IJST (2015) 39A1: 59-67
Iranian Journal of Science & Technology

http://ijsts.shirazu.ac.ir

Interval Type-2 Takagi-Sugeno fuzzy modeling of the chaotic systems based on variations in initial conditions

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Abstract

Chaotic systems are nonlinear dynamic systems, the main feature of which is high sensitivity to initial conditions. To initiate a design process in fuzzy model, chaotic systems must first be represented by T-S fuzzy models. In this paper, a new fuzzy modeling method based on sector nonlinearity approach has been recommended for chaotic systems relating to initial condition variations using the interval type-2 Takagi–Sugeno (IT2 T-S) fuzzy model. Examining many famous chaotic systems, it can be seen that nonlinear terms in chaotic systems are composed of just one variable or more. In the process of constructing an IT2 T-S fuzzy model which represents the chaotic systems, authors will focus on nonlinear terms of the chaotic systems. The proposed interval type-2 Takagi-Sugeno fuzzy modeling method is employed for two kinds of nonlinear terms; at first, a uni-variable nonlinear term is presented and then a multi-variable one will be introduced. So, it will be shown how many famous chaotic systems are represented by IT2 T-S fuzzy model. Then the proposed approach is applied to Genesio-Tesi and Rossler systems. Numerical simulations are given to demonstrate the efficiency of the proposed method in MATLAB environment.

Keywords*:* Chaotic modeling; footprint of uncertainty; interval type-2 fuzzy system; lower and upper membership functions; sector nonlinearity

1. Introduction

Examining chaotic processes and their applications, we can see that chaotic modeling is a very attractive topic because they are naturally observed in many physical, chemical, electrical, and mechanical systems. However chaotic modeling encounters the serious challenge of analyzing complicated dynamics. A chaotic system is a highly complicated dynamic nonlinear system and its response exhibits an excessive sensitivity to the initial conditions. The sensitive matter of chaotic systems is generally called *butterfly effect*. Chaotic phenomena and its theories have been applied to many fields of science such as physical systems (Vincent et al., 2006;Wu et al., 2012; Li et al., 2013), ecological systems (Wackernagel and Rees, 2013; Yu et al., 2011), chemical process (Srivastava et al., 2013), secure communications (Yang, 2004; Hamiche et al., 2013; Yang and Zhu 2014), etc.

Chaotic theory has received increasing consideration over the past decades, but general modeling tools have always encountered considerable analytical and numerical difficulties in observing the nature of chaotic systems. So, as we know, the fuzzy systems are appropriate for processes in which qualitative criteria are considered. Moreover, fuzzy systems are appropriate methods for modeling complex nonlinear systems such as chaotic systems. Because of mathematical analysis simplicity, the T–S fuzzy systems as a powerful tool for modeling and designing fuzzy controller are preferred among several kinds of fuzzy system methods (Takagi and Sugeno 1985; Lian et al., 2001; Zhang et al., 2007). Science the membership functions of type-1 fuzzy sets are varied because of uncertain information, the control problem consists of limited ability. Zhang et al., (2005), Dong (2007), Pourkargar and Shahrokhi (2011), Lam and Leung (2006), Lian et al. (2002), etc. have used the type-1 fuzzy model for modeling chaotic systems. Some of these papers are based on fuzzy modeling for just one type of chaotic systems, for example Pourkargar and Shahrokhi (2011) considered fuzzy modeling for Lorenz system. As type-1 fuzzy systems (T1 FSs) are certain (because the fact that its membership grades are crisp values), there are restrictions in T1 FSs for modeling uncertain systems (Wu and Wan 2006;

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Received: 21 February 2014 / Accepted: 8 October 2014

Hagras 2004; Hagras 2007; Tan and Chua 2007; Coupland and John 2007).

In recent years, type-2 fuzzy logic systems have received much attention as a powerful tool for nonlinear control. Jammeh (2004), Mendel et al. (2006), Roopaei (2011), and Leal-Ramírez (2011) have proposed a type-2 fuzzy logic system (FLS) to deal with uncertain grades of membership using type-2 fuzzy sets. Melin et al (2013) have designed optimal controllers for autonomous mobile robots. Castillo and Melin (2012) have reviewed the design and optimization of IT2 fuzzy controllers. Denisse et al. (2009) have combined type-2 fuzzy system and optimization methods. The superiority of IT2 fuzzy sets over type-1 in dealing with uncertain grades of membership have been shown in various applications (Wu, 2006; Castillo et al., 2007; Du et al., 2012). Many reported results for various applications have shown that IT2 FLSs are better than type-1 fuzzy subject with ability to handle uncertainties (Wu 2012; Castillo and Melin 2008).

