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Some notes on differential hyperrings

L. Kamali Ardekani and B. Davvaz*

Department of Mathematics, Yazd University, Yazd, Iran
E-mail: bdavvaz@yahoo.com & *davvaz@yazd.ac.ir*

Abstract

In this paper, we introduce the notion of derivation on Krasner hyperrings as follows: the function $d: R \to R$ is a derivation on a Krasner hyperring R if for all $x, y \in R$, $d(x+y) = d(x) + d(y)$ and $d(x \cdot y) \in d(x) \cdot y + x \cdot d(y)$. Then, we investigate some fundamental properties of derivation on Krasner hyperings and prime Krasner hyperrings. Also, we introduce differential Krasner hyperingsand discuss some related properties.

***Keywords**:* Krasner hyperring; prime Krasner hyperring; hyperideal; derivation; differential hyperring

1. Derivation of Krasner hyperrings

Let H be a non-empty set and let $P^*(H)$ be the set of all non-empty subsets of H. A hyperoperation on H is a map $\circ: H \times H \to P^*(H)$ and the couple $(H, \circ)$ is called a hypergroupoid. If A and B are non-empty subsets of H, then we denote $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$. A hypergroupoid $(H, \circ)$ is called a semihypergroup if for all x, y, z of H we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that $\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v$. There are different kinds of hyperrings. The most comprehensive reference for hyperrings is Davvaz and Leoreanu-Fotea's book (2007). Other references are (Davvaz, 2009; Davvaz and Salasi, 2006; Davvaz and Vougiouklis, 2007; Mirvakili et al., 2008; Mirvakili and Davvaz, 2010; Mirvakili and Davvaz, 2012; Nakassis, 1988). A Krasner hyperring (Krasner, 1983) is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms: (1) $(R, +)$ is a canonical hypergroup, i.e., (i) $(R, +)$ is a semihypergroup, i.e. $x + (y + z) = (x + y) + z$, for all $x, y, z \in R$, (ii) $x + y = y + x$, for all $x, y \in R$, (iii) There exists $0 \in R$ such that $0 + x = \{x\}$, for all $x \in R$, (iv) For all $x \in R$ there exists a unique element $x' \in R$ such that $0 \in x + x'$, (we write $-x$ for x' and we call it the opposite of x), (v) $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$, for all $x, y, z \in R$; (2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$; (3) The multiplication is distributive with respect to the hyperoperation $+$.

Throughout this paper, by a hyperring we mean a Krasner hyperring.

A hyperring $(R, +, \cdot)$ is called commutative, if $(R, \cdot)$ is a commutative semigroup. The meaning of center of R is $Z(R) = \{x \in R \mid x \cdot y = y \cdot x, \text{for all } y \in R\}$. A hyperring $(R, +, \cdot)$ is called hyperfield, if $(R \setminus \{0\}, \cdot)$ is a group. If $(R \setminus \{0\}, \cdot)$ is a monoid, then the identity element of this monoid is called unit element of hyperring $(R, +, \cdot)$. A hyperring $(R, +, \cdot)$ is called hyperdomain, if R is a commutative hyperring with unit element and $xy = 0$ implies that $x = 0$ or $y = 0$, for all $x, y \in R$.

A non-empty subset A of a hyperring $(R, +, \cdot)$ is called subhyperring of R if $(A, +, \cdot)$ is itself a hyperring. The subhyperring A of R is normal in R if and only if $x + A - x \subseteq A$, for all $x \in R$. A non-empty subset I of a hyperring R is called a

*Corresponding author
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left (respectively, right) hyperideal if and only if (1) $u, v \in I$ imply that $u - v \subseteq I$, for all $u, v \in I$, (2) $u \in I$ and $r \in R$ imply that $r \cdot u \in I$ (respectively, $u \cdot r \in I$). Also, I is called a hyperideal if I is both a left and a right hyperideal.

A good homomorphism between two hyperrings $(R_1, +_1, \cdot_1)$ and $(R_2, +_2, \cdot_2)$ is a map $f: R_1 \to R_2$ such that for all $x, y \in R_1$, we have

$f(x +_1 y) = f(x) +_2 f(y)$,

$f(x \cdot_1 y) = f(x) \cdot_2 f(y)$ and $f(0) = 0$.

Let $f: R_1 \to R_2$ be a good homomorphism. The kernel of f is the set $kerf = \{x \in R_1 \mid f(x) = 0\}$. It is inconsequential that $kerf$ is a hyperideal of R_1.

The concept of derivation on rings was introduced by Posner (1957), also see (Khadjiev and Çallialp, 1998; Kolchin, 1973; Soytürk, 1994; Wang, 1994). Differential rings, differential fields, and differential algebras are rings, fields, and algebras equipped with a derivation, which is a unary function that is linear and satisfies the Leibniz product rule. In (2000), Chvalina and Chvalinova gave a construction of hyperstructures determined by quasi-orders defined by means of derivation operators on differential rings. In (2012), Davvaz et al. introduced the concept of (3,3)-ary differential rings as a generalization of differential rings. Then, they gave a construction of hyperstructures determined by (3,3)-ary differential rings. In (2013), Asokkumar presented the definition of derivation in hyperrings.

We recall the definition of derivation in hyperrings (Asokkumar, 2013).

Definition 1.1. Let $(R, +, \cdot)$ be a hyperring. The function $d: R \to R$ is called a derivation if for all $x, y \in R$,

(1) $d(x + y) = d(x) + d(y)$,

(2) $d(x \cdot y) \in d(x) \cdot y + x \cdot d(y)$.

By the above definition for every derivation d on hyperring R, we have $d(0) = 0$ and $d(-x) = -d(x)$, for all $x \in R$.

Example 1. Let $R = \{0, 1, 2\}$. Consider the following tables:

$+$	0	1	2
0	0	1	2
1	1	1	R
2	2	R	2

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

So, $(R, +, \cdot)$ is a hyperring (Davvaz and Leoreanu-Fotea, 2007). It is easy to check that the function $d: R \to R$ defined by $d(0) = 0, d(1) = 2$ and $d(2) = 1$ is a derivation.

