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Modified chain least squares method and some numerical results

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Abstract

Recently, in order to increase the efficiency of least squares method in numerical solution of ill-posed problems, the chain least squares method is presented in a recurrent process by Babolian et al. Despite the fact that the given method has many advantages in terms of accuracy and stability, it does not have any stopping criterion and has high computational cost. In this article, the attempt is to decrease the computational cost of chain least squares method by introducing the modified least squares method based on stopping criterion. Numerical results show that the modified method has high accuracy and stability and because of its low computational cost, it can be considered as an efficient numerical method.

***Keywords**:* Chain least squares; Lagrange multipliers method; Ill- posed problem; Integral equations; Singular second order initial value differential equations

1. Introduction

Least squares method is one of the efficient methods in the numerical solution of many engineering and physics problems (Aksan et al. 2006; Alexander and George, 1990; Ching and Suh-Yuh, 2002; Jagalur-Mohan et al. 2013; Jannike and Hugo, 2012; Jinming, 2012; King and Krueger, 2003; Laeli and Maalek, 2012). In order to increase the efficiency of this method in numerical solution of some ill-posed problems, the chain least squares method is presented in recurrent form by Babolian et al. (2014). In this approach, by reducing an n-term least squares problem to the $(n-1)$-term ones and continuation of this trend up to the last stage (1-term problem), the efficiency of the least squares method in numerical solution of ill-posed problems has been significantly increased (Babolian et al. 2014). Thus, for solving an n-term problem by chain least squares method, we have to continue the recurrent process up to the last stage. In this article, the attempt is to prevent the continuation of the recurrent process up to the last stage by providing a logical and experimental stopping criterion. Besides decreasing the computational cost of chain least squares method, the definition of the stopping criterion maintains the stability and accuracy of this method. This stopping criterion is based on the convergence of intermediate matrix elements of least squares method to zero. This is inspired by the convergence of the Galerkin method in numerical solution of Fredholm integral equation of the second kind (Delves and Mohamed, 1985). It should be mentioned that by intermediate matrices, we mean the coefficient matrix of system of equation corresponding to the chain least squares method in turning k-term problem $(k = n, \dots, 2)$ to $(k-1)$-term one.

In the second step, considering the main role of the artificial trajectories in the definition of chain least squares method (Babolian et al. 2014), in order to decrease the computational cost of this method, a new process is introduced in defining of artificial trajectories. According to the kinds of problems solved by the chain least squares method, at least one of the artificial trajectories is decreased. In the new trend, instead of reducing n-term problem to $(n-1)$-term one, the attempt is to change n-term problem to $(n-l)$-term one $(l \geq 2)$ in such a way that the computational cost of this method is decreased. By presenting numerical examples in each section, the stability and accuracy of the new method will be shown.

Firstly, a review of chain least squares method has been given, then the modified chain least squares method is presented. Finally, the efficiency of the modified methods is investigated by solving several ill-posed functional equations.

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2. A Review of the Chain Least Squares Method

Let $f \in L^2[a,b]$ and $\{L_i\}_{i=1}^{\infty}$ be a basis of $L^2[a,b]$ and

$$f_n(s) = \sum_{i=1}^{n} a_i L_i(s), \quad s \in [a,b],$$

be an ordinary least squares approximation of f in the basis $\{L_i\}_{i=1}^{n}$. For determining the unknown coefficients $\{a_i\}_{i=1}^{n}$ we must solve the following minimization problem

$$min_{a_1,\dots,a_n} \mathrm{e}(a_1,\dots,a_n). \qquad (1)$$

In order to determine the solution of (1), it is sufficient to solve the following normal equations

$$\frac{\partial}{\partial a_i} e(a_1,\dots,a_n) = 0, \quad i = 1,\dots,n, \qquad (2)$$

where

$$e(a_1,\dots,a_n) = \int_a^b \left[\sum_{i=1}^{n} a_i L_i(s) - f(s)\right]^2 ds.$$

It is possible that solving (2) becomes an ill-posed problem. In other words, the condition number of (2) is large and its solution is determined with large error (Datta, 2010). In order to get the approximate solution with high accuracy in chain least squares method (Babolian et al. 2014), it is supposed that solution of (2) is true for the following conditions

$$g_i(a_1,\dots,a_n) = 0, \quad i = 1,\dots,n-1.$$

In which, artificial constrains g_i are defined as follows (for scalars $\{r_i\}_{i=1}^{n-1}$ belonging to $\mathbb{R}$) (Babolian et al., 2014)

$$g_i(a_1,\dots,a_n) = a_i - a_{i+1} - r_i, \quad i = 1,\dots,n-1. \qquad (3)$$

Therefore the minimization problem (1) is equivalent to

$$\begin{array}{ll} min & e(a_1,\dots,a_n) \\ s.t & g_i(a_1,\dots,a_n) = 0, \quad i = 1,\dots,n-1. \end{array} \qquad (4)$$

