IJST (2015) 39A2: 213-220
Iranian Journal of Science & Technology

http://ijsts.shirazu.ac.ir

Comultiplication lattice modules

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Abstract

Let M be a lattice module over the multiplicative lattice L. M is said to be a comultiplication L-module if for every element N of M there exists an element $a \in L$ such that $N = (0_M :_M a)$. Our objective is to investigate properties of comultiplication lattice modules.

***Keywords**:* Multiplicative lattice; lattice modules; comultiplication lattice modules

1. Introduction

A multiplicative lattice L is a complete lattice in which there is defined a commutative, associative multiplication which distributes over arbitrary joins and has a compact greatest element 1_L (least element 0_L) as a multiplicative identity (zero). Let L be a multiplicative lattice and $a \in L$, $L/a = \{b \in L : a \leq b\}$be a multiplicative lattice with multiplication $c \circ d = cd \vee a$. Multiplicative lattices have been studied (Jayaram and Johnson, 1995, 1997, 1998; Johnson, 2002, 2003, 2004; Johnson and Johnson, 2003).

An element $a \in L$ is said to be proper if $a < 1$. An element $p < 1$ in L is said to be prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$. An element $m < 1$ in L is said to be maximal if $m < x \leq 1$ implies $x = 1$. It is easily seen that maximal elements are prime.

If a, b belong to L, $(a :_L b)$ is the join of all $c \in L$ such that $cb \leq a$. An element e of L is called meet principal if $a \wedge be = ((a :_L e) \wedge b))e$ for all $a, b \in L$. An element e of L is called join principal if $((ae \vee b) :_L e) = a \vee (b :_L e)$ for all $a, b \in L$. $e \in L$ is said to be principal if e is both meet principal and join principal. $e \in L$ is said to be weak meet (join) principal if $a \wedge e = e(a :_L e)$ $(a \vee (0_L :_L e) = (ea :_L e))$ for all $a \in L$. An element a of a multiplicative lattice L is called compact if $a \leq \vee b_\alpha$ implies $a \leq b_{\alpha_1} \vee b_{\alpha_2} \vee \ldots \vee b_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. If each element of L is a join of principal (compact) elements of L, then L is called a PG-lattice (CG-lattice).

Let M be a complete lattice. Recall that M is a lattice module over the multiplicative lattice L, or simply an L-module in case there is a multiplication between elements of L and M, denoted by lB for $l \in L$ and $B \in M$, which satisfies the following properties:
i. $(lb)B = l(bB)$;
ii. $(\vee_\alpha l_\alpha)(\vee_\beta B_\beta) = \vee_{\alpha,\beta} l_\alpha B_\beta$;
iii. $1_L B = B$;
iv. $0_L B = 0_M$; for all l, l_α, b in L and for all B, B_β in M.

Let M be an L-module. If N, K belong to M, $(N :_L K)$ is the join of all $a \in L$ such that $aK \leq N$. If $a \in L$, then $(0_M :_M a)$ is the join of all $H \in M$ such that $aH = 0_M$. An element N of M is called meet principal if $\left(b \wedge (B :_L N)\right)N = bN \wedge B$ for all $b \in L$ and for all $B \in M$. An element N of M is called join principal if $b \vee (B :_L N) = \left((bN \vee B) :_L N\right)$ for all $b \in L$ and for all $B \in M$. N is said to be principal if it is both meet principal and join principal. In a special case, an element N of M is called weak meet principal (weak join principal) if $(B :_L N)N = B \wedge N$ $\left((bN :_L N) = b \vee (0_M :_L N)\right)$ for all $B \in M$ (for all $b \in L$). N is said to be weak principal if N is both weak meet principal and weak join principal.

Let M be an L-module. An element N in M is called compact if $N \leq \vee_a B_a$ implies $N \leq B_{\alpha_1} \vee B_{\alpha_2} \vee \ldots \vee B_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. The greatest element of M will be denoted by 1_M. If each element of M is a join of principal (compact) elements of M, then M is called a PG-lattice (CG-lattice).

Let M be an L-module. An element $N \in M$ is said to be proper if $N < 1_M$. If $Ann(M) = (0_M :_L 1_M) = 0_L$, then M is called a faithful L-module. If $cm =$

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Received: 1 August 2013 / Accepted: 25 November 2014

0_M implies $m = 0_M$ or $c = 0_L$ for any $c \in L$ and $m \in M$, M is called a torsion-free L-module.

For various characterizations of lattice modules, the reader is referred to Nakkar and Al-Khouja (1989), Nakkar and Anderson (1988) and Scott Culhan (2005). In this paper we study comultiplication lattice modules over a multiplicative lattice and generalize the important results for comultiplication modules over commutative rings, obtained by Ansari-Toroghy and Farshadifar (2007, 2011), Shaniafi and Smith (2011) to the lattice modules over multiplicative lattice.

2. Comultiplication Lattice Modules

Definition 1.
i. (Callıalp and Tekir, 2011) An L-module M is called a multiplication lattice module if for every element $N \in M$ there exists an element $a \in L$ such that $N = a1_M$
ii. Let M be a lattice L-module. M is said to be a comultiplication L-module if for every element N of M there exists an element $a \in L$ such that $N = (0_M :_M a)$.

Lemma 1. Let M be a lattice L-module. Then, M is a comultiplication lattice L-module if and only if $N = (0_M :_M (0_M :_L N))$ for every element N in M.