Example 2. Let $Q^+ = \{x \in Q \mid x \geq 0\}$, where Q is the set of rational numbers. The binary hyperoperation $+$ and the binary operation $\cdot$ are defined as follows:

$$x + x = \{y \in Q^+ \mid y \leq x\}, \text{ for all } x \in Q^+,$$
$$x + y = \max\{x, y\}, \text{ for all } x, y \in Q^+, x \neq y,$$
$$x \cdot y = xy, \text{ for all } x, y \in Q^+.$$

Then, $(Q^+, +, \cdot)$ is a hyperring (Corsini, 1993). The function $d: Q^+ \to Q^+$ defined by $d(x) = x$, for all $x \in Q^+$, is a derivation, since for all $x, y \in Q^+$,

$$d(x) + d(y) = d(x + y),$$
$$d(x \cdot y) = xy \in \{t \in Q^+ \mid t \leq xy\} = xy + xy$$
$$= d(x) \cdot y + x \cdot d(y).$$

Example 3. Let $(G, \cdot, e)$ be a finite group with m elements, $m > 3$, and define a hyperaddition and a multiplication on $H = G \cup \{0\}$, by

$$x + 0 = 0 + x = \{x\}, \forall x \in H,$$
$$x + x = \{x, 0\}, \forall x \in G,$$
$$x + y = y + x = H \setminus \{x, y\}, \forall x, y \in G, x \neq y,$$
$$x \otimes 0 = 0 \otimes x = 0, \forall x \in H,$$
$$x \otimes y = x \cdot y, \forall x, y \in G.$$

Then, $(H, +, \otimes)$ is a hyperring (Davvaz and Leoreanu-Fotea, 2007; Nakassis, 1988). The function $d: H \to H$ defined by $d(x) = x$, for all $x \in H$, is a derivation, since

$$d(x) + d(y) = d(x + y), \forall x, y \in H,$$
$$d(x \otimes 0) = d(0) = 0 \in \{0\} = 0 + 0$$
$$= d(x) \otimes 0 + x \otimes d(0), \forall x \in H,$$
$$d(x \otimes y) = d(x \cdot y) = x \cdot y \in \{x \cdot y, 0\}$$

$= x\cdot y + x\cdot y$
$= d(x)\otimes y + x\otimes d(y), \forall x, y\in G.$

Example 4. Consider Example 3 and let $(G,\cdot,e)$ be an abelian group, which has no elements of order 2. Then, the function $d_1 : H\to H$ defined by

$$d_1(x)=\begin{cases}0 & x=0\\ x^{-1} & for\ all\ x\in G\end{cases}$$

is a derivation function, since

$$d_1(x+0)=d_1(0+x)=\{d_1(x)\}=\{x^{-1}\}$$
$$= d_1(x)+d_1(0)$$
$$= d_1(0)+d_1(x), \forall x\in H;$$
$$d_1(x+x)=\{d_1(x), d_1(0)\}=\{x^{-1},0\}$$
$$= x^{-1}+x^{-1}$$
$$= d_1(x)+d_1(x), \forall x\in G;$$
$$d_1(x+y)=H\setminus\{x^{-1},y^{-1}\}=x^{-1}+y^{-1}$$
$$= d_1(x)+d_1(y), \forall x,y\in G, x\neq y.$$

Hence, the first condition of the definition of derivation is valid. Also, we have

$$d_1(x\otimes 0)=d_1(0)=0\in\{0\}$$
$$= x^{-1}\otimes 0 + x\otimes 0$$
$$= d_1(x)\otimes 0 + x\otimes d_1(0), \forall x\in H;$$
$$d_1(x)\otimes y + x\otimes d_1(y) = x^{-1}\otimes y + x\otimes y^{-1}$$
$$= x^{-1}\cdot y + x\cdot y^{-1}, \forall x,y\in G.$$

By the above relations, in order to prove the second condition of the definition of derivation, it is enough to show that $d(xy)=x^{-1}y^{-1}\in x^{-1}\cdot y + x\cdot y^{-1}$, for all $x,y\in G$. We have $x^{-1}\cdot y + x\cdot y^{-1} = H\setminus\{x^{-1}\cdot y, x\cdot y^{-1}\}$, since G has no elements of order 2. If $x^{-1}\cdot y^{-1}=x^{-1}\cdot y$, then $y=y^{-1}$ and if $x^{-1}\cdot y^{-1}=x\cdot y^{-1}$ then $x=x^{-1}$. Hence, $x^{-1}y^{-1}\notin\{x^{-1}\cdot y, x\cdot y^{-1}\}$, since G has no elements of order 2. Therefore,

$$x^{-1}y^{-1}\in H\setminus\{x^{-1}\cdot y, x\cdot y^{-1}\}=x^{-1}\cdot y + x\cdot y^{-1}$$

The following example shows that the identity function is not always a derivation.

Example 5. Let $(G,\cdot,e)$ be a group. Define a hyperaddition and a multiplication on $H=G\cup\{0\}$ as follows:

$$x+0=0+x=\{x\}, \forall\ x\in H,$$
$$x+x=H\setminus\{x\}, \forall\ x\in G,$$
$$x+y=y+x=\{x,y\}, \forall\ x,y\in G, x\neq y,$$
$$x\otimes 0=0\otimes x=0, \forall\ x\in H,$$
$$x\otimes y = x\cdot y, \forall\ x,y\in G.$$

Then, $(H,+,\otimes)$ is a hyperring (Corsini, 1993). The function $d: H\to H$ defined by $d(x)=x$, for all $x\in H$, is not a derivation, since $d(x)\otimes y + x\otimes d(y) = x\cdot y + x\cdot y = H-\{x\cdot y\}$, for all $x,y\in G$. So,

$$d(x\otimes y)=x\cdot y\notin d(x)\otimes y + x\otimes d(y).$$

In a hyperring, we may use xy instead of $x\cdot y$.