In this paper, to deal with some problems still existing in control of chaotic systems via type-1 T-S fuzzy models, we propose an IT2 T-S fuzzy model based on sector nonlinearity to represent many well-known chaotic systems relating to variability in initial conditions covered by the lower and upper membership functions of the interval type-2 fuzzy sets. This model can be used to control and synchronize chaotic systems using type-2 fuzzy system. The features of the proposed method can be explained as follow:

Motivated by potential applications in modeling chaotic phenomena such as chaos synchronization, communication, physical theory and system, control of chaotic dynamic have received increased interest. Since chaotic systems are inherently complicated systems, fuzzy modeling provides an appropriate representation for such systems. Among several types of fuzzy methods, we use T-S fuzzy model because of the simplicity of the mathematical analysis and the fact that it provides an exact representation. Generally, the type-2 FLS can be considered as an infinite set of type-1 FLSs. So, extra information, including initial condition variations, can be captured by the type-2 FLS. For chaotic systems that evolve within a bounded region of the state space and initial condition variations, the type-2 T–S fuzzy model can represent the nonlinear dynamics by lower and upper membership functions of interval type-2 fuzzy sets. Also, because of computational burden of defuzzification and type reduction for general type-2 FLS, we employ the IT2 sets to decrease computational complexity. So, IT2 T-S modeling is used because type-2 fuzzy systems are able to model structured uncertainties such as perturbations in initial conditions such as FOU in fuzzy rules by choosing an appropriate membership function. In the procedure of constructing an IT2 T-S fuzzy model which represents the chaotic system, we focus on nonlinear terms of the chaotic system. The proposed IT2 T-S fuzzy modeling method is employed for two kinds of nonlinear terms; at first, a uni-variable nonlinear term is presented and then a multi-variable one will be introduced. In this paper a fuzzy modeling method is introduced for chaotic systems with several initial conditions using the IT-2 T-S fuzzy model. So, the main advantage of proposed method is simplicity in mathematical computations and parametric uncertainties modeling.

The rest of this paper is organized as follows. In section 2, the IT2 T–S fuzzy model for different famous chaotic systems will be presented. Numerical examples are given in section 3. Finally, Conclusions are presented in section 4.

2. Fuzzy modeling of chaotic systems using IT2 T-S

Chaotic systems are nonlinear dynamical systems the main feature of which is high sensitivity to initial conditions. For this purpose, we consider variability in initial conditions to model some classical chaotic systems based on sector nonlinearity approach (Ohtake et al., 2003) via IT2 T-S fuzzy model. This approach is one of the approaches that makes T-S models for fuzzy control design, as it can obtain an exact representation of a nonlinear system. To initiate a design process based on fuzzy model, chaotic systems must first be represented by T-S fuzzy models. In this section we show IT2 T-S fuzzy modeling of chaotic systems since it seems to be appropriate for chaotic systems modeling with structural variations of system. Examining many famous chaotic systems such as Henon (Hénon 1978), Genesio-Tesi (Genesio and Tesi 1992), Rossler system (Rössler 1976), Lorenz system (Lorenz, 1963), etc, we found that nonlinearities in chaotic systems are composed of just one variable or have the same single variable. This variable can be used as output of chaotic system in many applications such as secure communication. If it is taken as the premise variable of type 2 fuzzy rules and the initial condition is varied in a certain bound then a fuzzy dynamical model with lower and upper membership functions can be obtained to represent chaotic systems. Now, we will illustrate how to represent many famous chaotic systems by the IT2 T–S fuzzy model.

We consider some classical chaotic dynamic systems as follows:

$$\dot{\mathbf{x}} = f(\mathbf{x}(\mathrm{t})) + C \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in R^n$ is the state vector, $f(\mathbf{x}(t))$ is a smooth nonlinear matrix function with appropriate dimension and C is a constant vector. The elements of matrix function $f(\mathbf{x}(t))$ are assumed to be bounded. The scheduling variables are chosen as $Z_\alpha(.) \in [\underline{Z}_\alpha, \overline{Z}_\alpha]$, $\alpha = 1,2,\dots,q$ where Z_α denote the non-constant terms in $f(\mathbf{x}(t))$ and $\underline{Z}_\alpha$ and $\overline{Z}_\alpha$ are the minimum and maximum of Z_α respectively. The following describe the IT2 TS fuzzy model with p rules (Lam and Leung 2011):