By the Lagrange multipliers method (Ito and Kunisch, 2008) there exist real scalars $\{\lambda_i\}_{i=1}^{n-1}$ such that the problem (4) is equivalent to

$$\begin{cases} \vec{\nabla} e = \sum_{i=1}^{n-1} \lambda_i \vec{\nabla} g_i \\ g_i(a_1 \dots, a_n) = 0, \quad i = 1,\dots,n-1. \end{cases} \qquad (5)$$

In which, $\vec{\nabla}$ is Gradian operator. From (5) one gets

$$\begin{cases} 2c_{11}a_1 + 2c_{12}a_2 + \cdots + 2c_{1n}a_n - 2f_1 = \lambda_1 \\ 2c_{21}a_1 + 2c_{22}a_2 + \cdots + 2c_{2n}a_n - 2f_2 = \lambda_2 - \lambda_1 \\ \quad \vdots \qquad \cdots \qquad \vdots \\ 2c_{n1}a_1 + 2c_{n2}a_2 + \cdots + 2c_{nn}a_n - 2f_n = -\lambda_{n-1} \\ a_1 - a_2 = r_1 \\ \quad \ddots \qquad \vdots \\ a_{n-1} - a_n = r_{n-1}, \end{cases}$$

where

$$c_{ij} = \int_a^b L_i(s) L_j(s) ds, \quad i,j \in \{1,\dots,n\},$$

$$f_i = \int_a^b L_i(s) f(s) ds, \quad i = 1,\dots,n.$$

By summing the first n equations, (for removing $\{\lambda_i\}_{i=1}^{n-1}$) we have

$$\begin{cases} d_1 a_1 + d_2 a_2 + \cdots + d_n a_n = h \\ a_1 - a_2 = r_1 \\ \quad \cdots \\ a_{n-1} - a_n = r_{n-1}, \end{cases} \qquad (6)$$

where

$$h = \sum_{i=1}^{n} f_i, \quad d_j = \sum_{i=1}^{n} c_{ij}, \quad j \in \{1,\dots,n\}.$$

Finally, by (6) the coefficients $\{a_i\}_{i=1}^{n}$ are determined as follows:

$$a = DR,$$

where

$$a = (a_1,\dots,a_n)^T, \quad R = (r_1,\dots,r_{n-1},1)^T,$$

$$D = \frac{1}{N}\begin{pmatrix} N-t_1 & N-t_2 & \cdots & N-t_{n-1} & h \\ -t_1 & N-t_2 & \cdots & N-t_{n-1} & h \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ -t_1 & -t_2 & \cdots & N-t_{n-1} & h \\ -t_1 & -t_2 & \cdots & -t_{n-1} & h \end{pmatrix},$$

$$N = \sum_{i=1}^{n} d_i,$$

$$t_1 = d_1, \quad t_i = t_{i-1} + d_i, \quad i = 1,\dots,n-1.$$

Let

$$L(s) = (L_1(s),\dots,L_n(s)),$$

By the above assumptions, the n-term minimization problem (1) transforms to $(n-1)$-term one as follows:

$$\min_{r_1,\dots,r_{n-1}} E(r_1,\dots,r_{n-1}),$$

in which

$$E(r_1,\dots,r_{n-1}) = \int_a^b \left[\sum_{i=1}^{n-1} r_i p_i(s) - \bar{f}(s)\right]^2 ds,$$

and

$$p_i(s) = L(s)D^i, \ \ i = 1, \dots, n,$$

$$\bar{f}(s) = f(s) - p_n(s),$$

where

$$D^i = i\text{'}th\ column\ of\ D.$$

In chain least squares method (Babolian, 2014), the $(n-1)$-term problem is turned into the $(n-2)$-term one by the same trend. This process has been continued up to the final stage (1-term problem).

3. Conditional Chain Least Squares Method

Suppose that the purpose is to determine the least squares approximation of $f \in L^2[a,b]$ on the basis of $\{ L_i\}_{i=1}^n$. If this problem is to be solved by chain least squares method, then we have to solve n-term, $(n-1)$-term, …, 1-term problems. In other words, the chain least squares method will be continued up to the last stage and this process has very high computational cost.

In this section, the purpose is to present an experimental stopping criterion for chain least squares method in such a way that its recurrent process is finished before reaching to 1-term problem. By this trend, not only is the computational cost of this method decreased, but also its stability will be increased significantly. For explaining stopping criterion, suppose that the purpose is to determine the chain least squares of the following examples on the basis of $\{s^i\}_{i=0}^{14}$.

Example 1. $f(s) = e^s, \ \ s \in [0,1]$.

Example 2. $f(s) = \sin(s), \ \ s \in [0,1]$.