Proof: ⇐:Clear.
⇒: Suppose that M is a comultiplication lattice L-module and $N \in M$. Then there exists an $a \in L$ such that $N = (0_M :_M a)$. Thus we have $a \leq (0_M :_L N)$ so that $(0_M :_M (0_M :_L N)) \leq (0_M :_M a) = N$. It is clear that $N \leq \left(0_M :_M (0_M :_L N)\right)$. This implies $N = (0_M :_M (0_M :_L N))$.

Proposition 1. Let M be a lattice L-module. Then the followings are equivalent.
i. For any $K, N \in M$, $(0_M :_L K) \leq (0_M :_L N)$ implies that $N \leq K$.
ii. For any $K, N \in M$, $(K :_L N) = \left((0_M :_L N) :_L (0_M :_L K)\right)$.

Proof: (i)⇒(ii): For any $K, N \in M$, $(K :_L N) \leq \left((0_M :_L N) :_L (0_M :_L K)\right)$. Indeed $b = (K :_L N) \Rightarrow bN \leq K \Rightarrow b(0_M :_L K)N = 0_M \Rightarrow b(0_M :_L K) \leq (0_M :_L N) \Rightarrow b = (K :_L N) \leq \left((0_M :_L N) :_L (0_M :_L K)\right)$. Conversely, let $r = \left((0_M :_L N) :_L (0_M :_L K)\right)$. Then $r(0_M :_L K)N = 0_M \Rightarrow (0_M :_L K) \leq (0_M :_L rN)$, by (i), we have $rN \leq K$ and so $r \leq (K :_L N)$. (ii)⇒(i): Suppose that $(0_M :_L K) \leq (0_M :_L N)$.Then $(K :_L N) = ((0_M :_L N) :_L (0_M :_L K)) = 1_L$ by (ii) and so $N \leq K$.

Theorem 1. Let M be a lattice L-module. Suppose $\phi: L \to M$ is defined by $\phi(a) = (0_M :_M a)$ and $\psi: M \to L$ by $\psi(N) = (0_M :_L N)$ for all $a \in L$ and $N \in M$. Then,
i. $(\phi\psi\phi)(a) = (0_M :_M (0_M :_L (0_M :_M a))) = (0_M :_M a) = \phi(a)$ for all $a \in L$.
ii. $(\psi\phi\psi)(N) = (0_M :_L (0_M :_M (0_M :_L N))) = (0_M :_L N) = \psi(N)$ for all $N \in M$.

Proof: i. Suppose that $(0_M :_M a) = N$. Clearly, $(0_M :_M a) = N \leq \left(0_M :_M (0_M :_L N)\right)$. On the other hand, $aN = 0_M$ and so $a \leq (0_M :_L N)$. Therefore, $\left(0_M :_M (0_M :_L N)\right) \leq (0_M :_M a) = N$.
ii. Suppose that $b = (0_M :_L N)$.Clearly $b = (0_M :_L N) \leq \left(0_M :_L \left(0_M :_M (0_M :_L N)\right)\right)$. On the other hand,$bN = 0_M$and so $N \leq (0_M :_M b) = \left(0_M :_M (0_M :_L N)\right)$. Hence $(0_M :_L (0_M :_M (0_M :_L N))) \leq (0_M :_L N) = b$.

Corollary 1. Let M be a lattice L-module. Let us define $\phi: L \to M$ where $\phi(a) = (0_M :_M a)$, and $\psi: M \to L$ where $\psi(N) = (0_M :_L N)$ for all $a \in L$ and $N \in M$. The followings are equivalent.
i. M is a comultiplication lattice L-module.
ii. There exists $a \in L$ such that $N = (0_M :_M a) = \phi(a)$ for all $N \in M$.
iii. $\phi\psi$ is an identity map.
iv. ψ is one-to-one.
v. $(0_M :_L K) = (0_M :_L N)$ implies $K = N$.

Proposition 2. Let M be a comultiplication lattice L-module. If L is a Noetherian (Artinian) multiplicative lattice, then M is an Artinian (Noetherian) lattice L −module.

Proof: Let L be an Artinian multiplicative lattice. Suppose that $N_1 \leq N_2 \leq \ldots$ Then, $(0_M :_L N_1) \geq (0_M :_L N_2) \geq \cdots$ Since L is Artinian, there exists a positive integer k such that $(0_M :_L N_k) = (0_M :_L N_{k+1}) = \cdots$. Therefore, $N_k = (0_M :_M (0_M :_L N_k)) = (0_M :_M (0_M :_L N_{k+1})) = N_{k+1} = \ldots$. Consequently, M is a Noetherian lattice module. Similarly, if L is Noetherian, then M is Artinian lattice L-module.

Let L be a multiplicative lattice and M be an L-module. Suppose that $N \in M$. Consider the set $[0_M, N] = \{A \leq N : A \in M\}$. We say that $[0_M, N]$ is a submodule of M. If M is a comultiplication L-module, it is clear that $[0_M, N]$ is a comultiplication L-module.

Proposition 3. Let M be a comultiplication lattice L-module. If $(0_M :_M b) = 0_M$ for some $b \in L$, then $bY = Y$ for all $Y \in M$. In particular, $b1_M = 1_M$.

Proof: Let $b \in L$ and $Y \in M$. Since M is a comultiplication lattice module, it follows that $bY = (0_M :_M a)$ for some $a \in L$. Then $abY = 0_M$. Since $(0_M :_M b) = 0_M$, we have $aY = 0_M$. Consequently, $Y \leq (0_M :_M a) = bY$ and so $bY = Y$.

