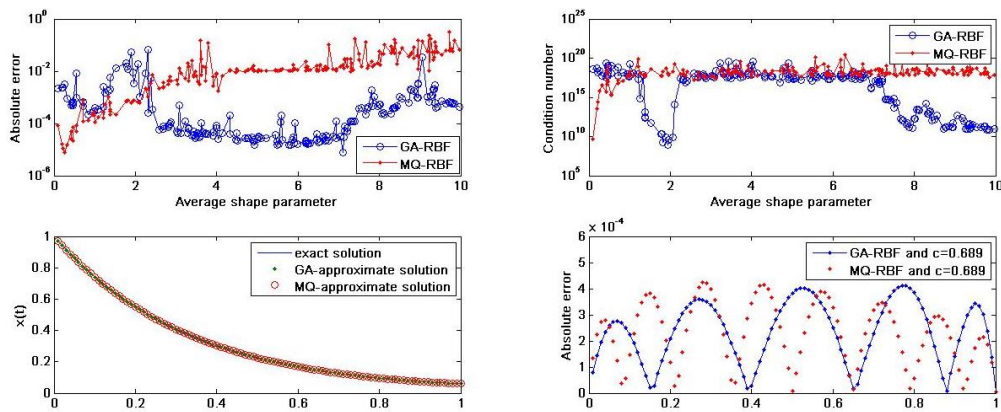
Fig. 3. (Example 3) Accuracy and numeical stability of the solution as N=10 loboto center points
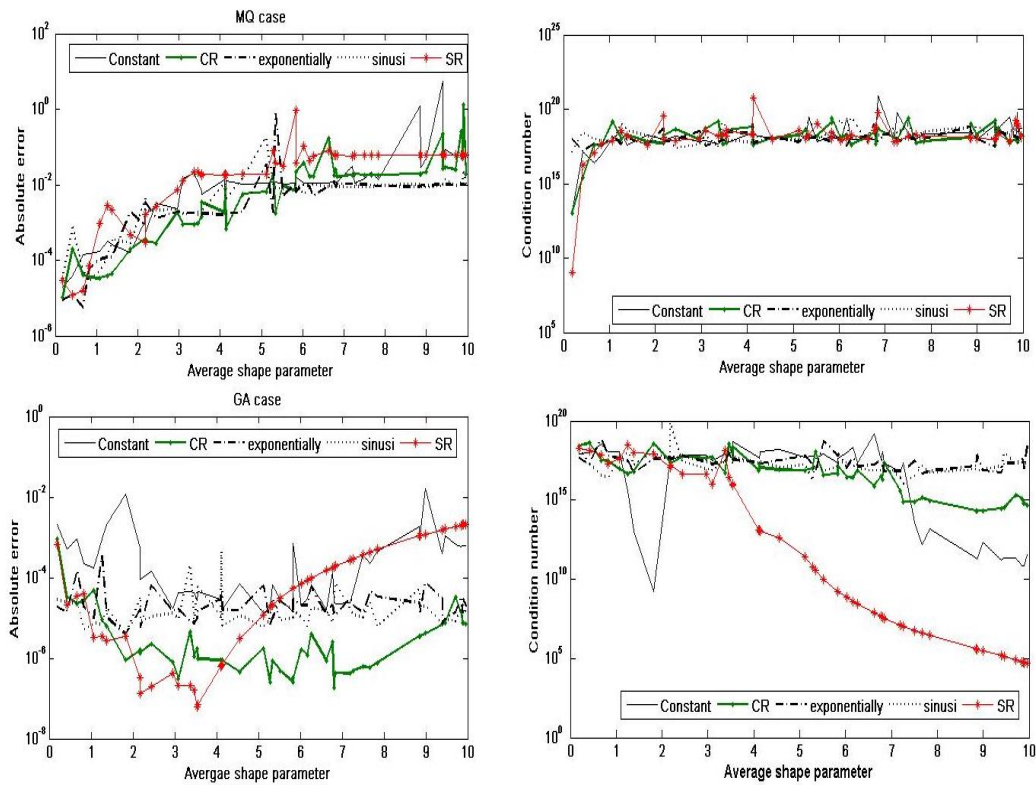

Fig. 4. (Example 3) Comparison between various VSPs for MQ-RBF (up) and GA-RBF (down)
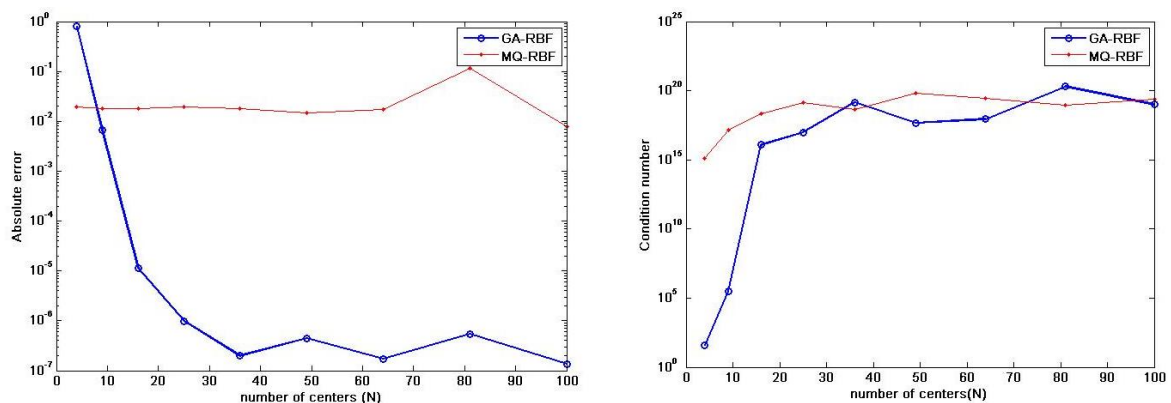
Fig. 5. (Example 3) Behaviour of Absolute error when number of centers (N) increases

5. Conclusion

This paper dealt with the constrained variational problem with fixed and free end points. Using computational properties of RBFs a RBF-direct method for solving this type of problems was developed and it essentially reduced the constrained variational problem to a static optimization problem through which its well-definedness under certain conditions was proved by a theorem.