In order to determine these approximations, we have to solve (15)-term, (14)-term, …, 1-term problems for each example. Suppose that the linear system of equations of k-term problems ($k = 15, \dots, 1$) is shown by $A_k a_k = F_k$. A_k and F_k matrices are called intermediate matrices of chain least squares method. Also, we define

$$MA_k = \ Max_{1 \le i,j \le k}(A_k)_{i,j}, \ \ k = 15, \dots, 1,$$

$$MF_k = Max_{1 \le j \le k}(F_k)_j, \ \ k = 15, \dots, 1.$$

To show the behavior of $\{MA_k\}_{k=1}^{15}$ and $\{MF_k\}_{k=1}^{15}$, these parameters are computed for Examples 1 and 2 and the numerical results are presented in Tables 1 and 2, respectively.

Table 1. Values of MA_k and MF_k for Example 1

k	MA_k	MF_k
15	3.45×10^{-02}	1.71×10^{-01}
14	9.65×10^{-03}	7.49×10^{-02}
13	9.65×10^{-04}	5.10×10^{-03}
12	6.45×10^{-05}	3.01×10^{-05}
11	3.18×10^{-06}	2.07×10^{-05}
10	1.19×10^{-07}	1.19×10^{-06}
9	3.41×10^{-09}	3.35×10^{-08}
8	7.29×10^{-11}	5.12×10^{-10}
7	1.14×10^{-12}	3.87×10^{-12}
6	1.26×10^{-14}	6.02×10^{-15}
5	9.33×10^{-17}	1.04×10^{-16}
4	4.26×10^{-19}	6.52×10^{-19}
3	1.06×10^{-21}	1.35×10^{-21}

Table 2. Values of MA_k and MF_k for Example 2

k	MA_k	MF_k
15	3.45×10^{-02}	5.35×10^{-02}
14	9.65×10^{-03}	2.03×10^{-02}
13	9.65×10^{-04}	4.92×10^{-04}
12	6.45×10^{-05}	4.80×10^{-04}
11	3.18×10^{-06}	5.92×10^{-05}
10	1.19×10^{-07}	3.37×10^{-06}
9	3.41×10^{-09}	1.40×10^{-07}
8	7.29×10^{-11}	3.16×10^{-09}
7	1.14×10^{-12}	4.11×10^{-11}
6	1.26×10^{-14}	2.58×10^{-13}
5	9.33×10^{-17}	1.09×10^{-16}
4	4.26×10^{-19}	7.04×10^{-18}
3	1.06×10^{-21}	3.01×10^{-20}

According to the presented numerical results in Tables 1 and 2, it is concluded that

$$\lim_{k \to 1} MA_k = 0, \ \ \lim_{k \to 1} MF_k = 0.$$

In other words, the following conclusion is obtained experimentally.

Conclusion 1. In determining the chain least squares approximation of $f \in L^2[a,b]$ in the basis $\{L_i\}_{i=1}^n$, the elements of intermediate matrices A_k and F_k converge to zero.

It should be mentioned that such a state occurs in approximating the solution of the Fredholm integral equation of second kind on an orthogonal basis by Galerkin method (Delves and Mohamed, 1985). In other words, if $Ba = b$ is the linear system of equations corresponding to this problem, then we have

$$\lim_{n \to \infty} b_n = 0, \ \ \lim_{n \to \infty} B_{nj} = 0, \ \ j = 1, \dots, n,$$

in which, $\{B_{nj}\}_{j=1}^n$ are the elements of n'th row of matrix B.

Now suppose that the purpose is to determine the chain least squares approximation of $f \in L^2[a,b]$ in the basis $\{L\}_{i=1}^n$. Also suppose that

$$A_k a_k = F_k, \ \ k = n+1, \dots, 1,$$

is the intermediate linear system of equations of this method. By the conclusion 1, it is logical that we continue the algorithm of chain least squares up to the stage that ($eps = 2.26 \times 10^{-16}$)

$$MA_k > eps \text{ and } MF_k > eps.$$

Because otherwise, the continuation of chain method will not only improve the solution of the problem, but also the accuracy of the obtained approximations will be decreased by introducing round off errors and additional noises to the problem's solution.

Note 1. In the chain least squares method, for determining the approximation of function f in the basis of $\{L_i\}_{i=1}^n$, we encounter $(n+1)$-term problem in the first stage, n-term problem in the second stage, …, $(n-k+1)$-term problem in the k'th stage. In other words, this method is done in $(n+1)$ stages.

By the above explanations and according to the fact that in every stage $MA_k \cong MF_k$(Tables 1 and 2), the following stopping criterion is presented for chain least squares method.