Rule i:
If
$Z_1(\mathbf{x}(t))$ *is* $\tilde{M}_1^i$ *AND* ... *AND* $Z_q(\mathbf{x}(t))$ *is* $\tilde{M}_q^i$
Then:

$$\dot{\mathbf{x}} = A_i\mathbf{x}(t) + b_i$$

Here $\tilde{M}_\alpha^i$ is an interval type-2 fuzzy set of rules i subject to the function $Z_\alpha(\mathbf{x}(t))$, $\alpha = 1,2,\dots,q$; $i = 1,2,\dots,p$; q is a positive number; $\mathbf{x}(t) \in R^n$ is the state vector; $A_i \in R^{n\times n}$ and $b_i \in R^{n\times 1}$ are the known system matrices. The firing strength of the i-th rule resides in the following interval sets:

$$\tilde{\omega}_i(\mathbf{x}(t)) = \left[\omega_i^L(\mathbf{x}(t)), \omega_i^U(\mathbf{x}(t))\right], i = 1,2,\dots,p \quad (2)$$

where

$$\omega_i^L(\mathbf{x}(t)) = \prod_{\alpha=1}^{q} \underline{\mu}_{\tilde{M}_\alpha^i}(Z_\alpha(\mathbf{x}(t))) \ge 0 \quad (3)$$

$$\omega_i^U(\mathbf{x}(t)) = \prod_{\alpha=1}^{q} \overline{\mu}_{\tilde{M}_\alpha^i}(Z_\alpha(\mathbf{x}(t))) \ge 0 \quad (4)$$

In which $\omega_i^L(\mathbf{x}(t))$ and $\omega_i^U(\mathbf{x}(t))$ are the lower and upper grades of membership, respectively. The functions $\underline{\mu}_{\tilde{M}_\alpha^i}(Z_\alpha(\mathbf{x}(t)))$ and $\overline{\mu}_{\tilde{M}_\alpha^i}(Z_\alpha(\mathbf{x}(t)))$ are the lower and upper membership functions, respectively. So, the IT2 T-S fuzzy model can be shown as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} \omega_i^L(\mathbf{x}(t))\underline{v}_i(\mathbf{x}(t))(A_i\mathbf{x}(t) + b_i) + \sum_{i=1}^{p} \omega_i^U(\mathbf{x}(t))\overline{v}_i(\mathbf{x}(t))(A_i\mathbf{x}(t) + b_i)$$
$$= \sum_{i=1}^{p} \omega_i(\mathbf{x}(t))(A_i\mathbf{x}(t) + b_i) \quad (5)$$

where

$$\begin{cases} \omega_i(\mathbf{x}(t)) = \omega_i^L(\mathbf{x}(t))\underline{v}_i(\mathbf{x}(t)) + \omega_i^U(\mathbf{x}(t))\overline{v}_i(\mathbf{x}(t)) \ge 0 \\ \forall i, \sum_{i=1}^{p} \omega_i(\mathbf{x}(t)) = 1 \end{cases} \quad (6)$$

Here $\underline{v}_i(\mathbf{x}(t)) \ge 0$ and $\overline{v}_i(\mathbf{x}(t)) \ge 0$ are nonlinear functions and in which $\forall i$, $\underline{v}_i(\mathbf{x}(t)) + \overline{v}_i(\mathbf{x}(t)) = 1$. For simplicity, $\underline{v}_i(\mathbf{x}(t))$ and $\overline{v}_i(\mathbf{x}(t))$ can be define equal to 0.5 (Liang and Mendel 2000).

We assume that the premise variables are independent of input. This assumption is used to avoid an extra computation subject to defuzzification procedure. Additionally, we consider another assumption as follows:

Assumption 1. Considering the boundedness of chaotic systems, it is supposed that the fuzzy set is chosen in the state space as the following set:

$$\Omega = \{\mathbf{x}(t) \in \mathbb{R}^n : \|\mathbf{x}(t)\| \le \beta\} \quad (7)$$

For chaotic systems, the existence of the parameter β is normal.

3. Numerical simulations

In the process of structuring an IT2 T-S fuzzy model which represents the nonlinear system (1), we will focus on nonlinear terms of the nonlinear system. The IT2 T-S fuzzy modeling method is employed for two kinds of nonlinear terms; at first we consider a uni-variable nonlinear term and then a multi-variable one will be introduced.

Case 1. There is just one variable in a nonlinear term (The Genesio-Tesi map)
This system is given by:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -cx_1 - bx_2 - ax_3 + x_1^2 \\ y = x_1 \end{cases} \quad (8)$$

where $a, b, c > 0$ and $ab < c$. Selecting the parameters of system as $(a, b, c) = (1, 2.8, 5)$, the chaotic attractors are shown in Fig. 1.