Lemma 1.2. Let d be a derivation on a hyperring R. For all $x,y\in R$, define $x^0 y = y$ and $d^0(x)=x$. Then, for all $n\in N$ and $x,y\in R$,

(1) If R is commutative, then $d(x^n)\in n(x^{n-1}.d(x))$.

(2) $d^n(xy)\in\sum_{i=0}^{n}\binom{n}{i}d^{n-i}(x)d^i(y)$,

where d^n denotes the derivation of order n.

Proof: (1) The proof follows easily by induction.
(2) It is inconsequential that the statement is valid for $n=1$. Now, let the statement be valid for $n=k-1$ (induction hypothesis). We have

$$d^k(xy)=d\left(d^{k-1}(xy)\right)$$
$$\in d\left(\sum_{i=0}^{k-1}\binom{k-1}{i}d^{k-i-1}(x)d^i(y)\right)$$
$$\subseteq\sum_{i=0}^{k-1}\binom{k-1}{i}d^{k-i}(x)d^i(y)$$
$$+\sum_{i=0}^{k-1}\binom{k-1}{i}d^{k-i-1}(x)d^{i+1}(y)$$
$$=\sum_{i=0}^{k}\binom{k}{i}d^{k-i}(x)d^i(y).$$

Lemma 1.3. Let R be a hyperring and $[x,y]$ denotes the set $xy-yx$, for all $x,y\in R$. Then, for all $x,y,z\in R$, we have,

(1) $[x+y,z]=[x,z]+[y,z]$,
(2) $[xy,z]\subseteq x[y,z]+[x,z]y$,
(3) If $x\in Z(R)$, then $[xy,z]=x[y,z]$,
(4) If d is a derivation of R, then $d[x,y]\subseteq[d(x),y]+[x,d(y)]$.

Proof: For $x, y, z \in R$,

$(1)\,[x+y,z]=(x+y)z-z(x+y)$
$=xz-zx+yz-zy$
$=[x,z]+[y,z].$

$(2)\,[xy,z]=xyz-zxy$
$\subseteq xyz-xzy+xzy-zxy$
$=x(yz-zy)+(xz-zx)y$
$=x[y,z]+[x,z]y.$

(3) If $x \in Z(R)$, then we have

$[xy,z]=xyz-zxy=xyz-xzy$
$=x(yz-zy)=x[y,z].$

(4) Suppose that d is a derivation of R. Then,

$d[x,y]=d(xy-yx)=d(xy)-d(yx)$
$\subseteq d(x)y+xd(y)-d(y)x-yd(x)$
$=d(x)y-yd(x)+xd(y)-d(y)x$
$=[d(x),y]+[x,d(y)].$

Theorem 1.4. Let d be a derivation on a hyperring R and n be the smallest natural number such that $d^n(R)=0$. Then, for all $y \in R$, $d(y)=0$ or there is $0<k<n$ such that $0 \in n(d^{n-1}(x_0)d^k(y))$, where $0 \neq x_0 \in R$ is a fixed element.

Proof: Suppose that n is the smallest natural number such that $d^n(R)=0.$ Then, $d^{n-1}(R)\neq 0$. So, there is $0\neq x_0 \in R$ such that $d^{n-1}(x_0)\neq 0$. Let $d(y)\neq 0$, where $y\in R$. Then, there is $0<k<n$ such that $d^k(y)\neq 0$ and $d^{k+1}(y)=0$. By Lemma 1.2, we have

$$0=d^n\left(x_0 d^{k-1}(y)\right)$$
$$\in \sum_{i=0}^{n}\binom{n}{i} d^{n-i}(x_0)d^{k+i-1}(y)$$
$$=d^n(x_0)d^{k-1}(y)+n(d^{n-1}(x_0)d^k(y))$$
$$+\sum_{i=0}^{n-2}\binom{n}{i+2}(d^{n-i-2}(x_0)d^{k+i+1}(y))$$
$$=n(d^{n-1}(x_0)d^k(y)).$$

Theorem 1.5. Let d be a good homomorphism and derivation on a hyperring R. Then, for all $x, y\in R$,

$$d(x)yd(x)\in\left(d(x)yx+xd(yx)-xd(yx)\right)$$
$$\cap\left(xyd(x)+d(xy)x-d(xy)x\right).$$

Proof: We have, for all $x, y \in R$,

$$d(x)d(y)=d(xy)\in d(x)y+xd(y). \quad (1)$$

Replace y by yx, in (1),

$d(xy)d(x)=d(x)d(y)d(x)=d(x)d(yx)$
$=d(xyx)\in d(x)yx+xd(yx).$

On the other hand,

$d(xy)d(x)\in d(x)yd(x)+xd(y)d(x)$
$=d(x)yd(x)+xd(yx)$.

So, $d(x)yd(x)\in d(x)yx+xd(yx)-xd(yx).$

Now, we replace x by yx in (1),

$d(y)d(xy)=d(y)d(x)d(y)=d(yx)d(y)$
$=d(yxy)\in d(yx)y+yxd(y).$

On the other hand,

$d(y)d(xy)\in d(y)d(x)y+d(y)xd(y)$
$=d(yx)y+d(y)xd(y).$

So, $d(y)xd(y)\in yxd(y)+d(yx)y-d(yx)y$.

By changing the role of x and y, we have $d(x)yd(x)\in xyd(x)+d(xy)x-d(xy)x$. This completes the proof.

2. Derivation of prime Krasner hyperrings

In this section, we study the concept of derivation on prime hyperrings.

Definition 2.1. A hyperring R is called prime if $xRy=0$ implies that either $x=0$ or $y=0.$ Also, R is called semiprime if $xRx=0$ implies that $x=0.$ Obviously, every prime hyperring is a semiprime hyperring but the converse is not always true.

Example 6. Every hyperdomain is prime.