Conclusion 2. In determining the approximation of $f \in L^2[a,b]$ in the basis $\{L_i\}_{i=1}^n$ by chain least squares method, the algorithm of this method in the k'th stage will be continued when $MA_k \geq eps$. We name this new method, conditional chain least squares method **(CCLSM).**

Therefore, if the chain least squares method is continued up to the k'th stage ($k \in N$), then n-term, $(n-1)$-term, …, $(n-k+1)$-term problems will be solved by this method and we will not need to solve $(n-k)$-term,…,1-term problems. So its computational cost will be decreased significantly. In order to compare the accuracy of conditional chain least squares method with ordinary least squares method **(OLSM)** and chain least squares method **(CLSM)**, the examples 1 and 2 are approximated by these methods in the basis $\{s^i\}_{i=0}^n$. The numerical results are given in Tables 3 and 4.

Table 3. Maximum absolute errors for Example 1

n	OLSM	CLSM	CCLSM
1	1.55×10^{-01}	1.55×10^{-01}	1.55×10^{-01}
2	1.49×10^{-02}	1.49×10^{-02}	1.49×10^{-02}
3	1.05×10^{-03}	1.05×10^{-03}	1.05×10^{-03}
4	5.76×10^{-05}	5.76×10^{-05}	5.76×10^{-05}
5	2.59×10^{-06}	2.59×10^{-06}	2.59×10^{-06}
6	9.39×10^{-08}	9.39×10^{-08}	9.39×10^{-08}
7	3.32×10^{-09}	3.29×10^{-09}	3.29×10^{-09}
8	3.07×10^{-10}	9.65×10^{-11}	9.65×10^{-11}
9	1.28×10^{-09}	2.53×10^{-12}	2.53×10^{-12}
10	6.22×10^{-09}	5.95×10^{-14}	5.95×10^{-14}
11	2.51×10^{-08}	1.66×10^{-15}	1.66×10^{-15}
12	8.49×10^{-08}	1.77×10^{-15}	1.77×10^{-15}
13	2.49×10^{-07}	2.22×10^{-15}	1.33×10^{-15}
14	8.22×10^{-07}	8.88×10^{-16}	1.77×10^{-15}

Table 4. Maximum absolute errors for Example 2

n	OLSM	CLSM	CCLSM
1	4.61×10^{-02}	4.61×10^{-02}	4.61×10^{-02}
2	7.46×10^{-03}	7.46×10^{-03}	7.46×10^{-03}
3	3.10×10^{-04}	3.10×10^{-04}	3.10×10^{-04}
4	2.94×10^{-05}	2.94×10^{-05}	2.94×10^{-05}
5	7.64×10^{-07}	7.64×10^{-07}	7.64×10^{-07}
6	5.12×10^{-08}	5.12×10^{-08}	5.12×10^{-08}
7	9.70×10^{-10}	9.66×10^{-10}	9.66×10^{-10}
8	9.08×10^{-11}	5.01×10^{-11}	5.01×10^{-11}
9	2.97×10^{-10}	7.42×10^{-13}	7.42×10^{-13}
10	1.66×10^{-09}	3.13×10^{-14}	3.13×10^{-14}
11	6.99×10^{-09}	3.79×10^{-16}	3.79×10^{-16}
12	2.07×10^{-08}	3.33×10^{-16}	3.33×10^{-16}
13	3.09×10^{-08}	2.22×10^{-16}	2.22×10^{-16}
14	8.52×10^{-08}	3.63×10^{-16}	3.63×10^{-16}

For example, in determining the approximation of $f(s) = e^s, s \in [0,1]$ in the basis $\{s^i\}_{i=0}^{14}$ by **CCLSM**, this method is continued up to the 11'th stage. So solving 4-term,…,1-term problems are avoided.

According to the numerical results in Tables 3 and 4, it is clear that the conditional chain least squares method besides decreasing the computational cost, has the desirable accuracy and stability.

4. Modified Chain Least Squares Method

Suppose that the aim is to obtain the numerical solution of an ill-posed problem. Fredholm integral equation of the first kind (Bitsadze, 1995; Delves and Mohamed, 1985) and determination of the least squares approximation of an arbitrary function on the basis $\{s^i\}_{i=0}^n$ (Datta, 2010; Kincaid and Ward, 2002) are some examples of this kind of ill-posed problem.

If ill-posed problems are approximated by least squares method on the basis $\{s^i\}_{i=0}^n$ then we will encounter $(n+1)$-term problem. Let $Aa = F$ be the corresponding linear system of equations of this problem in the basis $\{s^i\}_{i=0}^n$, by increasing n, the condition number of the matrix A will be enlarged (Datta, 2010; Kincaid and Ward, 2002) and the solution of the corresponding system is determined by large errors. In the chain least squares method for solving n-term problem (Babolian et al. 2014), $(n-1)$ artificial trajectories are defined. By continuing this process up to 1-term problem, we overcome the ill-posedness and accurate approximate solutions are obtained.