Proposition 4. Let M be a comultiplication lattice L-module. If p is a maximal element of L and $(0_M :_M p) \neq 0_M$, then $(0_M :_M p)$ is minimal in M.

Proof: Suppose that $N \leq (0_M :_M p)$. Since M is a comultiplication lattice L-module, there exists an element a of L such that $N = (0_M :_M a)$. Since $N \leq (0_M :_M p)$, we have $pN = 0_M$ and so $p \leq (0_M :_L N)$. Since p is maximal, $p = (0_M :_L N)$ or $(0_M :_L N) = 1_L$. If $p = (0_M :_L N)$, then $N = (0_M :_M (0_M :_L N)) = (0_M :_M p)$. If $(0_M :_L N) = 1_L$, then $N = 0_M$. Therefore, $(0_M :_M p)$ is minimal in M.

Proposition 5. Let M be a comultiplication PG-lattice L-module with 1_M compact. If $p \in L$ is prime and $(0_M :_M p) = 0_M$, then there exists $c \in L$ such that $c \nleq p$ and $c1_M = 0_M$.

Proof: Since 1_M is compact, then $1_M = \bigvee_{i=1}^{n} Y_i$ where Y_i s are principal elements of M. Since $(0_M :_M p) = 0_M$, $pY_i = Y_i$ for all $i \in \{1,2,\dots,n\}$ by Proposition 3. Then $p \vee (0_M :_L Y_i) = (pY_i :_L Y_i) = 1_L$ and so $(0_M :_L Y_i) \nleq p$ for all $i \in \{1,2,\dots,n\}$. Therefore, $c = \prod_{i=1}^{n} (0_M :_L Y_i) \nleq p$ and $c1_M = 0_M$.

Corollary 2. Let M be a comultiplication PG-lattice L-module with 1_M compact. If M is faithful, then $(0_M :_M p) \neq 0_M$ for some prime element $p \in L$.

Corollary 3. If M is a comultiplication PG-lattice L-module with 1_M compact and $(0_M :_M a) = 0_M$ for some $a \in L$, then $1_L = a \vee (0_M :_L 1_M)$.

Proof: Suppose that $1_L \neq a \vee (0_M :_L 1_M)$. Then there exists a maximal element $p \in L$ such that $a \vee (0_M :_L 1_M) \leq p$. Thus we have $(0_M :_M p) \leq (0_M :_M a) = 0_M$. Hence $(0_M :_M p) = 0_M$. There exists an element $c \in L$, $c \nleq p$ such that $c \leq (0_M :_L 1_M)$ by Proposition 5. Since $(0_M :_L 1_M) \leq p$, we have $c \leq p$. This is a contradiction. Consequently, $a \vee (0_M :_L 1_M) = 1_L$.

Proposition 6. Let M be a non-zero comultiplication PG-lattice L-module. Then, M has a minimal element. In particular, every nonzero element of M has a minimal element.

Proof: Suppose that Y is a nonzero principal element of M. Then $(0_M :_L Y) = a < 1_L$. Then there exists a maximal element p such that $a \leq p$. If $N = (0_M :_M p) = 0_M$, then $pY = Y$ by Proposition 3 and so $p \vee (0_M :_L Y) = (pY :_L Y) = 1_L$. Therefore, $a = (0_M :_L Y) \nleq p$. This is a contradiction. Hence $N = (0_M :_M p) \neq 0_M$. Therefore, N is a minimal element of M by Proposition 4.

Proposition 7. Let M be a non-zero comultiplication PG-lattice$L$-module. Then $K \in M$ is minimal if and only if $K = (0_M :_M p) \neq 0_M$ for some maximal element $p \in L$.

Proof: $\Leftarrow$: By Proposition 4.
$\Rightarrow$: Let K be a minimal principal element of M. Since M is a comultiplication lattice L-module, $K = (0_M :_M (0_M :_L K))$. We will show that $(0_M :_L K)$ is maximal. Let $c \in L$ such that $(0_M :_L K) \leq c$. Since K is minimal and $cK \leq K$, it follows that $cK = K$ or $cK = 0_M$. If $cK = K$, then $1_L = (cK :_L K) = c \vee (0_M :_L K) = c$. If $cK = 0_M$, then $c \leq (0_M :_L K)$ and so $c = (0_M :_L K)$.

Proposition 8. Let M be a comultiplication lattice L-module. Then, $(N :_M a) = \big((0_M :_M a) :_M (0_M :_L N)\big)$ for any $a \in L, N \in M$.

Proof: Let $K = (N :_M a)$. Then $aK \leq N \Rightarrow (0_M :_L N)aK = 0_M \Rightarrow (0_M :_L N)K \leq (0_M :_M a) \Rightarrow K = (N :_M a) \leq \big((0_M :_M a) :_M (0_M :_L N)\big)$.
Conversely, if $R = \big((0_M :_M a) :_M (0_M :_L N)\big)$, then $(0_M :_L N)R \leq (0_M :_M a) \Rightarrow (0_M :_L N)aR = 0_M \Rightarrow aR \leq \big(0_M :_M (0_M :_L N)\big) = N$. Consequently, $R \leq (N :_M a)$.

Theorem 2. Let L be a distributive lattice. Let M be a comultiplication lattice L-module and $(0_M :_M a) \vee (0_M :_M b) = (0_M :_M a \wedge b)$ for all $a, b \in L$. Then M is distributive.