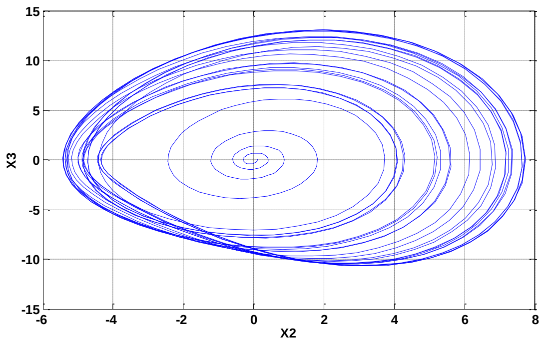

Fig. 1. chaotic attractors of Genesio-Tesi system

This system can be rewritten in the form (1), as:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c + x_1 & -b & -a \end{bmatrix}\mathbf{x} \quad (9)$$

If we consider $f(\mathbf{x}(\mathrm{t})) = x_1^2$ as a single scalar nonlinear function then nonlinear term depends just on one state variable $x_1(t)$. So, the scheduling variable i.e., the non-constant element in the matrix function can be defined as $Z_1(\mathbf{x}(t)) = x_1$.

As mentioned, many researchers have used type-1 TS fuzzy model via the sector nonlinearity technique for chaotic system, but type-1 TS fuzzy sets have limited ability of considering the uncertainty information. Now, we describe type one fuzzy model again and show this inability. When initial conditions are constant, we have, $Z_1(\mathbf{x}(t)) = x_1 \in (x_{1min}, x_{1max})$, i.e., lower and upper bound of $x_1(t)$ are constant. So, the membership functions are defined as follows:

$$\mu_{M_1^1}\left(Z_1(x(t))\right) = \frac{x_{1max} - x_1}{x_{1max} - x_{1min}}$$
$$\mu_{M_1^2}\left(Z_1(x(t))\right) = 1 - \mu_{M_1^1}\left(Z_1(x(t))\right) = \frac{x_1 - x_{1min}}{x_{1max} - x_{1min}}$$

The following type-1 fuzzy rule is used to describe the Genesio-Tesi system:

Rule i:

if $Z_1(\mathbf{x}(t))$ *is* M_1^i

Then: $\dot{\mathbf{x}} = A_i\mathbf{x}(t) \quad ; \quad i = 1,2 \qquad (10)$

where M_1^1 and M_1^2 are type-1 fuzzy set and:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c_1 + x_{1min} & -b & -a \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c_1 + x_{1max} & -b & -a \end{bmatrix} \qquad (11)$$

The type-1 TS fuzzy model can be defined as follows:

$$\dot{\mathbf{x}} = \sum_{i=1}^{2} \hat{\omega}_i(\mathbf{x}(t))(A_i\mathbf{x}(\mathrm{t})) \qquad (12)$$

where the normalized grades of membership are defined as:

$$\hat{\omega}_i(\mathbf{x}(t)) = \frac{\mu_{M_1^i}\left(Z_1(\mathbf{x}(t))\right)}{\mu_{M_1^1}\left(Z_1(\mathbf{x}(t))\right) + \mu_{M_1^2}\left(Z_1(\mathbf{x}(t))\right)}, i = 1,2. \qquad (13)$$

It should be noted that the initial conditions are assumed to be a constant. So, type-1 fuzzy model cannot consider initial conditions as uncertain. Instead, an IT2 T-S fuzzy model is proposed as follows.

By choosing initial condition for $x_1(t)$ in the interval $[0\ 1]$ with step 0.1, the lower and upper bounds of $x_1(t)$ can be obtained from simulation as Table (1). Other initial conditions of state variables are considered as constant. Note that this interval can be chosen wider.

The lower and upper membership functions should satisfy the following inequality:

$$\underline{\mu}_{\tilde{M}_1^i}\left(Z_1(\mathbf{x}(t))\right) \le \mu_{M_1^i}\left(Z_1(\mathbf{x}(t))\right) \le \overline{\mu}_{\tilde{M}_1^i}\left(Z_1(\mathbf{x}(t))\right);$$
$$i = 1,2 \qquad (14)$$

Table 1. Initial conditions and lower and upper bound of x_1 for Genesio-Tesi system