In numerical solution of an ill-posed problem, the first equations of the system $Aa = F$ do not have a main role in ill-posedness of this system (Delves and Mohamed, 1985). The ill-posedness appears mainly on final equations of this system. By this argument, in chain least squares method, artificial trajectories are defined in such a way that the structure of the few first equations of the system

$Aa = F$ is maintainted in decreasing the dimension of least squares problem and instead, some of the artificial trajectories are eliminated.

For example, if the first s artifical trajectories are eliminated, then in chain least squares method, n-term problem will be changed to $(n-s-1)$-term problem. In other words, in the new method, we will not have any need to solve $(n-1)$-term, …, $(n-s)$-term problems. So the computational cost of chain least squares method will be decreased significantly. If this work is done by maintaining accuracy and stability of this method, it will be very valuable. To explain this method, we act as follows.

Let f_n be the least squares approximation of $f \in L^2[a,b]$ in the basis $\{L_i\}_{i=1}^{n}$, i.e.,

$$f_n(s) = \sum_{i=1}^{n} a_i L_i(s)\,, \quad s \in [a,b]$$

Now, we must solve the following minimization problem

$$min_{a_1,\dots,a_n}\, e(a_1,\dots,a_n), \tag{7}$$

where

$$e(a_1,\dots,a_n) = \int_a^b [\sum_{i=1}^{n} a_i L_i(s) - f(s)]^2 ds. \tag{8}$$

Assume that the unknown coefficients $\{a_i\}_{i=1}^{n}$ holds in the following trajectories (for scalars $\{r_i\}_{i=1}^{n-s-1}$ belonging to $\mathbb{R}$)

$$g_i(a_1,\dots,a_n) = 0, \;\; i = 1,\dots,n-s-1, \tag{9}$$

where $n, s \in \mathbb{N}$, $0 \le s \le n-1$ and

$$g_i(a_1,\dots,a_n) = a_{s+i} - a_{s+i+1} - r_i, \quad i = 1,\dots,n-s-1.$$

In other words, we omit the following s-artificial trajectories

$$\begin{cases} a_1 - a_2 = r_1 \\ \quad\vdots \qquad \vdots \\ a_s - a_{s+1} = r_s \end{cases}$$

Then by (9) and (7) we have

$$\begin{cases} \min\limits_{a_1,\dots,a_n} e(a_1,\dots,a_n), \\ g_i(a_1 \dots, a_n) = 0, \;\; i = 1,\dots,n-s-1. \end{cases}$$

now by the Lagrange multipliers method (Ito and Kunisch, 2008) one gets

$$\begin{cases} \nabla\vec{e} = \sum\limits_{i=1}^{n-s-1} \lambda_i \vec{\nabla} g_i \\ g_i(a_1,\dots,a_n) = 0 \;\; i = 1,\dots,n-s-1 \end{cases}$$

where $\{\lambda_i\}_{i=1}^{n-s-1}$are real scalars. Since

$$\vec{\nabla} e = \left(\frac{\partial e}{\partial a_1},\dots,\frac{\partial e}{\partial a_n}\right),$$

we have

$$\begin{cases} c_{11}a_1 + c_{12}a_2 + \dots + c_{1n}a_n = f_1 \\ \quad\vdots \\ c_{s1}a_1 + c_{s2}a_2 + \dots + c_{sn}a_n = f_s \\ 2(c_{s+1,1}a_1 + c_{s+1,2}a_2 + \dots + c_{s+1,n}a_n - f_{s+1}) = \lambda_1 \\ 2(c_{s+2,1}a_1 + c_{s+2,2}a_2 + \dots + c_{s+2,n}a_n - f_{s+2} = \lambda_2 - \lambda_1 \\ \quad\vdots \\ 2(c_{n1}a_1 + c_{n2}a_2 + \dots + c_{nn}a_n - f_n = -\lambda_{n-s-1} \\ a_{s+1} - a_{s+2} = r_1 \\ \quad\ddots \\ a_{n-1} - a_n = r_{n-s-1} \end{cases}$$

In which

$$c_{ij} = \int_a^b L_i(s) L_j(s) ds\,, \;\; i,j \in \{1,\dots,n\},$$

$$f_i = \int_a^b L_i(s) f(s) ds\,, \;\; i \in \{1,\dots,n\},$$

by summing equations $(s+1)$, …, n (for removing $\{\lambda_i\}_{i=1}^{n-s-1}$) we have

$$\begin{cases} c_{11}a_1 + c_{12}a_2 + \dots + c_{1n}a_n = f_1 \\ \quad\vdots \\ c_{s1}a_1 + c_{s2}a_2 + \dots + c_{sn}a_n = f_s \\ d_1a_1 + d_2a_2 + \dots + d_na_n = h \\ a_{s+1} - a_{s+2} = r_1 \\ \quad\ddots \\ a_{n-1} - a_n = r_{n-s-1} \end{cases} \tag{10}$$