Proof: Let $X, Y, Z \in M$. There exist $a, b, c \in L$ such that $X = (0_M :_M a)$, $Y = (0_M :_M b)$, $Z = (0_M :_M c)$. Then,
$(X \vee Y) \wedge Z = \big((0_M :_M a) \vee (0_M :_M b)\big) \wedge (0_M :_M c) = (0_M :_M \mathrm{a} \wedge \mathrm{b}) \wedge (0_M :_M \mathrm{c}) = (0_M :_M a \wedge b) \wedge (0_M :_M c) = (0_M :_M (a \wedge b) \vee c) = \big(0_M :_M (a \wedge c) \vee (b \wedge c)\big) = (0_M :_M a \wedge c) \wedge (0_M :_M b \wedge c) = (X \vee Z) \wedge (Y \vee Z)$.

Corollary 4. Let L be a distributive lattice. Let M be a comultiplication lattice L-module and $a \vee b = 1_L$ for all $a, b \in L$. Then M is distributive.

Proof: If $a \vee b = 1_L$, then$(K :_M a \wedge b) = (K :_M a \wedge b)(a \vee b) = a(K :_M a \wedge b) \vee b(K :_M a \wedge b) \leq (K :_M b) \vee (K :_M a)$ for all $a, b, c \in L$.Note that $a(K :_M a \wedge b) \leq (K :_M b)$ and $b(K :_M a \wedge b) \leq (K :_M a)$.It is clear that $(K :_M b) \vee (K :_M a) \leq (K :_M a \wedge b)$. For$K = 0_M$, we have $(0_M :_M a) \vee (0_M :_M b) = (0_M :_M a \wedge b)$. The result follows from Theorem 2.

Proposition 9. Let M be a comultiplication lattice L-module and p, q be maximal elements of L. If

$(0_M :_M p) \neq 0_M$ and $(0_M :_M q) \neq 0_M$, then $(0_M :_M p) \vee (0_M :_M q) = (0_M :_M p \wedge q)$.

Proof: Let $0_M \neq (0_M :_M p) = N$. Since $pN = 0_M$ and p is maximal, we have $p = (0_M :_L N)$. Similarly, if $0_M \neq K = (0_M :_M q)$, then $q = (0_M :_L K)$. Since M is a comultiplication L-module, it follows that $N \vee K = \left(0_M :_M (0_M :_L N \vee K)\right) = \left(0_M :_M (0_M :_L N) \wedge (0_M :_L K)\right)$. Consequently, $(0_M :_M p) \vee (0_M :_M q) = (0_M :_M p \wedge q)$.

Definition 2. A lattice L-module M is said to be finitely cogenerated, if for every set $\{M_\lambda\}_{\lambda \in \Lambda}$ of elements of M, $\wedge_{\lambda \in \Lambda} M_\lambda = 0_M$ implies $\wedge_{i=1}^{m} M_{\lambda_i} = 0_M$ for some positive integer $m > 0$.

Theorem 3. Let M be a faithful comultiplication PG-lattice L-module.
i. 1_M is compact.
ii. $(0_M :_M a) \neq 0_M$ for all $a < 1_L$.
iii. $(0_M :_M p) \neq 0_M$ for all maximal elements $p \in L$.
iv. M is finitely cogenerated.
Then (i)⇒(ii)⇒(iii)⇒(iv).

Proof: (i)⇒(ii): Suppose that $(0_M :_M a) = 0_M$ and $a < 1_L$. Then$(0_M :_M p) = 0_M$ for all maximal elements$a \leq p$.This is a contradiction by Corollary 2. (ii)⇒(iii): Clear. (iii)⇒(iv): Let $N_\alpha = (0_M :_M a_\alpha)$. Supppose that $0_M = \wedge_{\alpha \in I} N_\alpha = \wedge_{\alpha \in I} (0_M :_M a_\alpha) = (0_M :_M \vee_{\alpha \in I} a_\alpha)$. Then $\vee_{\alpha \in I} a_\alpha = 1_L$. Indeed, if $\vee_{\alpha \in I} a_\alpha \leq p$ for some maximal element$p$, then $(0_M :_M p) \leq (0_M :_M \vee_{\alpha \in I} a_\alpha) = 0_M$. This is a contradiction with (iii). Since 1_L is compact, $\vee_{i=1}^{n} a_{\alpha_i} = 1_L$ and so $0_M = \wedge_{\alpha \in I} N_\alpha = (0_M :_M \vee_{i=1}^{n} a_{\alpha_i}) = \wedge_{i=1}^{n} (0_M :_M a_{\alpha_i}) = \wedge_{i=1}^{n} N_{\alpha_i}$ for some $n \geq 1$.

Let $Jac(L)$ denote the infimum of the maximal elements of L. Note that $Jac(L)$ is called the Jacobson radical of L (Nakkar and Al-Khouja, 1985).

Theorem 4. (A dual of Nakayama Lemma for comultiplication lattice modules) Let M be a comultiplication PG-lattice L-module and $a \in L$ such that $a \leq Jac(L)$. If $(0_M :_M a) = 0_M$, then $M = 0_M$.

Proof: Suppose that $M \neq 0_M$. Then, there exists a maximal element p such that $0_M \neq K = (0_M :_M p)$ is minimal in M by Proposition 6 and Proposition 7. Since $a \leq p$ and $(0_M :_M a) = 0_M$, we have $(0_M :_M p) = 0_M$. This is a contradiction.