Initial Conditions (x_1, x_2, x_3)	Lower and Upper bound of x_1
$[0.1\ \ 0.1\ \ 0.1]$	$[-2.65\ \ 5.8]$
$[0.2\ \ 0.1\ \ 0.1]$	$[-2.67\ \ 5.84]$
$[0.3\ \ 0.1\ \ 0.1]$	$[-2.66\ \ 5.8]$
$[0.4\ \ 0.1\ \ 0.1]$	$[-2.65\ \ 5.8]$
$[0.5\ \ 0.1\ \ 0.1]$	$[-2.67\ \ 5.83]$
$[0.6\ \ 0.1\ \ 0.1]$	$[-2.65\ \ 5.79]$
$[0.7\ \ 0.1\ \ 0.1]$	$[-2.66\ \ 5.78]$
$[0.8\ \ 0.1\ \ 0.1]$	$[-2.66\ \ 5.8]$
$[0.9\ \ 0.1\ \ 0.1]$	$[-2.65\ \ 5.8]$
$[1\ \ 0.1\ \ 0.1]$	$[-2.65\ \ 5.8]$

It can be deduced from eq. (14) that the lower and upper membership functions make the footprint of uncertainty (FOU) region that captures the uncertainties in initial conditions. Any type-1 fuzzy can be reconstructed based on the FOU. So, the IT2 T-S fuzzy model consists of infinite number of type-1 fuzzy models. By selecting values as assumption 1 and substituting numerical values, the lower and upper bounds of $\mu_{\tilde{M}_1^i}(Z_1(\mathbf{x}(t)))$ can be determined and the lower and upper membership functions are depicted in Table 2 based on sector nonlinearity.

Table 2. Lower and upper membership functions of Genesio-Tesi system

Lower membership function	Upper membership function
$\underline{\mu}_{\tilde{M}_1^1}\left(Z_1(\mathbf{x}(t))\right) = \frac{min\{Z_{1max}\} - Z_1(\mathbf{x}(t))}{max\{Z_{1max}\} - min\{Z_{1min}\}} = \frac{5.78 - x_1}{5.84 + 2.67}$	$\overline{\mu}_{\tilde{M}_1^1}\left(Z_1(\mathbf{x}(t))\right) = \frac{max\{Z_{1max}\} - Z_1(\mathbf{x}(t))}{min\{Z_{1max}\} - max\{Z_{1min}\}} = \frac{5.84 - x_1}{5.78 + 2.65}$
$\underline{\mu}_{\tilde{M}_1^2}\left(Z_1(\mathbf{x}(t))\right) = \frac{Z_1(\mathbf{x}(t)) - max\{Z_{1min}\}}{max\{Z_{1max}\} - min\{Z_{1min}\}} = \frac{x_1 + 2.65}{5.84 + 2.67}$	$\overline{\mu}_{\tilde{M}_1^2}\left(Z_1(\mathbf{x}(t))\right) = \frac{Z_1(\mathbf{x}(t)) - min\{Z_{1min}\}}{min\{Z_{1max}\} - max\{Z_{1min}\}} = \frac{x_1 + 2.67}{5.78 + 2.65}$

Figures 2 and 3 show the lower and upper membership functions and FOU of the IT2 T-S fuzzy model for Genesio-Tesi system.

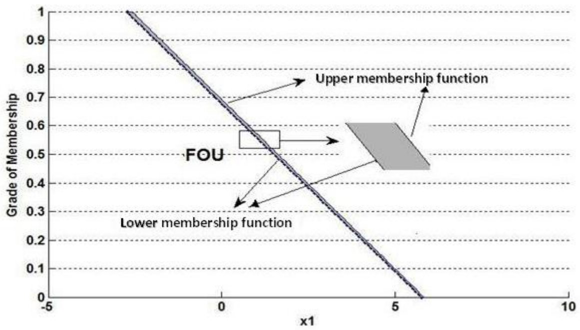

Fig. 2. Plot of $\mu_{\widetilde{M}_1^1}\left(Z_1(\mathrm{x}(t))\right)$, Lower membership function $\underline{\mu}_{\widetilde{M}_1^1}\left(Z_1(\mathrm{x}(t))\right)$, Upper membership function $\overline{\mu}_{\widetilde{M}_1^1}\left(Z_1(\mathrm{x}(t))\right)$ and footprint of uncertainty (grey area) for Genesio-Tesi system