In which

$$h = \sum_{i=s+1}^{n} f_i\,, \;\; d_j = \sum_{i=s+1}^{n} c_{ij}\,, \;\; j = 1,\dots,n.$$

if D be the inverse of the coefficient matrix of system (10) then $\{a_i\}_{i=1}^{n}$ are determined as follows

$$a = DF, \tag{11}$$

where

$$a = (a_1,\dots,a_n)^T,$$
$$F = (f_1,\dots,f_s,h,r_1,\dots,r_{n-s-1})^T.$$

Let

$$L(s) = (L_1(s),\dots,L_n(s))^T$$

from (8) and (11) we have

$$e(a_1,\dots,a_n) = \int_a^b [L(s)a - f(s)]^2 ds$$
$$= \int_a^b [L(s)DF - f(s)]^2 ds \tag{12}$$

Since

$$L(s)DF = L(s)D^1 f_1 + \cdots + L(s)D^s f_s + L(s)D^{s+1}h + L(s)D^{s+2}r_1 + \cdots + L(s)D^n r_{n-s-1},$$

where

$$D^i = i\text{'th column of } D,$$

and by taking

$$p_i(s) = L(s)D^{s+1+i}, \; i = 1, \dots, n-s-1.$$

From (11) we have

$$E(r_1, \dots, r_{n-s-1}) = e(a_1, \dots, a_n) = \int_a^b [\sum_{i=1}^{n-s-1} p_i(s)r_i - \bar{f}(s)]^2 \, ds,$$

where

$$\bar{f}(s) = f(s) - \sum_{i=1}^{s} L(s)D^i f_i - L(s)D^{s+1}h.$$

So the minimization problem (7) reduces to

$$\min_{r_1, r_2, \dots, r_{n-s-1}} \mathrm{E}(r_1, r_2, \dots, r_{n-s-1}).$$

Now, for solving the minimization problem with $(n-s-1)$-term we use chain least squares method which is introduced in section 3.

Definition 1. We call, "transformation of the n-term least squares problem with presented algorithm in this section to the $(n-s-1)$-term least squares problem and solving by conditional chain least squares" as Modified Chain Least Squares method **(MCLSM)**.

We expect that for small and logical values of s, the accuracy and stability of the above method is the same as CCLSM, but computational cost of MCLSM is less than CCLSM. To confirm this, we compare the least squares approximations of Examples 1 and 2 in the basis $\{s^i\}_{i=0}^n$ by OLSM and MCLSM and CPU times of these methods (CLSM, CCLSM, MCLSM) are compared for Example 1. We take $s = 3$ for MCLSM. Tables 5, 6 and 7 show that, the modified chain least squares method is almost equal to conditional chain least squares method but the CPU time of MCLSM is less than CLSM and CCLSM.

5. Numerical Solution of Some Ill-posed Functional Equations with Modified Chain Least Squares Method

In this section, two cases of ill-posed functional equations are investigated and some examples are solved by MCLSM.

Table 5. Comparison of maximum absolute errors of OLSM and MCLSM for Example 1

n	OLSM	MCLSM
4	5.76×10^{-05}	5.76×10^{-05}
5	2.59×10^{-06}	2.59×10^{-06}
6	9.39×10^{-08}	9.93×10^{-08}
7	3.32×10^{-09}	3.29×10^{-09}
8	3.07×10^{-10}	9.64×10^{-11}
9	1.28×10^{-09}	2.60×10^{-12}
10	6.22×10^{-09}	5.28×10^{-14}
11	2.51×10^{-08}	2.17×10^{-14}
12	8.49×10^{-08}	6.62×10^{-14}
13	2.49×10^{-07}	5.11×10^{-14}
14	8.22×10^{-07}	1.73×10^{-14}

Table 6. Comparison of maximum absolute errors of OLSM and MCLSM for Example 2

n	OLSM	MCLSM
4	2.94×10^{-05}	2.94×10^{-05}
5	7.64×10^{-07}	7.64×10^{-07}
6	5.12×10^{-08}	5.12×10^{-08}
7	9.70×10^{-10}	9.67×10^{-10}
8	9.08×10^{-11}	5.01×10^{-11}
9	2.97×10^{-10}	7.16×10^{-13}
10	1.66×10^{-09}	5.24×10^{-14}
11	6.99×10^{-09}	6.44×10^{-15}
12	2.07×10^{-08}	2.09×10^{-14}
13	3.09×10^{-08}	1.66×10^{-14}
14	8.52×10^{-08}	5.44×10^{-15}