Theorem 5. Let M be a comultiplication PG-lattice L-module and $\{N_\alpha\}_{\alpha \in \Lambda}$ be a collection of elements of M such that $\wedge_{\alpha \in \Lambda} N_\alpha = 0_M$. If $a = \vee_{\alpha \in \Lambda} (0_M :_L N_\alpha)$ and X.is a compact element of M, then $1_L = a \vee (0_M :_L X)$.

Proof: If X is compact and for $a = \vee_{\alpha \in \Lambda} (0_M :_L N_\alpha)$, $a \vee (0_M :_L X) \neq 1_L$, then there exists a maximal element p of L such that $a \vee (0_M :_L X) \leq p$. Then $(0_M :_M p) \leq (0_M :_M a) = (0_M :_M \vee_{\alpha \in \Lambda} (0_M :_L N_\alpha)) = \wedge_{\alpha \in \Lambda} N_\alpha = 0_M$. Hence $(0_M :_M p) = (0_M :_X p) = 0_M$. Since the submodule $[0_M, X]$ is comultiplication and X is compact, there exists an element $c \in L$, $c \nleq p$ such that $c \leq Ann_L(X)$ by Proposition 5. But this is a contradiction, because $Ann_L(X) = (0_M :_L X) \leq p$. So $a \vee (0_M :_L X) = 1_L$.

Corollary 5. Let M be a comultiplication PG-lattice L-module. If X is a compact element of M, then the submodule $[0_M, X]$ is finitely cogenerated.

Proof: Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be set of elements such that $X_\lambda \leq X$ with $\wedge_{\lambda \in \Lambda} X_\lambda = 0_M$. By Theorem 5, $1_L = a \vee (0_M :_L X)$ with $a = \vee_{\lambda \in \Lambda} (0_M :_L X_\lambda)$. But $(0_M :_L X) \leq (0_M :_L X_\lambda)$ for all $\lambda \in \Lambda$. Hence $1_L = a = \vee (0_M :_L X_\lambda)$. Since 1_L is compact, it follows that $1_L = a = \vee_{i=1}^{n} (0_M :_L X_{\lambda_i})$ for some $n \geq 1$. Since the submodule $[0_M, X]$ is a comultiplication module, we have $X_\lambda = (0_M :_M (0_M :_L X_\lambda))$ for all $\lambda \in \Lambda$. Hence we obtain $0_M = (0_M :_M 1_L) = (0_M :_M \vee_{i=1}^{n} (0_M :_L X_{\lambda_i})) = \wedge_{i=1}^{n} (0_M :_M (0_M :_L X_{\lambda_i})) = \wedge_{i=1}^{n} X_{\lambda_i}$.

Definition 3. (Callıalp and Tekir, 2011) Let M be an L-module. If 1_M is a principal element in M, then M is called a cyclic lattice module.

Theorem 6. Let M be a PG-lattice module.
i. If M is a multiplication L-module such that M has a faithful $[0_M, N]$ submodule and N is principal in M, then $a1_M < 1_M$ for every element $a \in L$ with $a < 1_L$.
ii. If M is a faithful cyclic comultiplication L-module, then $(0_M :_M a) \neq 0_M$ for every $a \in L$ with $a < 1_L$.
iii. If M is a comultiplication L-module, then for every element $a \in L$ with $a1_M < 1_M$, there exists a maximal element $p \in L$ with $a \vee (0_M :_L 1_M) \leq p$ such that $(0_M :_M p)$ is a minimal in M.

Proof: i. Since M is a multiplication L-module, there exists $b \in L$ such that $N = b1_M$. If there exists $a \in L$ with $a < 1_L$ such that $a1_M = 1_M$, then $N = b1_M = a(b1_M) = aN$. Since N is principal, $a \vee (0_M :_L N) = (aN :_L N) = 1_L$ and so $a = 1_L$.This is a contradiction.
ii. If there exists $a \in L$ with $a < 1_L$ such that $(0_M :_M a) = 0_M$, then $a1_M = 1_M$ by Proposition 3. Since 1_M is principal and $(0_M :_L 1_M) = 0_M$, we have $1_L = (a1_M :_L 1_M) = a \vee (0_M :_L 1_M) = a$. This is a

contradiction.

iii. Let $a \in L$ with $a1_M < 1_M$. Then $(0_M :_M a) \neq 0_M$ by Proposition 3. There exists a minimal element K in M with $K \leq (0_M :_M a)$ by Proposition 6. Hence there exists a maximal element $p \in L$ such that $K = (0_M :_M p) \neq 0_M$ by Proposition 7. It follows that $(0_M :_L 1_M) \leq p$. Indeed, $(0_M :_L 1_M) \leq (0_M :_L K) = \left(0_M :_L (0_M :_M p)\right) \geq p$. Since p is maximal, $\left(0_M :_L (0_M :_M p)\right) = p$, so $(0_M :_L 1_M) \leq p$. The proof will be completed if we show that $a \leq p$. Suppose that $a \nleq p$. Then $a \vee p = 1_L$. Since $K \leq (0_M :_M a)$, it follows that $0_M = (0_M :_M 1_L) = (0_M :_M a \vee p) = (0_M :_M a) \wedge (0_M :_M p) = K$. Hence $K = 0_M$. This is a contradiction. We obtain $a \leq p$.

Definition 4. (Nakkar and Anderson, 1988) Let M be an L-module. An element $N < 1_M$ in M is said to be primary, if $aX \leq N$ implies $X \leq N$ or $a^k 1_M \leq N$ for some $k \geq 0$ i.e. $a^k \leq (N :_L 1_M)$ for every $a \in L, X \in M$.

