Comultiplication lattice modules

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Abstract

Let $M$ be a lattice module over the multiplicative lattice $L$. $M$ is said to be a comultiplication $L$-module if for every element $N$ of $M$ there exists an element $a \in L$ such that $N = (0_M \leq a)$. Our objective is to investigate properties of comultiplication lattice modules.

Keywords: Multiplicative lattice; lattice modules; comultiplication lattice modules

1. Introduction

A multiplicative lattice $L$ is a complete lattice in which there is defined a commutative, associative multiplication which distributes over arbitrary joins and has a compact greatest element $1_L$ (least element $0_L$) as a multiplicative identity (zero). Let $L$ be a multiplicative lattice and $a \in L$, $L/a = \{b \in L : a \leq b\}$ be a multiplicative lattice with multiplication $c \cdot d = c d/a$. Multiplicative lattices have been studied (Jayaram and Johnson, 1995, 1997, 1998; Johnson, 2002, 2003, 2004; Johnson and Johnson, 2003).

An element $a \in L$ is said to be proper if $a < 1$. An element $p < 1$ in $L$ is said to be prime if $a b \leq p$ implies $a \leq p$ or $b \leq p$. An element $m < 1$ in $L$ is said to be maximal if $m \leq x \leq 1$ implies $x = 1$. It is easily seen that maximal elements are prime.

If $a, b$ belong to $L$, $(a; b)$ is the join of all $c \in L$ such that $c b \leq a$. An element $e$ of $L$ is called meet principal if $a \wedge b = (a; e)(e \wedge b)$ for all $a, b \in L$. An element $e$ of $L$ is called join principal if $(a \vee b) = a \vee (b; e)$ for all $a, b \in L$. $e \in L$ is said to be principal if $e$ is both meet principal and join principal $e \in L$ is said to be weak meet (join) principal if $a \wedge e = e(a; e)$ ($a \vee (b; e)$) for all $a \in L$. An element $a$ of a multiplicative lattice $L$ is called compact if $a \leq b$ implies $a \leq b_0 \wedge b_0 \wedge \ldots \wedge b_0$ for some subset $\{a_1, a_2, \ldots, a_n\}$. If each element of $L$ is a join of principal (compact) elements of $L$, then $L$ is called a $PG$-lattice ($CG$-lattice).

Let $M$ be a complete lattice. Recall that $M$ is a lattice module over the multiplicative lattice $L$, or simply an $L$-module in case there is a multiplication between elements of $L$ and $M$, denoted by $lB$ for $l \in L$ and $B \in M$, which satisfies the following properties:

i. $(b)B = l(bB)$;

ii. $(V_{a\leq l}B)(V_{b\leq l}B') = V_{a\leq l,b\leq l}B'B$;

iii. $1_lB = B$;

iv. $0_lB = 0_M$ for all $l, l \leq a$, $b$ in $L$ and for all $B, B'$ in $M$.

Let $M$ be an $L$-module. If $N, K$ belong to $M$, $(M; l)K$ is the join of all $a \in L$ such that $aK \leq N$. If $a \in L$, then $(0_M \leq a)$ is the join of all $H \in M$ such that $aH = 0_M$. An element $N$ of $M$ is called meet principal if $(b \wedge (B; l)) = b \wedge N \wedge B$ for all $b \in L$ and for all $B \in M$. An element $N$ of $M$ is called join principal if $(b \vee (B; l)) = (b \vee N \vee B)$ for all $b \in L$ and for all $B \in M$. $N$ is said to be principal if it is both meet principal and join principal. In a special case, an element $N$ of $M$ is called weak meet principal (weak join principal) if $(B; l)N = B \wedge N \wedge (b \vee N \vee B)$ for all $b \in L$. $N$ is said to be weak principal if $N$ is both weak meet principal and weak join principal.

Let $M$ be an $L$-module. An element $N$ in $M$ is called compact if $N \leq V_{a\leq l}B_a$ implies $N \leq B_{a_1} \vee B_{a_2} \vee \ldots \vee B_{a_n}$ for some subset $\{a_1, a_2, \ldots, a_n\}$. The greatest element of $M$ will be denoted by $1_M$. If each element of $M$ is a join of principal (compact) elements of $M$, then $M$ is called a $PG$-lattice ($CG$-lattice).

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0_M implies \( m = 0_M \) or \( c = 0_L \) for any \( c \in L \) and \( m \in M \). \( M \) is called a torsion-free \( L \)-module.

For various characterizations of lattice modules, the reader is referred to Nakkar and Al-Khouja (1989), Nakkar and Anderson (1988) and Scott Culhan (2005). In this paper we study comultiplication lattice modules over a multiplicative lattice and generalize the important results for comultiplication modules over commutative rings, obtained by Ansari-Toroghy and Farshadifar (2007, 2011), Shaniati and Smith (2011) to the lattice modules over multiplicative lattice.