Table 7. Comparison of CPU times (CLSM, CCLSM, MCLSM) for Example1

n	CLSM	CCLSM	MCLSM
4	01.34	00.86	00.13
5	02.58	01.20	00.38
6	04.07	01.42	00.83
7	02.98	02.18	01.31
8	03.33	03.37	01.52
9	04.91	04.74	02.16
10	06.94	06.79	03.37
11	09.63	09.44	05.09
12	13.10	12.70	06.91
13	16.70	16.60	09.68
14	21.90	21.30	12.90

It should be mentioned that all of these computations are done in Matlab 2011a with 16 significant digits and Gaussian quadrature rule of order 16 is used for computing the related integrals

Case 1.
In this case, Fredholm integral equations of the first kind are investigated. These equations appear in many physical problems (Balanis, 1989). Because of the ill-poseness of these functional equations, these equations are investigated by many researchers (Babolian and Delves, 1979; Babolian et al. 2007; Groetsch, 1984; Maleknejad et al. 2006; Nashed, 1976). The general form of the integral equations of the first kind is as follows.

$$\int_a^b k(s,t)x(t)dt = f(s), \;\; s \in [a,b]. \tag{13}$$

in which f and k are known functions and x is unknown function. For solving these equations with least squares method on the basis $\{t^i\}_{i=0}^n$, let

$$x(t) \cong \sum_{i=0}^n a_i t^i, \ \ t \in [a,b], \tag{14}$$

by putting approximate solution (14) in (13) we have

$$\sum_{i=0}^n a_i q_i(s) = f(s) + r_n(s), \ \ s \in [a,b],$$

where $r_n(s)$ is residual function and

$$q_i(s) = \int_a^b k(s,t) t^i \, dt, \ \ s \in [a,b].$$

For determining unknown coefficients $\{a_i\}_{i=0}^n$, it is sufficient to determine least squares of f in the set $\{q_i\}_{i=0}^n$.

In this section, we determine unknown coefficients $\{a_i\}_{i=0}^n$with (OLSM), (CLSM), (MCLSM) methods. We take $s = 1$ for MCLSM.

Example 3.

$$\int_0^1 e^{st} x(t) dt = \frac{e^{s+1} - 1}{s+1}, \ \ s \in [0,1].$$

Example 4.

$$\int_1^2 \cos(st)\, x(t) dt = \frac{-\cos(s) + 2cos^2(s) - s(\sin(s) - 4\cos(s)\sin(s) + 1)}{s^2}, \ \ s \in [1,2].$$

with the exact solutions e^t and t respectively. The maximum absolute errors of examples 3 and 4 are reported in Tables 8 and 9.

Table 8. Maximum absolute errors of (OLSM), (CLSM), (MCLSM) for Example 3

n	OLSM	CLSM	MCLSM
1	1.42×10^{-01}	1.42×10^{-01}	----------------
2	1.46×10^{-02}	1.46×10^{-02}	1.46×10^{-02}
3	1.05×10^{-03}	1.05×10^{-03}	1.05×10^{-03}
4	6.45×10^{-02}	5.94×10^{-05}	5.94×10^{-05}
5	2.27×10^{-01}	2.71×10^{-06}	2.74×10^{-06}
6	2.21×10^{-02}	1.94×10^{-06}	8.58×10^{-06}
7	5.82×10^{-02}	2.17×10^{-04}	4.89×10^{-04}
8	5.89×10^{-2}	3.42×10^{-04}	3.46×10^{-04}
9	3.18×10^{-02}	1.86×10^{-04}	2.36×10^{-04}
10	2.19×10^{-00}	5.45×10^{-04}	1.49×10^{-04}

Table 9. Maximum absolute errors of (OLSM), (CLSM), (MCLSM) for Example 4

n	OLSM	CLSM	MCLSM
1	0	0	----------------
2	6.61×10^{-11}	6.77×10^{-15}	3.38×10^{-14}
3	8.61×10^{-07}	1.79×10^{-13}	2.62×10^{-14}
4	4.75×10^{-04}	2.83×10^{-11}	6.67×10^{-12}
5	1.07×10^{-03}	3.72×10^{-09}	9.23×10^{-10}
6	2.32×10^{-03}	3.21×10^{-06}	1.13×10^{-05}
7	1.98×10^{-03}	6.06×10^{-07}	7.50×10^{-06}
8	1.19×10^{-02}	1.09×10^{-06}	7.30×10^{-06}
9	1.74×10^{-03}	1.77×10^{-05}	1.25×10^{-05}
10	1.15×10^{-02}	2.04×10^{-05}	1.60×10^{-04}

Case 2.