Example 7. All of the hyperrings in Examples 1, 2, 3 and 5 are prime and semiprime hyperrings.

Example 8. Let $(R,+,\cdot)$ be a hyperring. Set $M=\left\{\begin{pmatrix} x & y \\ 0 & 0\end{pmatrix} \mid x,y\in R\right\}$ and define the hyperoperation $\oplus$ and operation $\otimes$ on M as

$$\begin{pmatrix} x_1 & y_1 \\ 0 & 0\end{pmatrix}\oplus\begin{pmatrix} x_2 & y_2 \\ 0 & 0\end{pmatrix}$$

$$=\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a \in x_1 + x_2, b \in y_1 + y_2\right\},$$ and

$$\begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} x_2 & y_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_1 \cdot x_2 & x_1 \cdot y_2 \\ 0 & 0 \end{pmatrix},$$

where $x_1, x_2, y_1, y_2 \in R$. Then, $(M, \oplus, \otimes)$ is a hyperring. The hyperring $(M, \oplus, \otimes)$ is not semiprime hyperring, since for all $x, y \in R$ and $0 \neq b \in R$, we have

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \bar{0},$$ but $$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \neq \bar{0}.$$

Put $$M' = \left\{\begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} \mid a \in R\right\}.$$ Then, $(M', \oplus, \otimes)$ is a prime (respectively, semiprime) hyperring if and only if R is a prime (respectively, semiprime) hyperring.

The following example shows that a semiprime hyperring is not a prime hyperring, in general.

Example 9. Let $R = \{e, a, b, c, d, f\}$. Consider the following tables:

$+$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	a	$\{e,a,b\}$	d	d	$\{c,d,f\}$
b	b	$\{e,a,b\}$	b	f	$\{c,d,f\}$	f
c	c	d	f	e	a	b
d	d	d	$\{c,d,f\}$	a	a	$\{e,a,b\}$
f	f	$\{c,d,f\}$	f	b	$\{e,a,b\}$	b

$\cdot$	e	a	b	c	d	f
e	e	e	e	e	e	e
a	e	a	b	e	a	b
b	e	a	b	e	a	b
c	e	e	e	c	c	c
d	e	a	b	c	d	f
f	e	a	b	c	d	f

It is easy to check that $(R, +, \cdot)$ is a semiprime hyperring. But $(R, +, \cdot)$ is not a prime hyperring, since $aRc = e$ and $a, c \neq e$.

Example 10. Let $R = \{e, a, b, c\}$. Consider the following tables:

$+$	e	a	b	c
e	e	a	b	c
a	a	$\{e,a\}$	c	$\{b,c\}$
b	b	c	$\{e,b\}$	$\{a,c\}$
c	c	$\{b,c\}$	$\{a,c\}$	R

$\bullet$	e	a	b	c
e	e	e	e	e
a	e	e	e	e
b	e	a	b	c
c	e	a	b	c

It is easy to check that $(R, +, \cdot)$ is a hyperring. R is not semiprime, since $aRa = e$ but $a \neq e$.

Lemma 2.2. Let I be a non-zero hyperideal on a prime hyperring R. Then, for all $x, y \in R$,

(1) If $Ix = 0$ or $xI = 0$, then $x = 0$,
(2) If $xIy = 0$, then $x = 0$ or $y = 0$,
(3) If $x \in Z(R)$ and $xy = 0$, then $x = 0$ or $y = 0$,
(4) If $x \in R$ such that $[I, x] = 0$, then $x \in Z(R)$,
(5) If $x \in Z(R)$ and $xy \in Z(R)$, then $x = 0$ or $y \in Z(R)$.

Proof: (1) Suppose that $Ix = 0$. Then, $uRx \subseteq Ix = \{0\}$, for all $u \in I$. So $x = 0$, since R is prime and $I \neq 0$. In the case $xI = 0$, the proof is similar.
(2) Suppose that $xIy = 0$, then $xIRy \subseteq xIy = \{0\}$. Thus, $xIRy = 0$. Hence, $xI = 0$ or $y = 0$, since R is prime. So by (1), $x = 0$ or $y = 0$.
(3) Suppose that $x \in Z(R)$ and $xy = 0$. Then for all $r \in R$,
$0 = r0 = rxy = xry$. Therefore, $xRy = 0$ and this implies that $x = 0$ or $y = 0$, since R is prime.
(4) By Lemma 1.3 (2), we have $0 = [utr, x] \subseteq ut[r, x] + [ut, x]r = ut[r, x]$, for

all $u \in I$ and $t, r \in R$. Therefore, for all $s \in [r, x]$, we have $uts = 0$, which means that $uRs = 0$. Hence, $s = 0$, since R is prime and $I \neq 0$. This shows that $x \in Z(R)$.

(5) Suppose that $xy \in Z(R)$. Then, $0 \in [xy, r]$, for all $r \in R$. Therefore, $0 \in [xy, r] = xyr - rxy = xyr - xry = x[y, r]$. So, $0 = t0 \in tx[y, r] = xt[y, r]$, for all $t \in R$. This implies that $0 \in xR[y, r]$. Hence, $x = 0$ or $0 \in [y, r]$, for all $r \in R$, since R is prime. Then, $x = 0$ or $y \in Z(R)$.

Lemma 2.3. Let d be a derivation on a prime hyperring R and I be a non-zero hyperideal on R. Then, for all $x \in R$,

(1) If $d(I) = 0$, then $d = 0$,

(2) If $d(I)x = 0$ or $xd(I) = 0$, then $x = 0$ or $d = 0$,

(3) If $d(R)x = 0$ or $xd(R) = 0$, then $x = 0$ or $d = 0$.

Proof: (1) For all $u \in I$ and $x \in R$, we have $0 = d(ux) \in d(u)x + ud(x) = ud(x)$. Therefore, $Id(x) = 0$, which implies that $d = 0$, by Lemma 2.2 (1).