2. Comultiplication Lattice Modules

Definition 1.

i. (Callalp and Tekir, 2011) An \( L \)-module \( M \) is called a multiplication lattice module if for every element \( N \in M \) there exists an element \( a \in L \) such that \( N = a1_M \).

ii. Let \( M \) be a lattice \( L \)-module. \( M \) is said to be a comultiplication \( L \)-module if for every element \( N \) of \( M \) there exists an element \( a \in L \) such that \( N = (0_M; M \cdot a) \).

Lemma 1. Let \( M \) be a lattice \( L \)-module. Then, \( M \) is a comultiplication lattice \( L \)-module if and only if \( N = (0_M; M \cdot N) \) for every element \( N \) in \( M \).

Proof: \( \Rightarrow \): Suppose that \( M \) is a comultiplication lattice \( L \)-module and \( N \in M \). Then there exists an \( a \in L \) such that \( N = (0_M; M \cdot a) \). Thus we have \( a \leq (0_M; M \cdot N) \) so that \( (0_M; M \cdot N) \leq (0_M; M \cdot a) \). It is clear that \( N \leq (0_M; M \cdot N) \). This implies \( N = (0_M; M \cdot N) \).

Proposition 1. Let \( M \) be a lattice \( L \)-module. Then the followings are equivalent.

i. For any \( K, N \in M \), \( (0_M; L \cdot K) \leq (0_M; L \cdot N) \) implies \( N \leq K \).

ii. For any \( K, N \in M \), \( (K; L \cdot N) = ((0_M; L \cdot N); (0_M; K \cdot L)) \).

Proof: (i) \( \Rightarrow \) (ii): For any \( K, N \in M \), \( (K; L \cdot N) \leq ((0_M; L \cdot N); (0_M; K \cdot L)) \). Indeed \( b = (K; L \cdot N) \Rightarrow bn \leq K \Rightarrow b(0_M; L \cdot N) = 0_M \Rightarrow b(0_M; L \cdot K) \leq (0_M; L \cdot N) \Rightarrow b = (K; L \cdot N) \leq ((0_M; L \cdot N); (0_M; K \cdot L)) \).

Conversely, let \( r = ((0_M; L \cdot N); (0_M; K \cdot L)) \). Then \( r(0_M; L \cdot N) = 0_M \Rightarrow (0_M; L \cdot K) \leq (0_M; L \cdot rN) \), by (i), we have \( rN \leq K \) and so \( r \leq (K; L \cdot N) \). (ii) \( \Rightarrow \) (i): Suppose that \( (0_M; L \cdot K) \leq (0_M; L \cdot N) \). Then \( (K; L \cdot N) = ((0_M; L \cdot N); (0_M; K \cdot L)) = 1_L \) by (ii) and so \( N \leq K \).

Theorem 1. Let \( M \) be a lattice \( L \)-module. Suppose \( \phi: L \rightarrow M \) is defined by \( \phi(a) = (0_M; M \cdot a) \) and \( \psi: M \rightarrow L \) by \( \psi(N) = (0_M; L \cdot N) \) for all \( a \in L \) and \( N \in M \). Then,

i. \( (\phi \psi)(a) = (0_M; M \cdot (0_M; L \cdot (0_M; L \cdot a))) = (0_M; M \cdot a) = \phi(a) \) for all \( a \in L \).

ii. \( (\psi \phi)(N) = (0_M; L \cdot (0_M \cdot M \cdot (0_M; L \cdot N))) = (0_M; L \cdot N) = \psi(N) \) for all \( N \in M \).

Proof: i. Suppose that \( (0_M; M \cdot a) = N \). Clearly, \( (0_M; M \cdot a) = N \leq (0_M; M \cdot N) \). On the other hand, \( aN \in M \) and so \( a \leq (0_M; L \cdot N) \). Therefore, \( (0_M; M \cdot (0_M; L \cdot M \cdot N)) \leq (0_M; M \cdot a) = N \).

ii. Suppose that \( b = (0_M; L \cdot N) \). Clearly \( b = (0_M; L \cdot N) \leq (0_M; L \cdot (0_M; M \cdot (0_M; L \cdot N))) \). On the other hand, \( bM = 0_M \) and so \( N \leq (0_M; M \cdot b) = (0_M; M \cdot (0_M; L \cdot N)) \). Hence \( (0_M; L \cdot (0_M; M \cdot (0_M; L \cdot N))) \leq (0_M; L \cdot N) = b \).