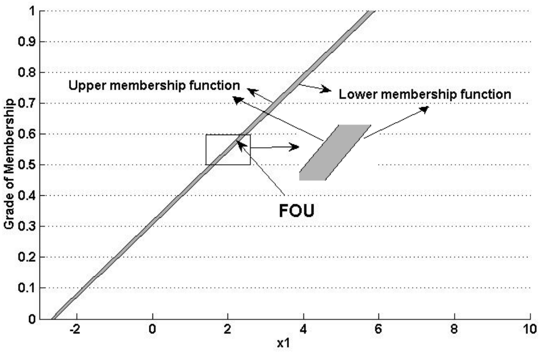

Fig. 3. Plot of $\mu_{\widetilde{M}_1^2}\left(Z_1(\mathrm{x}(t))\right)$, Lower membership function $\underline{\mu}_{\widetilde{M}_1^2}\left(Z_1(\mathrm{x}(t))\right)$, Upper membership function $\overline{\mu}_{\widetilde{M}_1^2}\left(Z_1(\mathrm{x}(t))\right)$ and footprint of uncertainty (grey area) for Genesio-Tesi system

Then an IT2 T-S fuzzy model with 2 rules of the following format is used to describe the Genesio-Tesi system:

Rule i:
if $Z_1(\mathbf{x}(t))$ *is* $\widetilde{M}_1^i$
Then: $\dot{\mathbf{x}} = A_i\mathbf{x}(t)$ *for* $i = 1,2$

So, IT2 T-S fuzzy model can be defined as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{2}\left(\omega_i^L(\mathbf{x}(t))\underline{v}_i(\mathbf{x}(t)) + \omega_i^U(\mathbf{x}(t))\overline{v}_i(\mathbf{x}(t))\right)\left(A_i\mathbf{x}(t)\right) = \sum_{i=1}^{2}\widetilde{\omega}_i(\mathbf{x}(t))\left(A_i\mathbf{x}(t)\right) \qquad (15)$$

where

$$\widetilde{\omega}_i(\mathbf{x}(t)) = \omega_i^L(\mathbf{x}(t))\underline{v}_i(\mathbf{x}(t)) + \omega_i^U(\mathbf{x}(t))\overline{v}_i(\mathbf{x}(t)) \qquad (16)$$

Based on this IT2 T-S fuzzy model, considering variability in initial conditions, an IT2 fuzzy controller with two rules can be employed to chaos synchronization and other applications. Figure 4 shows the difference between fuzzy model and original system with any value in the interval set of initial condition.

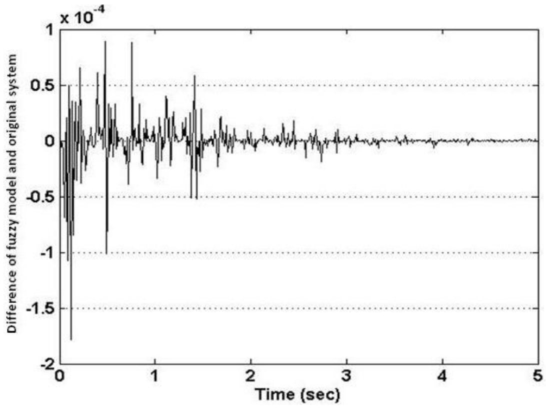

Fig. 4. Modeling error for Genesio-Tesi system

Based on defined FOU in Figs. 2 and 3, the IT2 T-S fuzzy model can be considered as a collection of type-1 T-S fuzzy models. In other words, a type-1 fuzzy model can be defined for every initial condition within the mentioned interval. For example, we have simulated fuzzy model of Genesio-Tesi system in the form of eq. (8) for one value of initial condition. As it is evident in Fig. 4, the difference between fuzzy model and original system becomes less than 0.5×10^{-4}after about 1.5 sec., i.e. the fuzzy model can represent the original system. It should be noted that the IT2 TS fuzzy model serves as a mathematical tool to facilitate the design of the IT2 fuzzy controller. In this system, based on IT2 T-S fuzzy model, an IT2 fuzzy controller with two rules can be employed for chaos synchronization and other applications.

Case 2. There is multi-variable in a nonlinear term (The Rossler system)

Consider the following Rossler system:

$$\begin{cases}\dot{x}_1 = -x_2 - x_3\\ \dot{x}_2 = x_1 + 0.2x_2\\ \dot{x}_3 = 0.2 + x_3(x_1 - 5)\\ y = x_1\end{cases} \qquad (17)$$

This system can be rewritten in the form (1), as:

$$\dot{\mathbf{x}} = \begin{bmatrix}0 & -1 & -1\\ 1 & 0.2 & 0\\ 0 & 0 & x_1 - 5\end{bmatrix}\mathbf{x} + \begin{bmatrix}0\\ 0\\ 0.2\end{bmatrix} \qquad (18)$$

The chaotic attractor of the fuzzy Rossler system is shown in Fig. 5.