(2) Suppose that $d(I)x = 0$. Then, $0 = d(yu)x \in d(y)ux + yd(u)x = d(y)ux$, for all $u \in I$ and $y \in R$. Therefore, $d(y)Ix = 0$, which implies that $d = 0$ or $x = 0$, by Lemma 2.2 (2). In the case $xd(I) = 0$, the proof is similar.

(3) In (2), substitute R with I.

Definition 2.4. Let R be a hyperring and d be a derivation on R. Then, $x \in R$ is called aconstant element associated to d if $d(x) = 0$. We denote by $C_d(R)$, the set of all of constant elements of R associated to derivation d. It is insignificant that $C_d(R)$ is a subhyperring of R.

Theorem 2.5. Let d be a derivation on a prime hyperring R such that $d(R) \subseteq Z(R)$. Also, let there be a constant element $c \in R$ associated to d such that $c \notin Z(R)$. Then, $d = 0$.

Proof: There is $x_0 \in R$ such that $cx_0 \neq x_0 c$, since $c \notin Z(R)$. We have

$d(xc) \in d(x)c + xd(c) = d(x)c$, for all $x \in R$. So, $d(x)c = d(xc) \in Z(R)$. Therefore, $d(x)cx_0 = x_0 d(x)c = d(x)x_0 c$. This means that $0 \in d(x)[c, x_0]$. Then, there is $t \in [c, x_0]$ such that $d(x)t = 0$. So, $d(x) = 0$ or $t = 0$, by Lemma 2.2 (3). If $t = 0$, then $0 \in [c, x_0] = cx_0 - x_0 c$ and this is a contradiction. So, $d(x) = 0$, for all $x \in R$.

Lemma 2.6. Let H and K be canonical subhypergroups of canonical hypergroup $(G, +, 0)$. Then, $H \cup K$ is a canonical subhypergroup of G if and only if $H \subseteq K$ or $K \subseteq H$.

Proof: If $H \subseteq K$ or $K \subseteq H$, then it is clear that $H \cup K$ is a canonical subhypergroup of G. Now, suppose that $H \cup K$ is a canonical subhypergroup of G and $H \not\subset K$ and $K \not\subset H$. Then, there are $a, b \in H \cup K$ such that $a \in H \setminus K$ and $b \in K \setminus H$. Also, we have $a + b \subseteq H \cup K$, since $H \cup K$ is a canonical subhypergroup. Now, one of two following cases happens: Case 1: $(a + b) \cap H \neq \emptyset$, then there exists $x \in (a + b) \cap H$. So, $b \in x - a \subseteq H$ and this is a contradiction. Case 2: $(a + b) \cap K \neq \emptyset$, in this case there exists $y \in (a + b) \cap K$. So, $a \in y - b \subseteq K$ and this is a contradiction.

Theorem 2.7. Let d be a non-zero derivation on a prime hyperring R and I be a non-zero hyperideal on R. Then,

(1) If $I \subseteq Z(R)$, then R is commutative,

(2) If $0 \in [u, R]Id(u)$, for all $u \in I$, then R is commutative.

Proof: (1) We have

$rsu = rus = (ru)s = s(ru) = sru,$ for all $r, s \in R$ and $u \in I$. So,

$0 \in rsu - sru = [r, s]u$. Therefore, $0 \in [r, s]$, for all $r, s \in R$, by Lemma 2.2 (1).

(2) Since $0 \in [u, r]Id(u)$, for $u \in I$ and $r \in R$, hence $0 \in [u, r]$ or $d(u) = 0$, by Lemma 2.2 (2). Put $A = \{u \in I \mid d(u) = 0\}$ and

$B=\{u\in I \mid u\in Z(R)\}$. It is clear that A and B are canonical subhypergroups of I and $I=A\cup B$. So, $I=A$ or $I=B$, by Lemma 2.6. If $I=A$ that is $d(I)=0$, then $d=0$, by Lemma 2.3 (1) and this is a contradiction. Therefore, $I=B$ that is $I\subseteq Z(R)$. This implies that R is commutative, by (1).

Definition 2.8. A hyperring R is called $n-$torsion free, where $n\in N$, if $0\in nx=\underbrace{x+x+\cdots+x}_{n}$, where $x\in R$, implies that $x=0$.

Example 11. In Example 1, R is a $2-$torsion free hyperring. In Example 9, R is a $3-$torsion free hyperring but R is not a $2-$torsion free hyperring, since $e\in 2c$ but $c\neq e$.

Theorem 2.9. Let I be a non-zero hyperideal of $2-$torsion free hyperring R. Then,
(1) If d is a derivation of R such that $d^2(I)=0$, then $d=0$.
(2) If d_1 and d_2 are derivations of R such that $d_1d_2(I)=0$, then $d_1=0$ or $d_2=0$.

Proof: (1) By Lemma 1.2, we have for all $u,v\in I$,

$$0=d^2(uv)\in d^2(u)v+2d(u)d(v)+ud^2(v)$$
$$=2d(u)d(v).$$

So, $d(u)d(v)=0$, since R is a $2-$torsion free hyperring. Therefore, $d=0$, by Lemma 2.3 (1) and (2).
(2) We have for all $u,v\in I$,

$$0=d_1d_2(uv)\in d_1(d_2(u)v+ud_2(v))$$
$$\subseteq d_1d_2(u)v+d_2(u)d_1(v)$$
$$+d_1(u)d_2(v)+ud_1d_2(v)$$
$$=d_2(u)d_1(v)+d_1(u)d_2(v).$$

By replacing u by $d_2(u)$ in the above equation, we get

$$0\in d_2^2(u)d_1(v)+d_1d_2(u)d_2(v)=d_2^2(u)d_1(v),$$

that is $d_2^2(u)d_1(v)=0$. So, $d_1=0$ or $d_2^2(I)=0$, by Lemma 2.3 (1) and (2). Therefore, $d_1=0$ or $d_2=0$, by (1).