Corollary 1. Let \( M \) be a lattice \( L \)-module. Let us define \( \phi: L \rightarrow M \) where \( \phi(a) = (0_M; M \cdot a) \), and \( \psi: M \rightarrow L \) where \( \psi(N) = (0_M; M \cdot N) \) for all \( a \in L \) and \( N \in M \). The followings are equivalent.

i. \( M \) is a comultiplication lattice \( L \)-module.

ii. There exists \( a \in L \) such that \( N = (0_M; M \cdot a) = \phi(a) \) for all \( N \in M \).

iii. \( \phi \psi \) is an identity map.

iv. \( \psi \) is one-to-one.

v. \( (0_M; L \cdot K) = (0_M; L \cdot M \cdot N) \) implies \( K = N \).

Proposition 2. Let \( M \) be a comultiplication lattice \( L \)-module. If \( L \) is a Noetherian (Artinian) multiplicative lattice, then \( M \) is an Artinian (Noetherian) lattice \( L \)-module.

Proof: Let \( L \) be an Artinian multiplicative lattice. Suppose that \( N_1 \leq N_2 \leq \cdots \). Then, \( (0_M; L \cdot N_1) \geq (0_M; L \cdot N_2) \geq \cdots \). Since \( L \) is Artinian, there exists a positive integer \( k \) such that \( (0_M; L \cdot N_k) = (0_M; L \cdot (0_M; L \cdot N_{k+1})) = \cdots \). Therefore, \( N_k = (0_M; M \cdot (0_M; L \cdot N_k)) = (0_M; M \cdot (0_M; L \cdot N_{k+1})) = N_{k+1} = \cdots \). Consequently, \( M \) is a Noetherian lattice module. Similarly, if \( L \) is Noetherian, then \( M \) is Artinian lattice \( L \)-module.

Let \( L \) be a multiplicative lattice and \( M \) be an \( L \)-module. Suppose that \( N \in M \). Consider the set \( [0_M; N] = \{ A \leq N : A \in M \} \). We say that \( [0_M; N] \) is a submodule of \( M \). If \( M \) is a comultiplication \( L \)-module, it is clear that \( [0_M; N] \) is a comultiplication \( L \)-module.

Proposition 3. Let \( M \) be a comultiplication lattice \( L \)-module. If \( (0_M; L \cdot b) = 0_M \) for some \( b \in L \), then \( bY = Y \) for all \( Y \in M \). In particular, \( b1_M = 1_M \).

Proof: Let \( b \in L \) and \( Y \in M \). Since \( M \) is a comultiplication lattice module, it follows that \( bY = (0_M; L \cdot a) \) for some \( a \in L \). Then \( abY = 0_M \). Since \( (0_M; L \cdot b) = 0_M \), we have \( aY = 0_M \). Consequently, \( Y \leq (0_M; L \cdot a) = bY \) and so \( bY = Y \).
Proposition 4. Let $M$ be a comultiplication lattice $L$-module. If $p$ is a maximal element of $L$ and $(0_{M;M} p) \neq 0_M$, then $(0_{M;M} p)$ is minimal in $M$.

Proof: Suppose that $N \leq (0_{M;M} p)$. Since $M$ is a comultiplication lattice $L$-module, there exists an element $a$ of $L$ such that $N = (0_{M;M} a)$. Since $N \leq (0_{M;M} p)$, we have $pN = 0_M$ and so $p \leq (0_{M;M} N)$. Since $p$ is maximal, $p = (0_{M;M} N)$ or $(0_{M;M} N) = 1_L$. If $p = (0_{M;M} N)$, then $N = (0_{M;M} (0_{M;M} N)) = (0_{M;M} p)$. If $(0_{M;M} N) = 1_L$, then $N = 0_M$. Therefore, $(0_{M;M} p)$ is minimal in $M$.

Proposition 5. Let $M$ be a comultiplication $PG$-lattice $L$-module with $1_M$ compact. If $p \in L$ is prime and $(0_{M;M} p) = 0_M$, then there exists $c \in L$ such that $c \leq p$ and $1_M = c$.

Proof: Since $1_M$ is compact, then $1_M = \bigvee_{i=1}^n Y_i$ where $Y_i$ s are principal elements of $M$. Since $(0_{M;M} p) = 0_M$, $pY_i = Y_i$ for all $i \in \{1, 2, \ldots, n\}$ by Proposition 3. Then $p \vee (0_{M;M} Y_i) = (pY_i; Y_i) = 1_L$ and so $(0_{M;M} Y_i) \leq p$ for all $i \in \{1, 2, \ldots, n\}$. Therefore, $c = \prod_{i=1}^n (0_{M;M} Y_i) \leq p$ and $1_M = c$.

Corollary 2. Let $M$ be a comultiplication $PG$-lattice $L$-module with $1_M$ compact. If $M$ is faithful, then $(0_{M;M} p) \neq 0_M$ for some prime element $p \in L$.

Corollary 3. If $M$ is a comultiplication $PG$-lattice $L$-module with $1_M$ compact and $(0_{M;M} a) = 0_M$ for some $a \in L$, then $1_L = a \vee (0_{M;M} 1_M)$.