Definition 5. (Nakkar and Anderson, 1988) Let M be an L-module. Let B be an arbitrary element of M. A finite family $\{Q_i\}_{i=1}^n$ of elements of M such that Q_i is P_i-primary for any $i \in \{1,2,\ldots,n\}$ and $B = \wedge_{i=1}^n Q_i$, is called a primary decomposition of B in M. If no Q_i contains $Q_1 \wedge Q_2 \wedge \ldots \wedge Q_{i-1} \wedge Q_{i+1} \wedge \ldots \wedge Q_n$ and if the elements $P_1, P_2, \ldots, P_n$ are all distinct, then the primary decomposition is said to be reduced (irredundant).

An L-module M is called a K-lattice if it is a CG-lattice and for any compact element $h \in L$ and any compact element $H \in M$, the element hH is compact. Let L be a K-lattice in which the greatest element 1_L is compact and let M be a K-lattice. Clearly for an arbitrary element B of M, any primary decomposition of B can be simplified to a reduced one **(**Nakkar and Anderson, 1988).

Theorem 7. Let L be a K-lattice and let M be a K-lattice. Let M be a comultiplication lattice L-module. If 0_M has a primary decomposition, then every element of M has a primary decomposition.

Proof: Let $0_M = \wedge_{i=1}^n P_i$ be irredundant primary decomposition. Assume that $N \in M$. Then there exists an $a \in L$ such that $N = (0_M :_M a)$. Therefore, $N = (0_M :_M a) = \wedge_{i=1}^n (P_i :_M a)$. We will show that $(P_i :_M a)$ is a primary element of M for each $i = 1,2,\ldots,n$. Suppose that $bX \leq (P_i :_M a)$, where $b \in L$ and $X \in M$. Hence $abX \leq P_i$. Since P_i is primary, there exists a positive integer n such that $b^n 1_M \leq P_i$ or $aX \leq P_i$. Hence $X \leq (P_i :_M a)$ or $b^n 1_M \leq P_i \leq (P_i :_M a)$.

Theorem 8. Let M be a lattice L-module. Then the followings are equivalent.

i. M is a comultiplication module.

ii. For every element $N \in M$ and each element $c \in L$ with $N < (0_M :_M c)$, there exists an element $b \in L$ such that $c < b$ and $N = (0_M :_M b)$.

iii. For every element $N \in M$ and each element $c \in L$ with $N < (0_M :_M c)$, there exists an element $b \in L$ such that $c < b$ and $N \leq (0_M :_M b)$.

Proof: (i)⇒(ii). Let $N < (0_M :_M c)$ where $N \in M$, $c \in L$. Since M is a comultiplication module, we have $N = (0_M :_M (0_M :_L N))$. Let $b = c \vee (0_M :_L N)$. Since $N = \left(0_M :_M (0_M :_L N)\right) < (0_M :_M c)$, it follows that $(0_M :_L N) \nleq c$. Hence $c < b$ and we have $(0_M :_M b) = (0_M :_M c) \wedge \left(0_M :_M (0_M :_L N)\right) = \left(0_M :_M (0_M :_L N)\right) = N$. (ii)⇒(iii). Clear.

(iii)⇒(i). Suppose that M is not a comultiplication module. It is clear that $1_M = (0_M :_M 0_L)$. There exists $N < 1_M$ such that $N \neq (0_M :_M c)$ for all $c \in L$. Suppose that $\Omega = \{c \in L : N < (0_M :_M c)\}$. Since $0_L \in \Omega$, we have $\Omega \neq \emptyset$. Let $\{c_i\}$ be a chain in Ω. Since $N < (0_M :_M c_i)$, we have $c_i N = 0_M$ and so $(\vee c_i) N = \vee (c_i N) = 0_M$. Therefore $N \leq (0_M :_M \vee c_i)$. But $N < (0_M :_M \vee c_i)$ from above. Therefore, $\vee c_i \in \Omega$. There exists a maximal element of Ω by Zorn's Lemma. Let c be a maximal element of Ω. Since $N < (0_M :_M c)$, there exists $b > c$ such that $N \leq (0_M :_M b)$ by (iii). Since $N \neq (0_M :_M b)$, we have $b \in \Omega$. Since $b > c$, this is a contradiction.

Definition 6. Let M be a comultiplication lattice L-module. An element $0_M \neq N \in M$ is said to be second element in M, if for each $a \in L$, $aN = N$ or $aN = 0_M$.

Proposition 10. Let M be a comultiplication lattice L-module. If $(0_M :_L N) = p$ is prime in L for $N \in M$, then N is second element in M.

Proof: Let $p = (0_M :_L N)$ be prime element of L for $N \in M$. If $aN \neq 0_M$ for $a \in L$, then $0_M \neq K = aN \leq N$. Suppose that $0_M \neq K = aN < N = (0_M :_M (0_M :_L N)) = (0_M :_M p)$. By Theorem 8 (ii), there exists an element $b \in L$ such that $p < b$ and $K = aN = (0_M :_M b)$. It follows that $baN = 0_M$, and so $ba \leq p = (0_M :_L N)$. Since p is prime and $b \nleq p$, we have $a \leq p$ and so $aN = 0_M$. This is a contradiction. Consequently, $K = aN = N$.

Corollary 9. Let M be a comultiplication lattice L-module and $N \in M$. Then the followings are equivalent.

i. N is a second element in M.

ii. $(0_M :_L N)$ is a prime element in L.