In the next lemma and theorem, R will be a hyperring such that the center of it, i. e. $Z(R)$ is a ring.

Example 12. In Examples 1 and 10, the center of hyperring R is a ring, since in both $Z(R)=\{0\}$. It is clear that in Example 8, the center of M' is a ring if and only if the center of R is a ring. In Example 9, $Z(R)=\{e,c\}\cong Z_2$. So, $Z(R)$ is a ring.

Lemma 2.10. Let R be a hyperring such that the center of it i. e. $Z(R)$, is a ring. Also, let d be a derivation on R. Then, $d(x)\in Z(R)$, for all $x\in Z(R)$.

Proof: Suppose that $x\in Z(R)$. Then, $d(xr)=d(rx)$, for all $r\in R$. So,

$$0\in d(xr)-d(rx)\subseteq d(x)r+xd(r)$$
$$-d(r)x-rd(x)$$
$$=d(x)r+xd(r)-xd(r)-rd(x)$$
$$=d(x)r+(x-x)d(r)-rd(x)$$
$$=d(x)r-rd(x).$$

Therefore, $d(x)r=rd(x)$, for all $r\in R$.

Theorem 2.11. Let R be a prime hyperring such that the center of it i. e. $Z(R)$, is a ring. Also, let I be a non-zero hyperideal of R. Then, in every following cases R is commutative.
(1) If d is a derivation such that $d^2\neq 0$ and $d(R)\subseteq Z(R)$,
(2) If R is a $2-$torsion free hyperring and d is a non-zero derivation such that $d(I)\subseteq Z(R)$,
(3) If for all subset A of R, $0\in 3!$ implies that $0\in A$ and d is a non-zero derivation such that $d(I)\subseteq I$ and $d^2(I)\subseteq Z(R)$,
(4) If for all subset A of R, $0\in 3!$ implies that $0\in A$ and d_1,d_2 are non-zero derivations such that $d_2(I)\subseteq I$, $d_1d_2(I)\subseteq Z(R)$ and $d_1d_2^2(I)=0$.

Proof: (1) Suppose that $d(R)\subseteq Z(R)$. Then, $[d(x),y]=0$, for all $x,y\in R$. Replace x by xz, where $z\in R$. Hence,

$$0=[d(xz),y]\subseteq[d(x)z,y]+[xd(z),y]$$
$$=d(x)[z,y]+d(z)[x,y],$$ by Lemma 1.3 (3).

By replacing z by $d(z)$, we get

$$0\in d(x)[d(z),y]+d^2(z)[x,y]=d^2(z)[x,y].$$

So, $d^2(z)=0$ or $0\in[x,y]$, by Lemma 2.2 (3). Hence, R is commutative, since $d^2\neq 0$.
(2) If for all $x\in Z(R)$, we have $d(x)=0$. Then, $d(Z(R))=0$ and so $d^2(I)=0$. So, $d=0$, by Theorem 2.9 (1) and this is a contradiction. Hence, there is $x_0\in Z(R)$ such that $d(x_0)\neq 0$. By Lemmas 1.3 (3) and 2.10, we have for all $u\in I$ and $y\in R$,

$$\begin{aligned}0=[d(ux_0),y] &\subseteq [d(u)x_0+ud(x_0),y]\\ &= d(u)[x_0,y]+d(x_0)[u,y]\\ &= d(x_0)[u,y].\end{aligned}$$

that is $0\in d(x_0)[u,y]$. So, there is $t\in[u,y]$ such that $0=d(x_0)t$. Therefore, by Lemmas 2.2 (3) and 2.10, we get $t=0$, since $d(x_0)\neq 0$. This means that $I\subseteq Z(R)$. So, R is commutative, by Theorem 2.7 (1).
(3) Suppose that $u\in I$, then by Lemmas 1.3 (3) and 2.10, we have for all $y\in R$,

$$\begin{aligned}0 &\in [d^2(d(u)d(u)),y]\\ &\subseteq [d(d^2(u)d(u)+d(u)d^2(u)),y]\\ &= 2[d(d^2(u)d(u)),y]\\ &\subseteq 2[d^3(u)d(u)+d^2(u)d^2(u),y]\\ &= 2d^3(u)[d(u),y]+d^2(u)[d^2(u),y]\\ &= 2d^3(u)[d(u),y].\end{aligned}$$

Hence, $0\in d^3(u)[d(u),y]$, by hypothesis. So, $d^3(u)=0$ or $0\in[d(u),y]$, by Lemmas 2.2 (3) and 2.10. Therefore, $d^3(u)=0$ or $d(u)\in Z(R)$. Suppose that $d^3(u)=0$. Then, by Lemmas 1.2 and 1.3 (3), we have for all $y\in R$,

$$\begin{aligned}0 &= [d^2(ud(u)),y]\\ &\subseteq [d^2(u)d(u)+2d(u)d^2(u)+ud^3(u),y]\\ &= 3[d(u)d^2(u),y]=3d^2(u)[d(u),y].\end{aligned}$$

Hence, $0\in d^2(u)[d(u),y]$, by hypothesis. So, $d^2(u)=0$ or $0\in[d(u),y]$, by Lemma 2.2 (3). Therefore, $d^2(u)=0$ or $d(u)\in Z(R)$, for all $u\in I$.