Proof: Suppose that $1_L \neq a \vee (0_{M;M} 1_M)$. Then there exists a maximal element $p \in L$ such that $a \vee (0_{M;M} 1_M) \leq p$. Thus we have $(0_{M;M} p) \leq (0_{M;M} a) = 0_M$. Hence $(0_{M;M} p) = 0_M$. There exists an element $c \in L$ such that $c \leq (0_{M;M} 1_M)$ by Proposition 5. Since $(0_{M;M} 1_M) \leq p$, we have $c = p$. This is a contradiction. Consequently, $a \vee (0_{M;M} 1_M) = 1_L$.

Proposition 6. Let $M$ be a non-zero comultiplication $PG$-lattice $L$-module. Then, $M$ has a minimal element. In particular, every nonzero element of $M$ has a minimal element.

Proof: Suppose that $Y$ is a non-zero principal element of $M$. Then $(0_{M;M} Y) = a < 1_L$. Then there exists a maximal element $p$ such that $a \leq p$. If $N = (0_{M;M} p) = 0_M$, then $pY = Y$ by Proposition 3 and so $p \vee (0_{M;M} Y) = (pY; Y) = 1_L$. Therefore, $a = (0_{M;M} Y) \leq p$. This is a contradiction. Hence $N = (0_{M;M} p) \neq 0_M$. Therefore, $N$ is a minimal element of $M$ by Proposition 4.

Proposition 7. Let $M$ be a non-zero comultiplication $PG$-lattice $L$-module. Then $K \in M$ is minimal if and only if $K = (0_{M;M} p) \neq 0_M$ for some maximal element $p \in L$.

Proof: $\Rightarrow$: By Proposition 4.

$\Rightarrow$: Let $K$ be a minimal principal element of $M$. Since $M$ is a comultiplication lattice $L$-module, $K = (0_{M;M} K)$. We will show that $(0_{M;M} K)$ is maximal. Let $c \in L$ such that $(0_{M;M} K) \leq c$. Since $K$ is minimal and $cK \leq K$, it follows that $cK = K$ or $cK = 0_M$. If $cK = K$, then $0_M = (cK; K) = cV(0_{M;M} K)$. If $cK = 0_M$, then $c \leq (0_{M;M} K)$ and so $c = (0_{M;M} K)$.

Proposition 8. Let $M$ be a comultiplication lattice $L$-module. Then, $(N;M a) = ((0_{M;M} a);M (0_{M;M} N))$ for all $a \in L$, $N \in M$.

Proof: Let $K = (N;M a)$. Then $aK \leq N \Rightarrow (0_{M;M} N) aK = 0_M \Rightarrow (0_{M;M} N) K \leq (0_{M;M} a) \Rightarrow K = (N;M a) \leq ((0_{M;M} a);M (0_{M;M} N))$. Conversely, if $R = ((0_{M;M} a);M (0_{M;M} N))$, then $(0_{M;M} N) R \leq (0_{M;M} a) \Rightarrow (0_{M;M} L) aR = 0_M \Rightarrow aR \leq (0_{M;M} (0_{M;M} N)) = N$. Consequently, $R \leq (N;M a)$.

Theorem 2. Let $L$ be a distributive lattice. Let $M$ be a comultiplication lattice $L$-module and $(0_{M;M} a) V (0_{M;M} b) = (0_{M;M} a \wedge b)$ for all $a, b \in L$. Then $M$ is distributive.

Proof: Let $X, Y, Z \in M$. There exist $a, b, c \in L$ such that $X = (0_{M;M} a)$, $Y = (0_{M;M} b)$, $Z = (0_{M;M} c)$. Then, $(XVY) \wedge Z = ((0_{M;M} a) V (0_{M;M} b)) \wedge (0_{M;M} c) = (0_{M;M} a \wedge b) \wedge (0_{M;M} c) = (0_{M;M} a \wedge b) \wedge (0_{M;M} a \wedge c) = (0_{M;M} a \wedge c) = (0_{M;M} (a \wedge c) \wedge (0_{M;M} b \wedge c) = (XVZ) \wedge (YVZ)$.

Corollary 4. Let $L$ be a distributive lattice. Let $M$ be a comultiplication lattice $L$-module and $aVb = b^L$ for all $a, b \in L$. Then $M$ is distributive.

Proof: If $aVb = 1_L$, then $(K;M a \wedge b) = (K;M a \wedge b) (aVb) = a(K;M a \wedge b) V b(K;M a) \wedge b \leq (K;M b) V (K;M a) \wedge b \leq (K;M b) \wedge b \leq (K;M a)$. It is clear that $(K;M a) V (K;M a) \leq (K;M a \wedge b)$. For $K = 0_M$, we have $(0_{M;M} a) V (0_{M;M} b) = (0_{M;M} a \wedge b)$. The result follows from Theorem 2.