In this case, the singular second order initial value differential equations are investigated. These equations appear in some models of physical problems and are investigated by many researchers (Wazwaz, 2002; Kiymaz and Mirasyedioglu, 2005; Aslanov and Abu-Alshaikh, 2008). Since these equations have singular points, their numerical solutions are of paramount importance. The general form of these equations are as follows:

$$\begin{cases} p(t)y'' + q(t)y' + r(t)y = f(t), \ t \in [0,T], \\ y(0) = y_0, \ y'(0) = y_1, \end{cases} \tag{15}$$

where function p has some zeros in [0,T]. To approximate the solution in the basis $\{t^i\}_{i=0}^n$ with least squares method, let

$$y(t) \cong \sum_{i=0}^n a_i t^i \tag{16}$$

by using the initial values, the unknown parameters a_0 and a_1 are determined as follows:

$$a_0 = y_0, \ \ a_1 = y_1.$$

By putting (16) in (15) we have

$$\sum_{i=2}^n a_i L_i(t) = \bar{f}(t) + r_n(t), \tag{17}$$

where $r_n(t)$ is residual function and

$$L_i(t) = p(t)\big(i(i-1)\big)t^{i-2} + q(t) i t^{i-1} + r(t) t^i, \quad i = 2, \dots, n,$$

$$\bar{f}(t) = f(t) - (a_1 q(t) + (a_0 + a_1 t) r(t)).$$

For determining unknown coefficient $\{a_i\}_{i=2}^n$it is sufficient to determine least squares of f in the basis $\{L_i\}_{i=2}^n$.

Similar to case 1 we calculate unknown coefficients $\{a_i\}_{i=2}^n$with (OLSM), (CLSM), (MCLSM). We take $s = 2$ for MCLSM.

Example 5.

$$\begin{cases} t^2 y'' + (1+t)y' - sin(t)y = f(t), \ \ t \in [0,1]. \\ y(0) = 1, \ \ y'(0) = 1 \end{cases}$$

Example 6.

$$\begin{cases}(t-0.5)(t-0.7)y''+ty'+e^t y=f(t), & t\in[0,1],\\ y(0)=1, \quad y'(0)=0.\end{cases}$$

The right hand side of the above equations are so considered such that e^t, $\cos(t)$ is the solutions respectively.

The maximum absolute errors of (OLSM), (CLSM), (MCLS) for example 5, 6 are reported in Tables 10, 11.

Table 10. Maximum absolute errors of (OLSM), (CLSM), (MCLSM) for Example 5

n	OLSM	CLSM	MCLSM
3	1.33×10^{-02}	1.33×10^{-02}	----------------
4	6.38×10^{-04}	6.38×10^{-04}	----------------
5	2.53×10^{-05}	2.53×10^{-05}	2.53×10^{-05}
6	8.65×10^{-07}	8.65×10^{-07}	8.65×10^{-07}
7	2.51×10^{-08}	2.51×10^{-08}	2.51×10^{-08}
8	6.89×10^{-10}	6.89×10^{-10}	6.88×10^{-10}
9	1.54×10^{-11}	1.81×10^{-11}	1.80×10^{-11}
10	1.37×10^{-11}	3.71×10^{-13}	3.71×10^{-13}
11	7.41×10^{-11}	6.89×10^{-15}	7.10×10^{-15}
12	3.67×10^{-10}	4.44×10^{-16}	4.44×10^{-16}
13	3.21×10^{-10}	4.44×10^{-16}	4.44×10^{-16}
14	2.52×10^{-09}	4.44×10^{-16}	4.44×10^{-16}

6. Conclusions

According to the given numerical results, it is concluded that the presented conditional least squares and modified least squares methods, beside maintaining accuracy and stability, have low computational cost and this is a valuable advantage for the new methods. It should be mentioned that the parameter s introduced in modified chain least squares is chosen experimentally in such a way that by increasing the ill-posedness of the problem, this parameter is chosen close to 1. Of course, the presented method of section 4 can be expressed in other formats that will be discussed more in the next articles.

Table 11. Maximum absolute errors of (OLSM), (CLSM), (MCLSM) for Example 6

n	OLSM	CLSM	MCLSM
3	2.53×10^{-03}	2.53×10^{-03}	----------------
4	1.16×10^{-04}	1.16×10^{-04}	----------------
5	9.75×10^{-06}	9.75×10^{-06}	4.87×10^{-06}
6	2.12×10^{-07}	2.12×10^{-07}	1.06×10^{-07}
7	1.42×10^{-08}	1.42×10^{-08}	7.12×10^{-09}
8	1.99×10^{-10}	1.99×10^{-10}	9.96×10^{-11}
9	1.08×10^{-11}	1.08×10^{-11}	5.42×10^{-12}
10	2.61×10^{-12}	1.02×10^{-13}	5.10×10^{-14}
11	1.38×10^{-11}	5.55×10^{-15}	2.77×10^{-15}
12	1.11×10^{-10}	2.22×10^{-16}	1.11×10^{-16}
13	3.28×10^{-10}	2.22×10^{-16}	1.11×10^{-16}
14	2.45×10^{-10}	2.22×10^{-16}	1.11×10^{-16}

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