Put $A=\{u\in I \mid d(u)\in Z(R)\}$ and $B=\{u\in I \mid d^2(u)=0\}$. It is clear that A and B are canonical subhypergroups of I and $I=A\cup B$. So, $I=A$ or $I=B$, by Lemma 2.6. If $I=B$ that is $d^2(I)=0$, then $d=0$, by Theorem 2.9 (1) and this is a contradiction. So, $I=A$ that is $d(I)\subseteq Z(R)$. Now (2) completes the proof.
(4) By Lemma 1.3 (3), for all $u\in I$ and $x\in R$,

$$\begin{aligned}0&=[d_1d_2(d_2(u)d_2(u)),x]\\ &\subseteq [d_1(d_2^2(u)d_2(u)+d_2(u)d_2^2(u)),x]\\ &\subseteq 2[d_2^2(u)d_1d_2(u),x]\\ &= 2d_1d_2(u)[d_2^2(u),x].\end{aligned}$$

Hence, $0\in d_1d_2(u)[d_2^2(u),x]$, by hypothesis. So, $d_1d_2(u)=0$ or $d_2^2(u)\in Z(R)$, by Lemma 2.2 (3), for all $u\in I$. Put $A=\{u\in I \mid d_2^2(u)\in Z(R)\}$ and $B=\{u\in I \mid d_1d_2(u)=0\}$. It is clear that A and B are canonical subhypergroups of I and $I=A\cup B$. So, $I=A$ or $I=B$, by Lemma 2.6. If $I=B$ that is $d_1d_2(I)=0$, then $d_1=0$ or $d_2=0$, by Theorem 2.9 (2), this is a contradiction. So, $I=A$ that is $d_2^2(I)\subseteq Z(R)$. Now (3) completes the proof.

3. Differential Krasner hyperring

Definition 3.1. A hyperring R is called differentiable if there is at least a derivation on R. A hyperring R with all derivations is called differential hyperring. A hyperfield R is called differential hyperfield if R is differential hyperring. A subhyperring H of differential hyperring R is called differential subhyperring if for all derivation d of R, we have $d(h)\in H$, for all $h\in H$. A hyperideal I of differential hyperring R is called a differential hyperideal if I is differential subhyperring of R.

Example 13. For every differential hyperring R, $<0>_R$ is a differential hyperideal.

A differential hyperideal $I(\neq R)$ of a differential hyperring R is called prime, if for all $x,y\in R$, $xy\in I$ implies that $x\in I$ or $y\in I$. The intersection of all differential prime hyperideals of R that contains differential hyperideal I is called radical I and denoted by $Rad(I)$. If the differential hyperring R does not have any differential prime hyperideal containing

I, we define $Rad(I)=R$. A differential hyperideal I is called differential radical hyperideal if $Rad(I)=I$.

Let R be a differential hyperring, I is a differential hyperideal of R and Δ is the set of all derivations on R. Then, briefly we say that R is a Δ-hyperring and I is a Δ-hyperideal of R.

Let R and S be Δ_1 and Δ_2-hyperrings, respectively. By a differential good homomorphism of R into S, we mean a good homomorphism φ such that $d_2\varphi(x)=\varphi d_1(x)$, for all $x\in R$, $d_1\in\Delta_1$ and $d_2\in\Delta_2$.

Theorem 3.2. Let R and S be Δ_1 and Δ_2-hyperrings, respectively. Also, let $\varphi:R\to S$ be a differential good homomorphism. Then,

(1) $ker\varphi$ is a Δ_1-hyperideal,

(2) If I is a Δ_2-hyperideal of S, then $\varphi^{-1}(I)$ is a Δ_1-hyperideal of R.

Proof: It is inconsequential that $ker\varphi$ is a hyperideal of R. For all $d_1\in\Delta_1$, $d_2\in\Delta_2$ and $x\in ker\varphi$, we have

$\varphi d_1(x)=d_2\varphi(x)=d_2(0)=0$. So, $d_1(x)\in ker\varphi$.

The proof of the part (2) is similar.

Theorem 3.3. Let $(R,+,\cdot)$ be a Δ-hyperring and I and J be Δ-hyperideals of R. Then,

$IJ=\{x\mid x\in\sum_{i=1}^{n}a_ib_i,\ a_i\in I,b_i\in J,n\in N\}$ is also a Δ-hyperideal of R.

Proof: It is proved that IJ is a hyperideal (Davvaz and Leoreanu-Fotea, 2007; p. 78). If $x\in IJ$, then $x\in\sum_{i=1}^{n}a_ib_i$, for some $a_i\in I$, $b_i\in J$ and $n\in N$.

So, for all $d\in\Delta$, we have

$$d(x)\in d(\sum_{i=1}^{n}a_ib_i)=\sum_{i=1}^{n}d(a_ib_i)$$

$$=\sum_{i=1}^{n}(d(a_i)b_i+a_id(b_i))\subseteq IJ.$$

Theorem 3.4. Let R be a Δ-hyperring and P is a Δ-hyperideal of R. Then, $J=\{a\in R\mid ra\in P, \text{for all } r\in R\}$ is a Δ-hyperideal of R.

Proof: It is easy to check that $P\subseteq J$ and J is a hyperideal of R. We prove that J is differential. Suppose that $a\in J$. Then $ra\in P$, for all $r\in R$. So, $d(ra)\in d(P)\subseteq P$. On the other hand, $d(ra)\in d(r)a+rd(a)$. Therefore, $rd(a)\in -d(r)a+d(ra)\subseteq P$, for all $r\in R$. Hence, $rd(a)\in P$, for all $r\in R$, and this implies that $d(a)\in J$. So, J is a Δ-hyperideal.

Let $(R_1,+_1,\cdot_1)$ and $(R_2,+_2,\cdot_2)$ be Δ_1 and Δ_2-hyperrings, respectively. Then, $(R_1\times R_2,+,\cdot)$ is a hyperring, where for all $(a,b),(c,d)\in R_1\times R_2$ hyeroperation $+$ and operation $\cdot$ are defined as $(a,b)+(c,d)=\{(x,y)\mid x\in a+_1c, y\in b+_2d\}$ and $(a,b)\cdot(c,d)=(a\cdot_1c,b\cdot_2d)$. For all $d_1\in\Delta_1$ and $d_2\in\Delta_2$, we define the function $d_1\times d_2:R_1\times R_2\to R_1\times R_2$ as $(d_1\times d_2)(x,y)=(d_1(x),d_2(y))$, for all $(x,y)\in R_1\times R_2$. Then, $d_1\times d_2$ is a derivation on $R_1\times R_2$.