Proposition 9. Let $M$ be a comultiplication lattice $L$-module and $p, q$ be maximal elements of $L$. If
\((0_{M^L}p) \neq 0_M\) and \((0_{M^L}q) \neq 0_M\), then \((0_{M^L}p)V(0_{M^L}q) = (0_{M^L}p \land q)\).

**Proof:** Let \(0_M \neq (0_{M^L}p) = N\). Since \(pN = 0_M\) and \(p = 0\), we have \(p = (0_{M^L}N)\).

Similarly, if \(0_M \neq K = (0_{M^L}q)\), then \(q = (0_{M^L}K)\). Since \(M\) is a comultiplication \(L\)-module, it follows that \(N\) \(\land (0_{M^L}N) = (0_{M^L}(0_{M^L}N) \land (0_{M^L}K))\). Consequently, \((0_{M^L}p)V(0_{M^L}q) = (0_{M^L}p \land q)\).

**Definition 2.** A lattice \(L\)-module \(M\) is said to be finitely cogenerated, if for every set \(\{M_j\}_{j \in A}\) of elements of \(M\), \(\bigvee_{j \in A} M_j = 0_M\) implies \(\bigvee_{j=1}^m M_{j} = 0_M\) for some positive integer \(m > 0\).

**Theorem 3.** Let \(M\) be a faithful comultiplication \(PG\)-lattice \(L\)-module.

1. \(L\) is compact.
2. \((0_{M^L} \alpha) \neq 0_M\) for all \(\alpha < 1_L\).
3. \((0_{M^L}p) \neq 0_M\) for all maximal elements \(p \in L\).
4. \(M\) is finitely cogenerated.

**Proof:** (i)\(\Rightarrow\)(ii): Suppose that \((0_{M^L} \alpha) = 0_M\) and \(\alpha < 1_L\). Then \((0_{M^L}p) = 0_M\) for all maximal elements \(\alpha \leq p\). This is a contradiction by Corollary 2. (ii)\(\Rightarrow\)(iii): Clear. (iii)\(\Rightarrow\)(iv): Let \(N_\alpha = (0_{M^L}M_\alpha)\). Suppose that \(0_M = \bigvee_{\alpha \in A} N_\alpha = \bigvee_{\alpha \in A}(0_{M^L}M_\alpha) = (0_{M^L}V(\bigvee_{\alpha \in A}M_\alpha) = (0_{M^L}V(\bigvee_{\alpha \in A}M_\alpha) = 1_L\). Indeed, if \(V(\bigvee_{\alpha \in A}M_\alpha) \neq 0_M\) for some maximal element, then \((0_{M^L}p) \leq (0_{M^L}V(\bigvee_{\alpha \in A}M_\alpha)) = 0_M\). This is a contradiction with (iii). Since \(1_L\) is compact, \(V(\bigvee_{\alpha \in A}M_\alpha) = 1_L\). Hence we obtain \(0_M = (0_{M^L}1_L) = (0_{M^L}V(\bigvee_{\alpha \in A}M_\alpha)) = \bigvee_{\alpha \in A}(0_{M^L}M_\alpha) = \bigvee_{\alpha \in A}N_\alpha = \bigvee_{\alpha \in A}N_\alpha = 1_L\).

Let \(Jac(L)\) denote the infimum of the maximal elements of \(M\). Note that \(Jac(L)\) is called the Jacobson radical of \(L\) (Nakkar and Al-Khouja, 1985).

**Theorem 4.** (A dual of Nakayama Lemma for comultiplication lattice modules) Let \(L\) be a comultiplication \(PG\)-lattice \(L\)-module and \(\alpha \in L\) such that \(\alpha \leq Jac(L)\). If \((0_{M^L} \alpha) = 0_M\), then \(M = 0_M\).

**Proof:** Suppose that \(M \neq 0_M\). Then, there exists a maximal element \(p\) such that \(0_M \neq p = (0_{M^L}p)\) is minimal in \(M\) by Proposition 6 and Proposition 7. Since \(a \leq p\) and \((0_{M^L} \alpha) = 0_M\), we have \((0_{M^L}p) = 0_M\). This is a contradiction.

**Theorem 5.** Let \(M\) be a comultiplication \(PG\)-lattice \(L\)-module and \(\{N_\alpha\}_{\alpha \in A}\) be a collection of elements of \(M\) such that \(\bigvee_{\alpha \in A}N_\alpha = 0_M\). If \(a = V(\bigvee_{\alpha \in A}(0_{M^L}N_\alpha))\) and \(X\) is a compact element of \(M\), then \(1_L = aV(0_{M^L}X)\).