The high accuracy and performance of the approximation method in comparison with other existing methods was demonstrated through some examples including dynamic investment problem. Finally, to improve the performance of RBF method in scattered centers and equally distributed centers, a Cubic Root shape parameter strategy and a Square Root shape parameter strategy were introduced that successfully increase the accuracy and reliability of the solution, especially in GA-RBF. Meanwhile it was shown that using extended precision floating point arithmetic considerably increases accuracy of the method.

References

Babolian, E., Mokhtari, R., & Salmani, M. (2007). Using direct method for solving variational problems via triangular orthogonal functions. *Applied Mathematics & Computation*, 191, 206–217.

Chang, R. Y., & Wang, M. L. (1983). Shifted Legendre direct method for variational problem series. *Journal of Optimization Theory and Applications*, 39, 299–307.

Chen, C. F., & Hsiao, C. H., (1975). A Walsh series direct method for solving variational problems. *Journal of the Franklin Institue*, 300, 265–1373.

Chen, X., & Jung, J. H. (2012). Matrix stability of multiquadric radial basis function methods for hyperbolic problems with uniform centers. *Journal of Scientific Computing*, 51(3), 683–702.

Cheney, W. (2000). *An Introduction to Approximation Theory*. Second Edition. AMS Chelsea Publishing,

American Mathematical Society.

Clarke, F. (2013). *Functional Analysis, Calculus of Variations and Optimal Control*. Springer.

Fasshuar, G. (2007). *Meshfree Aproximation methods with MATLAB*. World Scientific Publishing.

Franke, C., & Shaback, R. (1998). Solving Partial Differential Equations by collocation using Radial Basis functions. *Applied Mathematics and Computation*, 93(1), 73–82.

Golbabai, A., & Rabeie, H. (2012). A meshfree method based on radial basis functions for the eigenvalues of transient Stokes equations. *Engineering Analysis with Boundary Elements*, Volume 36, Issue 11, 1555–1559.

Golbabai, A., & Saeedi, A. (2014). An improved RBF method for solving variational problems arising from dynamic economic models. *Computational Economics*, DOI 10.1007/s10614-014–9463-6.

Golbabai, A., & Seifollahi, S. (2006). Numerical Solution of the Second kind Integral Equations using Radial Basis Functions networks, *Applied Mathematics and Computation*, 174, 877–883.

Glabisz, W. (2004). Direct Walsh-wavelet packet method for variational problems. *Applied Mathematics and Computation*, 159, 769–781.

Garg, D., Hager, W., & Rao, A., (2011). Pseudospectral methods for solving infinite-horizon optimal control problems. *Automatica*, 47, 829–837.

Horng, I. R., & Chou, J. H. (1985). Shifted Chebyshev direct method for solving variational problems. *International Journal of systems science*, 16, 855–861.

Hoy, M., Livernoise, J., McKenna, C., Rees, R., & Stengos, T. (2001). *Mathematics for Economics*. 2nd edition, MIT press.

Hwang, C., & Shih, Y. P. (1985). Optimal control of delay systems via block-pulse functions. *Journal of Optimization Theory and Applications*, 101–112.

Lin, Y., & Yuan, M., (2006). Convergence rate on compactly supported RBF regularization. *Statistica Sinica*, 16, 425–439.

Madych. W. M. (1992). Miscellaneous error bounds for multiquadric and related interpolators. *Computers and Mathematics with Application.*, 24(12), 121138.

Madych, W. R., & Nelson N. A. (1992). Bounds on multivariate polynomials and exponential error estimates for multiquadric interpolation. *Journal of Approximation Theory*, 70(1), 94114.

Nocedal, J., & Wright, S. J. (1999). *Numerical Optimization*, Springer.

Razzaghi, M., & Ordokhani, Y. (2001). An application of rationalized Haar functions for variational problems. *Applied Mathematics and Computation*, 122, 353–364.

Sarra, S. A., & Sturgill, D. (2009). A random variable shape parameter strategy for radial basis function approximation. *Engineering Analysis with Boundary Elements*, (33), 1239–1245.

Schaback, R. (1995). Error estimates and condition numbers for radial basis function interpolation. *Advances in Computational Mathematics*, 3(3), 251-264.

Schaback, R. (2005). Multivariate interpolation by polynomials and radial basis functions. *Constructive Approximation*, 21(3), 293–317.

Stoer, J., & Bulirsch, R. (2002). *Introduction to Numerical Analysis.* 3rd edition, Springer.

Wendland, (2001). *Trends in Approximate Theory*, Vanderbilt University Press, Nashville, 427–436.

Yousefi, S. A., & Dehghan, M., (2010). The use of He's variational iteration method for solving variational problems. *International Journal of Computer Mathematics*, 87, 1299–1314.