Theorem 3.5. Let I be a Δ-radical hyperideal of commutative Δ-hyperring R. Then, $(I:r)=\{x\in R\mid xr\in I\}$, for all $r\in R$, is also a Δ-radical hyperideal.

Proof: Let $x,y\in(I:r)$. Then, $(x-y)r=xr-yr\subseteq I$. So, $x-y\subseteq(I:r)$. Now, suppose that $x\in(I:r)$ and $t\in R$. Then, $xtr=xrt\in It=I$. So, $xt\in(I:r)$. It shows that $(I:r)$ is a hyperideal. Let $x\in(I:r)$ and d is a derivation of R, then $d(x)rd(xr)\in d(x)rd(x)r+d(x)rxd(r)$. So, $(d(x)r)^2\in d(x)rd(xr)-d(x)rxd(r)\subseteq I.$ Therefore, $d(x)r\in Rad(I)=I$, which means that $d(x)\in(I:r)$. So, I is a Δ-hyperideal.

Obviously, $(I:r)\subseteq Rad((I:r))$. Let $x\in Rad((I:r))$. Then, there is $n\in N$ such that $x^n\in(I:r)$. Therefore, $x^nr\in I$. So, we have $(xr)^n=x^nr^n=r^{n-1}(x^nr)\in r^{n-1}I=I,$ since R is commutative. Hence, $xr\in Rad(I)=I$, which means $x \in (I:r)$. So, $(I:r)$ is a Δ-radical hyperideal.

If A is a normal hyperideal of hyperring R, then we define the relation
$x \equiv y(modA)$ if and only if $(x-y) \cap A \neq \emptyset$.
This relation is an equivalent relation and denoted by xA^*y (Davvaz and Leoreanu-Fotea, 2007).

Theorem 3.6. (Davvaz and Leoreanu-Fotea, 2007) Let R be a hyperring and A be a normal hyperideal of R. We define the hyperoperation $\oplus$ and the multiplication $\otimes$ on the set of classes $[R:A^*] = \{A^*(x) \mid x \in R\}$, as follows:

$$A^*(x) \oplus A^*(y) = \{A^*(z) \mid z \in A^*(x) + A^*(y)\};$$
$$A^*(x) \otimes A^*(y) = A^*(xy).$$

Then, $[R:A^*]$ is a hyperring.

Theorem 3.7. Let R, A and $[R:A^*]$ be as Theorem 3.6. Also, let R and A be differential and d be a derivation on R such that $A^*(d(x)) \subseteq d(A^*(x))$. Then, $\overline{d}:[R:A^*] \to [R:A^*]$ defined by $\overline{d}(A^*(x)) = A^*(d(x))$ is a derivation on $[R:A^*]$.

Proof: At first, we prove that $d(A^*(x)) \subseteq A^*(d(x))$.

Suppose that $d(s) \in d(A^*(x))$. Then,

$$s \in A^*(x) \Rightarrow (s-x) \cap A \neq \emptyset$$
$$\Rightarrow (d(s)-d(x)) \cap A \neq \emptyset$$
$$\Rightarrow d(s) \in A^*(d(x)).$$

So, $A^*(d(x)) = d(A^*(x))$, by hypothesis.

Now, it is clear that $\overline{d}$ is well defined. We have

$$\overline{d}(A^*(x)) \oplus \overline{d}(A^*(y)) = A^*(d(x)) \oplus A^*(d(y))$$
$$= \{A^*(z) \mid z \in A^*(d(x)) + A^*(d(y))\}$$
$$= A^*(d(x)) + A^*(d(y)).$$

On the other hand,

$$\overline{d}(A^*(x) \oplus A^*(y))$$
$$= \overline{d}(\{A^*(z) \mid z \in A^*(x) + A^*(y)\})$$
$$= \{A^*(d(z)) \mid z \in A^*(x) + A^*(y)\}$$
$$= A^*(d(A^*(x)) + d(A^*(y)))$$
$$= A^*(A^*(d(x)) + A^*(d(y)))$$
$$= A^*(d(x)) + A^*(d(y)).$$

So,

$$\overline{d}(A^*(x) \oplus A^*(y)) = \overline{d}(A^*(x)) \oplus \overline{d}(A^*(y)),$$

for all $A^*(x), A^*(y) \in [R:A^*]$. Also, we have

$$\overline{d}(A^*(x) \otimes A^*(y)) = \overline{d}(A^*(xy)) = A^*(d(xy))$$
$$\in A^*(d(x)y + xd(y))$$
$$= A^*(d(x)y) + A^*(xd(y))$$
$$= A^*(A^*(d(x)y) + A^*(xd(y)))$$
$$= A^*(d(x)y) \oplus A^*(xd(y))$$
$$= (A^*(d(x)) \otimes A^*(y))$$
$$\oplus (A^*(x) \otimes A^*(d(y)))$$
$$= (\overline{d}(A^*(x)) \otimes A^*(y))$$
$$\oplus (A^*(x) \otimes \overline{d}(A^*(y))).$$

So, $\overline{d}$ is a derivation on $[R:A^*]$.

Theorem 3.8. Let d be a derivation on Δ-hyperfield R such that $d(1) = 0$, where 1 is the unite element of R. Then, $C_d(R)$ is also a Δ-hyperfield.

Proof: Let $x, y \in C_d(R)$, then $d(x+y) = d(x) + d(y) = 0$.

So, $x + y \subseteq C_d(R)$. Also, we have $d(xy) \in d(x)y + xd(y) = 0$, which means that $xy \in C_d(R)$. Now, suppose that $0 \neq x \in C_d(R)$, then $d(x) = 0$. Since R is a hyperfield, there is $y \in R$ such that $xy = 1$.

We have $0 = d(1) = d(xy) \in d(x)y + xd(y) = xd(y)$, that is $xd(y) = 0$. So, $d(y) = 0$, since R is Δ-hyperfield and $x \neq 0$. This shows that $y \in C_d(R)$.

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