**Proof:** If \(X\) is compact and for \(a = V(\bigvee_{\alpha \in A}(0_{M^L}N_\alpha))\), \(aV(0_{M^L}X) \neq 1_L\), then there exists a maximal \(X\) \(p\) of \(M\) such that \(aV(0_{M^L}X) \neq p\). Then \((0_{M^L}p) \leq (0_{M^L}a) = (0_{M^L}V(\bigvee_{\alpha \in A}(0_{M^L}N_\alpha)))\). Hence \((0_{M^L}p) = (0_{M^L}X) = 0_M\). Since the submodule \([0_{M^L}X]\) is comultiplication and \(X\) is compact, there exists an element \(c \in L\), \(c \neq p\) such that \(c \neq Ann_L(X)\) by Proposition 5. But this is a contradiction, because \(Ann_L(X) = (0_{M^L}X) \leq p\). So \(aV(0_{M^L}X) = 1_L\).

**Corollary 5.** Let \(M\) be a comultiplication \(PG\)-lattice \(L\)-module. If \(X\) is a compact element of \(M\), then the submodule \([0_{M^L}X]\) is finitely cogenerated.
iii. Let $a \in L$ with $a1_M < 1_M$. Then $(0_{M^2;M}a) \neq 0_M$ by Proposition 3. There exists a minimal element $K$ in $M$ with $K \leq (0_{M^2;M}a)$ by Proposition 6. Hence there exists a maximal element $p$ in $L$ such that $K = (0_{M^2;M}p) \neq 0_M$ by Proposition 7. It follows that $(0_{M^2;1_M}) \subseteq (0_{M^2;1_M}) \subseteq (0_{M^2;K}) \subseteq (0_{M^2;L})$, which is a contradiction. 

Definition 4. (Nakkar and Anderson, 1988) Let $M$ be an L-module. An element $N < 1_M$ in $M$ is said to be primary, if $a \leq N$ implies $X \leq N$ or $a^k1_N \leq N$ for some $k \geq 0$ i.e. $a^k \leq (N;_N 1_M)$ for every $a \in L, X \in M$.

Definition 5. (Nakkar and Anderson, 1988) Let $M$ be an L-module. Let $B$ be an arbitrary element of $M$. A finite family $\{Q_i\}_{i=1}^n$ of elements of $M$ such that $Q_i$ is $P_i$-primary for any $i \in \{1, 2, \ldots, n\}$ and $B = \bigwedge_{i=1}^n Q_i$ is called a primary decomposition of $B$ in $M$. If no $Q_i$ contains $Q_1 \wedge Q_2 \wedge \ldots \wedge Q_{i-1} \wedge Q_{i+1} \ldots \wedge Q_n$ and if the elements $P_1, P_2, \ldots, P_n$ are all distinct, then the primary decomposition is said to be reduced (irredundant).

An L-module $M$ is called a K-lattice if it is a CG-lattice and for any compact element $h \in L$ and any compact element $h \in H$, the element $hH$ is compact. Let $L$ be a K-lattice in which the greatest element $1_L$ is irredundant and let $M$ be a K-lattice. Clearly for an arbitrary element $B$ of $M$, any primary decomposition of $B$ can be simplified to a reduced one (Nakkar and Anderson, 1988).

Theorem 7. Let $L$ be a K-lattice and let $M$ be a K-lattice. Let $M$ be a comultiplication lattice L-module. If $0_M$ has a primary decomposition, then every element of $M$ has a primary decomposition.

Proof: Let $0_M = \bigwedge_{i=1}^n P_i$ be irredundant primary decomposition. Assume that $N \in M$. Then there exists an $a \in L$ such that $N = (0_{M^2;M}a)$. Therefore, $N = (0_{M^2;M}a) = \bigwedge_{i=1}^n P_i_{M^2;M}a$. We will show that $(P_i_{M^2;M}a)$ is a primary element of $M$ for each $i = 1, 2, \ldots, n$. Suppose that $bX \leq (P_i_{M^2;M}a)$, where $b \in L$ and $X \in M$. Hence $abX \leq P_i$. Since $P_i$ is primary, there exists a positive integer $n$ such that $b^n1_M \leq P_i$ or $aX \leq P_i$. Hence $X \leq (P_i_{M^2;M}a)$ or $b^n1_M \leq P_i$. Therefore, $N \subseteq (P_i_{M^2;M}a)$, which is a contradiction. Consequently, $K = aN = N$.

Corollary 9. Let $M$ be a comultiplication lattice L-module and $N \in M$. Then the following are equivalent.

i. $N$ is a second element in $M$.
ii. $(0_{M^2;1}N)$ is a prime element in $L$.
Proof: (i)⇒(ii). Suppose that $N \not\subset M$ is a second element. Let $p = (0_{M:M} N)$. Suppose that $ab \leq p$ and $b \not\leq p$. Since $b \not\leq p$, we have $bN \not\subset M$. Since $N$ is a second element, we have $bN = N$ and so $0_M = (ab)N = a(bN) = aN$. Therefore, $a \leq p = (0_M;N)$. 