Proof: (i)⇒(ii). Suppose that $N \in M$ is a second element. Let $p = (0_M :_M N)$. Suppose that $ab \leq p$ and $b \nleq p$. Since $b \nleq p$, we have $bN \neq 0_M$. Since N is a second element, we have $bN = N$ and so $0_M = (ab)N = a(bN) = aN$. Therefore, $a \leq p = (0_M :_L N)$.
(ii)⇒(i). Proposition 10.

Proposition 11. Let M be a nonzero comultiplication PG-lattice L-module.
i. Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of elements of a module M with $\wedge M_\lambda = 0_M$. Then $N = \wedge_{\lambda \in \Lambda}(N \vee M_\lambda)$ for every $N \in M$.
ii. Let p be a minimal element in L and $(0_M :_M p) = 0_M$. Then M is simple.

Proof: i. Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of elements of a module M with $\wedge M_\lambda = 0_M$. Therefore, $N = \left(0_M :_M (0_M :_L N)\right) = \left(\wedge M_\lambda :_M (0_M :_L N)\right) = \wedge\left(M_\lambda :_M (0_M :_L N)\right)$ and $M_\lambda \leq \left(M_\lambda :_M (0_M :_L N)\right)$ and $N \leq (M_\lambda :_M (0_M :_L N))$ for $\lambda \in \Lambda$. Therefore, $N = \wedge\left(M_\lambda :_M (0_M :_L N)\right) \geq \wedge(M_\lambda \vee N) \geq N$.
ii. Let $0_M \neq X \in M$ be a principal element and let p be a minimal element in L such that $(0_M :_M p) = 0_M$. There exists $a \in L$ such that $X = (0_M :_M a)$. Then$X = (0_M :_M a) = \left((0_M :_M p) :_M a\right) = (0_M :_M ap)$. Since p is minimal, we have $0_L \leq ap \leq p$ and so $ap = 0_L$ or $ap = p$. If $ap = p,\ X = (0_M :_M ap) = (0_M :_M p) = 0_M$. This is a contradiction. Hence $ap = 0_L$. Therefore, $X = (0_M :_M ap) = 1_M$ is principal. Consequently, M is cyclic. Since $M = \{0_M, 1_M\}$, M is simple.

Let L be a multiplicative lattice. An element $a \in L$ is called zero-divisor if there exists an element $0_L \neq b \in L$ such that $ab = 0_L$. L is said to be a domain if it has only zero-divisor 0_L. Note that $Z(L)$ denote the set of zero divisors of L.

Lemma 2. Let M be a faithful comultiplication L-module. Then$W(M) = \{a \in L : a1_M < 1_M\} = Z(L)$.

Proof: Let $a \in W(M)$. Then $a1_M < 1_M$. Since M is comultiplication, $a1_M = (0_M :_M (0_M :_L a1_M))$. It is clear that $(0_M :_L a1_M) \neq 0_L$ and $(0_M :_L a1_M)a1_M = 0_M$. Since M is faithful, $(0_M :_L a1_M)a = 0_L$. We have $a \in Z(L)$. Conversely, let $a \in Z(L)$. There exists $0_L \neq b \in L$ such that $ab = 0_L$. Therefore, $(ab)1_M = b(a1_M) = 0_M \Rightarrow a1_M \leq (0_M :_M b) \neq 1_M$. Indeed, if $(0_M :_M b) = 1_M$, then $b1_M = 0_M$. Since M is faithful, we have $b = 0_L$. This is a contradiction. Therefore $a1_M \neq 1_M$ and so $a \in W(M)$.

Definition 7. (Nakkar and Anderson, 1988) Let M be an L-module. An element $N < 1_M$ in M is said to be prime, if $aX \leq N$ implies $X \leq N$ or $a1_M \leq N$ i.e.$a \leq (N :_L 1_M)$ for every $a \in L,\ X \in M$.

Definition 8. Let M be an L-module. M is said to be prime L-module if 0_M is prime element of M.
It is clear that 0_M is prime element in M if and only if $(0_M :_L 1_M) = (0_M :_L N)$ for all $0_M \neq N \in M$.

Definition 9. Let M be an L-module. M is said to be coprime L-module if $(0_M :_L 1_M) = (N :_L 1_M)$ for all $N \in M$.

Proposition 12. Let M be a L-module.
i. Let M be a comultiplication prime L-module. Then M is a simple L-module.
ii. If M is a multiplication coprime L-module, then M is a simple module.
iii. Let L be a domain and let M be a faithful multiplication and comultiplication L-module. Then M is simple.

Proof: i. Let $0_M \neq N \in M$. Since M is a prime L-module, we have $(0_M :_L 1_M) = (0_M :_L N)$ for all $0_M \neq N \in M$. Then $N = \left(0_M :_M (0_M :_L N)\right) = \left(0_M :_M (0_M :_L 1_M)\right) = 1_M$. Hence M is a simple.
ii. Let $N < 1_M$. Since M is coprime L-module, we have $(0_M :_L 1_M) = (N :_L 1_M)$. Since M is a multiplication L-module, $N = (N :_L 1_M)1_M = (0_M :_L 1_M)1_M = 0_M$. Therefore, M is a simple L-module.
iii. Let $N \in M$. Therefore, $N = (0_M :_M a)$ and $N = b1_M$ for some $a, b \in L$. So, $aN = ab1_M = 0_M$. Since M is faithful, $ab = 0_L$. Then $a = 0_L$ or $b = 0_L$ as L is a domain. Hence $N = 1_M$ or $N = 0_M$.