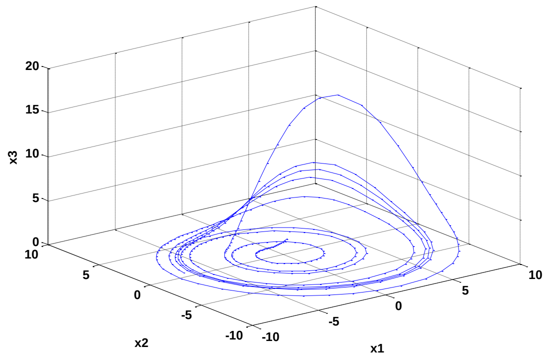

Fig. 5. The chaotic attractor of the fuzzy Rossler system

If we consider $f(\mathbf{x}(\mathrm{t})) = x_1x_3$ as a single scalar nonlinear function then nonlinear term depends on multi state variable $x_1(t)$ and $x_3(t)$. Here, we choose $Z_1(\mathbf{x}(t)) = x_1$ as premise variable of fuzzy rules, which satisfies $x_1 \in (x_{1min}, x_{1max})$. The scheduling variable in the matrix function is $Z_1(\mathbf{x}(t)) = x_1$.

When initial conditions are constant, then $Z_1(\mathbf{x}(t)) = x_1 \in (x_{1min}, x_{1max})$, i.e., lower and upper bound of $x_1(t)$ are constant. So, the membership functions are defined as follows:

$$\mu_{M_1^1}\left(Z_1(x(t))\right) = \frac{x_{1max} - x_1}{x_{1max} - x_{1min}}$$
$$\mu_{M_1^2}\left(Z_1(x(t))\right) = 1 - \mu_{M_1^1}\left(Z_1(x(t))\right) = \frac{x_1 - x_{1min}}{x_{1max} - x_{1min}} \quad (19)$$

The following type-1 fuzzy rule is used to describe the Rossler system:

Rule i:

if $Z_1(\mathbf{x}(t))$ *is* M_1^i

Then: $\dot{\mathbf{x}} = A_i\mathbf{x}(t) + b_i \quad ; \quad i = 1,2 \quad (20)$

where M_1^1 and M_1^2 are type-1 fuzzy set; with

$$A_1 = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 0 & 0 & x_{1min} - 5 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 0 & 0 & x_{1max} - 5 \end{bmatrix} b_1 = b_2 = \begin{bmatrix} 0 \\ 0 \\ 0.2 \end{bmatrix} \quad (21)$$

So, the type-1 TS fuzzy model can be defined as follows:

$$\dot{\mathbf{x}} = \textstyle\sum_{i=1}^{2} \hat{\omega}_i(\mathbf{x}(t))(A_i\mathbf{x}(\mathrm{t}) + b_i) \quad (22)$$

where the normalized grades of membership are defined as:

$$\hat{\omega}_i(\mathbf{x}(t)) = \frac{\mu_{M_1^i}\left(Z_1(\mathbf{x}(t))\right)}{\mu_{M_1^1}\left(Z_1(\mathbf{x}(t))\right) + \mu_{M_1^2}\left(Z_1(\mathbf{x}(t))\right)}, i = 1,2. \quad (23)$$

Here we assumed that the initial conditions are constant. By varying the initial condition for $x_1(t)$ in the interval [0 1] with step 0.1, the lower and upper bound of $x_1(t)$ can be obtained from simulation as in Table 3 and an IT2 T-S fuzzy model is proposed as follows. It should be noted that other initial conditions of state variables are considered as constant.

Table 3. Initial conditions and lower and upper bound of x_1 for Rossler system

Initial Conditions (x_1, x_2, x_3)	Lower and Upper bound of x_1
[0.1 1 1]	[−7.89 9.97]
[0.2 1 1]	[−7.73 9.84]
[0.3 1 1]	[−7.47 9.54]
[0.4 1 1]	[−8 10.09]
[0.5 1 1]	[−7.7 9.78]
[0.6 1 1]	[−7.85 9.94]
[0.7 1 1]	[−7.94 10.05]
[0.8 1 1]	[−8 10.09]
[0.9 1 1]	[−7.93 10.02]
[1 1 1]	[−7.75 9.83]

The lower and upper membership functions should satisfy the inequality (14). By considering values as assumption 1 and substituting numerical values, the lower and upper membership functions are defined based on sector nonlinearity in Table 4.