(ii)⇒(i). Proposition 10.

Definition 8. Let $M$ be an $L$-module. $M$ is said to be prime $L$-module if $0_M$ is prime element of $M$. It is clear that $0_M$ is prime element in $M$ if and only if $(0_{M:L} 1_M) = (0_M;N)$ for all $0_M \neq N \in M$.

Definition 9. Let $M$ be an $L$-module. $M$ is said to be coprime $L$-module if $(0_{M:L} 1_M) = (N;1_M)$ for all $N \in M$.

Proposition 11. Let $M$ be a nonzero comultiplication $PG$-lattice $L$-module.

i. Let $(M;A)\in A$ be a family of elements of a module $M$ with $\Lambda M = M$. Then $N = \Lambda (NVMX)$ for every $N \in M$.

ii. Let $p$ be a minimal element in $L$ and $(0_M;M p) = 0_M$. Then $M$ is simple.

Proof: i. Let $(M;A)\in A$ be a family of elements of a module $M$ with $\Lambda M = M$. Therefore, $N = (0_M;M (0_M;1_M N)) = \Lambda (M;M (0_M;1_M N))$ and $M \leq (M;M (0_M;1_M N))$ and $N \leq (0_M;M (0_M;1_M N))$ for $\lambda \in A$. Therefore, $N \leq \Lambda (M;M (0_M;1_M N)) \leq \Lambda (M;M (0_M;1_M N)) \geq N$.

ii. Let $0_M \neq X \in M$ be a principal element and let $p$ be a minimal element in $L$ such that $(0_M;M p) = 0_M$. There exists $a \in L$ such that $X = (0_M;M a)$. Then $X = (0_M;M a) = (0_M;M p);M a) = (0_M;M a)p$. Since $p$ is minimal, we have $0 \leq ap \leq p$ and so $ap = 0_L$ or $ap = p$. If $ap = p$, $X = (0_M;M ap) = (0_M;M p) = 0_M$. This is a contradiction. Hence $ap = 0_L$. Therefore, $X = (0_M;M ap) = 1_M$ is principal. Consequently, $M$ is cyclic. Since $M = (0_M;1_M)$, $M$ is simple.

Let $L$ be a multiplicative lattice. An element $a \in L$ is called zero-divisor if there exists an element $0_L \not\leq b \in L$ such that $ab = 0_L$. $L$ is said to be a domain if it has only zero-divisor $0_L$. Note that $Z(L)$ denote the set of zero divisors of $L$.

Lemma 2. Let $M$ be a faithful comultiplication $L$-module. Then $W(M) = \{a \in L : a1_M < 1_M\} = Z(L)$.

Proof: Let $a \in W(M)$. Then $1_M < 1_M$. Since $M$ is comultiplication, $1_M = (0_M;M (0_M;1_M a1_M))$. It is clear that $(0_M;1_M a1_M) \not\leq 0_L$ and $(0_M;1_M a1_M)a1_M = 0_M$. Since $M$ is faithful, $(0_M;1_M a1_M)a = 0_L$. We have $a \in Z(L)$. Conversely, let $a \in Z(L)$. There exists $0_L \not\leq b \in L$ such that $ab = 0_L$. Therefore, $(ab)1_M = b(a1_M) = 0_M = a1_M \leq (0_M;M b) \not\leq 1_M$. Indeed, if $(0_M;M b) = 1_M$, then $b1_M = 0_M$. Since $M$ is faithful, we have $b = 0_L$. This is a contradiction. Therefore $a1_M \not\leq 1_M$ and so $a \in W(M)$.

Definition 7. (Nakkar and Anderson, 1988) Let $M$ be an $L$-module. An element $N < 1_M$ in $M$ is said to be prime, if $aX \leq N$ implies $X \leq N$ or $a1_M \leq N$ i.e. $a \leq (N;1_M)$ for every $a \in L$, $X \in M$.

Proof: Assume that $M$ is not cyclic. Then, $M$ is...
tobation. Hence \((0_{M^∗:L}X) \neq 0_L\) for all principal \(X\). Since \(1_M\) is compact, \(1_M = VX_1\) implies that \(1_M = \bigvee_{i=1}^n X_i\) for some principal elements \(X_i\). Therefore, \(0_L = (0_{M^∗:L}1_M) = A_{i=1}(0_{M^∗:L}X_i) \geq \bigwedge_{i=1}^n (0_{M^∗:L}X_i) \neq 0_L\). This is a contradiction.

**Proposition 13.** Let \(L\) be a comultiplication PG-lattice and \(M\) be a faithful PG-lattice L-module. Then for each \(a \in L\), with \(a < 1_L\), \((0_{M^∗:M}a) \neq 0_M\) and \(a 1_M < 1_M\).