Definition 10.
i. Let M be a PG-lattice L-module. M is called a torsion module if $Ann(X) = (0_M :_L X) \neq 0_L$ for all principal elements $X \in M$.
ii. Let M be an L-module. M is called a domain if $Ann(N) = 0_L$ for all $0_M \neq N \in M$.

Theorem 9. If M is a comultiplication PG-lattice L-module, then M is cyclic or torsion.

Proof: Let M be a comultiplication PG-lattice module. Suppose that M is not a torsion L-module. Thus there exists a principal element $X \in M$ such that $(0_M :_L X) = 0_L$. Then $X = (0_M :_M (0_M :_L X)) = 1_M$. Hence, M is cyclic.

Corollary 7. Let M a faithful comultiplication PG-lattice L-module and 1_M compact. If L is domain, then M is cyclic.

Proof: Assume that M is not cyclic. Then, M is torsion. Hence $(0_M :_L X) \neq 0_L$ for all principal X. Since 1_M is compact, $1_M = \vee X_i$ implies that $1_M = \vee_{i=1}^{n} X_i$ for some principal elements X_i. Therefore, $0_L = (0_M :_L 1_M) = \wedge_{i=1}^{n}(0_M :_L X_i) \geq \prod_{i=1}^{n}(0_M :_L X_i) \neq 0_L$. This is a contradiction.

Proposition 13. Let L be a comultiplication PG-lattice and M be a faithful PG-lattice L-module. Then for each $a \in L$, with $a < 1_L$, $(0_M :_M a) \neq 0_M$ and $a1_M < 1_M$.

Proof: Let $a \in L$, with $a < 1_L$. Suppose that $(0_M :_M a) = 0_M$. Then $(0_L :_L a)1_M = 0_M$, for if $(0_L :_L a)1_M \neq 0_M$, there exists a principal element $x \in L$ such that $x \leq (0_L :_L a)$ and a principal element $Y \in M$ such that $xY \neq 0_M$. Since $ax = 0_L$, we have $axY = 0_M$. Then $xY \leq (0_M :_M a) = 0_M$. This is a contradiction. Since $(0_L :_L a)1_M = 0_M$, it follows that $(0_L :_L a) \leq Ann_L(M) = 0_L$. Since L is a comultiplication lattice, $a = \big(0_L :_L (0_L :_L a)\big) = 1_L$. This is a contradiction. Now suppose that $a1_M = 1_M$. Therefore $(0_L :_L a) = (0_L :_L 1_M) = 0_L$. Since L is a comultiplication lattice, $a = 1_L$. This is a contradiction.

Definition 11. Let M be an L-module and N a non-zero element of M. Then N is said to be large if for every element K in M such that $N \wedge K = 0_M$ implies $K = 0_M$.

Definition 12. Let M be an L-module and N be a proper element of M. Then N is said to be small element if for every element K in M such that $N \vee K = 1_M$ implies that $K = 1_M$.

Proposition 14. Let M be a faithful comultiplication PG-lattice L-module with 1_M compact. Then every non-zero element of M is large if and only if every element $a \in L$, with $a < 1_L$ is small.

Proof: ⇒: Suppose that every non-zero element of M is large and let $a \in L$, $a < 1_L$ such that $a \vee b = 1_L$ for some $0_L \neq b \in L$. Then $0_M = (0_M :_M a \vee b) = (0_M :_M a) \wedge (0_M :_M b)$. Since $a < 1_L$ we have $(0_M :_M a) \neq 0_M$ by Theorem 3. We know that $(0_M :_M a)$ is large. Hence $(0_M :_M b) = 0_M$. Therefore, we obtain $b = 1_L$.⇐: Suppose that $N \in M$ such that $K \wedge N = 0_M$ where $0_M \neq K \in M$. Since M is a comultiplication L-module, $K = (0_M :_M (0_M :_L K))$ and $N = (0_M :_M (0_M :_L N))$. Then $0_M = K \wedge N = (0_M :_M (0_M :_L K) \vee (0_M :_L N))$ and so $(0_M :_L K) \vee (0_M :_L N) = 1_L$. Since $0_M \neq K$, we have $(0_M :_L K) \neq 1_L$. Since $(0_M :_L K)$ is small, it follows that $(0_M :_L N) = 1_L$ and so $N = 0_M$.

Proposition 15. Let M be a faithful comultiplication PG-lattice L-module with 1_M compact. Then $N \in M$ is large if and only if there exists a small element $a \in L$ such that $N = (0_M :_M a)$.

Proof: ⇒: Suppose that $N \in M$ is large. Since M is a comultiplication L-module, $N = (0_M :_M a)$. Suppose $a \vee b = 1_L$ for some $0_L \neq b \in L$. Then $N \wedge (0_M :_M b) = (0_M :_M a) \wedge (0_M :_M b) = (0_M :_M a \vee b) = 0_M$. Since N is large, we have $(0_M :_M b) = 0_M$, hence by Theorem 3, we have $b = 1_L$. So a is small.
⇐: Suppose that $a \in L$ be a small element of L. Let $N = (0_M :_M a)$. Assume that $K \in M$ such that $N \wedge K = 0_M$. Since M is a comultiplication L-module, there exists $b \in L$ such that $K = (0_M :_M b)$. Then$0_M = N \wedge K = (0_M :_M a) \wedge (0_M :_M b) = (0_M :_M a \vee b)$ and so $a \vee b = 1_L$ by Theorem 3. Therefore $b = 1_L$. Hence $K = (0_M :_M b) = 0_M$. Consequently, N is large.

Acknowledgement

The authors thank the referee for his valuable and suggestions.

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