**Proof:** Let \(a \in L\), with \(a < 1_L\). Suppose that \((0_{M^∗:M}a) = 0_M\). Then \((0_{L^∗:L}a)1_M = 0_M\), for if \((0_{L^∗:L}a)1_M \neq 0_M\), there exists a principal element \(x \in L\) such that \(x \leq (0_{L^∗:L}a)\) and a principal element \(Y \in M\) such that \(xY \neq 0_M\). Since \(ax = 0_L\), we have \(axY = 0_M\). Then \(xY = (0_{L^∗:L}a) = 0_M\). This is a contradiction. Since \((0_{L^∗:L}a)1_M = 0_M\), it follows that \((0_{L^∗:L}a) \leq \text{Ann}_L(M) = 0_L\). Since \(L\) is a comultiplication lattice, \(a = (0_{L^∗:L}a) = 1_L\). This is a contradiction. Now suppose that \(a 1_M = 1_M\). Therefore \((0_{L^∗:L}a) = (0_{L^∗:L}1_M) = 0_L\). Since \(L\) is a comultiplication lattice, \(a = 1_L\). This is a contradiction.

**Definition 11.** Let \(M\) be an L-module and \(N\) a non-zero element of \(M\). Then \(N\) is said to be large if for every element \(K\) in \(M\) such that \(N \land K = 0_M\) implies \(K = 0_M\).

**Definition 12.** Let \(M\) be an L-module and \(N\) be a proper element of \(M\). Then \(N\) is said to be small if for every element \(K\) in \(M\) such that \(NVK = 1_M\) implies that \(K = 1_M\).

**Proposition 14.** Let \(M\) be a faithful comultiplication PG-lattice L-module with \(1_M\) compact. Then every non-zero element of \(M\) is large if and only if every element \(a \in L\), with \(a < 1_L\) is small.

**Proof:** Suppose that every non-zero element of \(M\) is large and let \(a \in L\), \(a < 1_L\) such that \(a \land Vb = 1_L\) for some \(0_L \neq b \in L\). Then \(0_M = (0_{M^∗:M}a \land Vb) = (0_{M^∗:M}a) \land (0_{M^∗:M}b)\). Since \(a < 1_L\) we have \((0_{M^∗:M}a) \neq 0_M\) by Theorem 3. We know that \((0_{M^∗:M}a)\) is large. Hence \((0_{M^∗:M}b) = 0_M\). Therefore, we obtain \(b = 1_L\). Suppose that \(N \in M\) such that \(K \land N = 0_M\) where \(0_M \neq K \in M\). Since \(M\) is a comultiplication L-module, \(K = (0_{M^∗:M}(0_{M^∗:L}K))\) and \(N = (0_{M^∗:M}(0_{M^∗:L}N))\). Then \(0_M = K \land N = (0_{M^∗:M}(0_{M^∗:L}K) \land (0_{M^∗:L}N))\) and so \((0_{M^∗:L}K) \land (0_{M^∗:L}N) = 1_L\). Since \(0_M \neq K\), we have \((0_{M^∗:L}K) \neq 1_L\). Since \((0_{M^∗:L}K)\) is small, it follows that \((0_{M^∗:L}N) = 1_L\) and so \(N = 0_M\).

**Proposition 15.** Let \(M\) be a faithful comultiplication PG-lattice L-module with \(1_M\) compact. Then \(N \in M\) is large if and only if there exists a small element \(a \in L\) such that \(N = (0_{M^∗:M}a)\).

**Proof:** Suppose that \(N \in M\) is large. Since \(M\) is a comultiplication L-module, \(N = (0_{M^∗:M}a)\). Suppose \(a \land Vb = 1_L\) for some \(0_L \neq b \in L\). Then \(N \land (0_{M^∗:M}b) = (0_{M^∗:M}a) \land (0_{M^∗:M}b) = (0_{M^∗:M}a) \land Vb = 0_M\). Since \(N\) is large, we have \((0_{M^∗:M}b) = 0_M\), hence by Theorem 3, we have \(b = 1_L\). So \(a\) is small.

\(\Leftrightarrow\): Suppose that \(a \in L\) be a small element of \(L\). Let \(N = (0_{M^∗:M}a)\). Assume that \(K \in M\) such that \(N \land K = 0_M\). Since \(M\) is a comultiplication L-module, there exists \(b \in L\) such that \(K = (0_{M^∗:M}b)\). Then \(0_M = N \land K = (0_{M^∗:M}a) \land (0_{M^∗:M}b) = (0_{M^∗:M}a \land Vb) = 0_M\) and so \(a \land Vb = 1_L\) by Theorem 3. Therefore \(b = 1_L\). Hence \(K = (0_{M^∗:M}b) = 0_M\). Consequently, \(N\) is large.

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**References**