Table 4. Lower and upper membership functions of Rossler system

Lower membership function	Upper membership function
$\underline{\mu}_{\tilde{M}_1^1}\left(Z_1(\mathbf{x}(t))\right) = \frac{min\{Z_{1max}\} - Z_1(\mathbf{x}(t))}{max\{Z_{1max}\} - min\{Z_{1min}\}} = \frac{9.54 - x_1}{10.09 + 8}$	$\overline{\mu}_{\tilde{M}_1^1}\left(Z_1(\mathbf{x}(t))\right) = \frac{max\{Z_{1max}\} - Z_1(\mathbf{x}(t))}{min\{Z_{1max}\} - max\{Z_{1min}\}} = \frac{10.09 - x_1}{9.54 + 7.47}$
$\underline{\mu}_{\tilde{M}_1^2}\left(Z_1(\mathbf{x}(t))\right) = \frac{Z_1(\mathbf{x}(t)) - max\{Z_{1min}\}}{max\{Z_{1max}\} - min\{Z_{1min}\}} = \frac{x_1 + 7.47}{10.09 + 8}$	$\overline{\mu}_{\tilde{M}_1^2}\left(Z_1(\mathbf{x}(t))\right) = \frac{Z_1(\mathbf{x}(t)) - min\{Z_{1min}\}}{min\{Z_{1max}\} - max\{Z_{1min}\}} = \frac{x_1 + 8}{9.54 + 7.47}$

Figures 6 and Fig. 7 depict the lower and upper membership functions of the IT2 T-S fuzzy model.

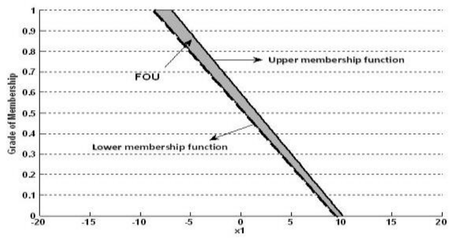

Fig. 6. Plot of $\mu_{\tilde{M}_1^1}\left(Z_1(\mathrm{x}(t))\right)$, Lower membership function $\underline{\mu}_{\tilde{M}_1^1}\left(Z_1(\mathrm{x}(t))\right)$ (dashed line), Upper membership function $\overline{\mu}_{\tilde{M}_1^1}\left(Z_1(\mathrm{x}(t))\right)$ (filled line) and footprint of uncertainty (grey area) for Rossler system

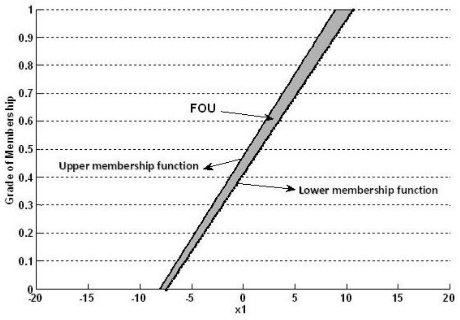

Fig. 7. Plot of $\mu_{\widetilde{M}_1^2}\left(Z_1\left(\mathrm{x}(t)\right)\right)$, Lower membership function $\underline{\mu}_{\widetilde{M}_1^2}\left(Z_1\left(\mathrm{x}(t)\right)\right)$ (dashed line), Upper membership function $\overline{\mu}_{\widetilde{M}_1^2}\left(Z_1\left(\mathrm{x}(t)\right)\right)$ (filled line) and footprint of uncertainty (grey area) for Rossler system

Then an IT2 T-S fuzzy model with 2 rules of the following format is used to describe the Rossler system:

Rule i:

if $Z_1(\mathbf{x}(t))$ *is* $\widetilde{M}_1^i$

Then: $\dot{\mathbf{x}} = A_i\mathbf{x}(t)$ *for* $i = 1,2$

The IT2 T-S fuzzy model can be defined as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{2}\left(\omega_i^L(\mathbf{x}(t))\underline{v}_i(\mathbf{x}(t)) + \omega_i^U(\mathbf{x}(t))\overline{v}_i(\mathbf{x}(t))\right)\left(A_i\mathbf{x}(t)\right) = \sum_{i=1}^{2}\widetilde{\omega}_i(\mathbf{x}(t))\left(A_i\mathbf{x}(t)\right) \quad (24)$$

where

$$\widetilde{\omega}_i(\mathbf{x}(t)) = \omega_i^L(\mathbf{x}(t))\underline{v}_i(\mathbf{x}(t)) + \omega_i^U(\mathbf{x}(t))\overline{v}_i(\mathbf{x}(t)) \quad (25)$$

Based on this IT2 T-S fuzzy model, an IT2 fuzzy controller with the above two rules can be employed for chaos synchronization and other applications. Figure 8 shows the difference of fuzzy model and original system with any value in the interval set of initial condition.

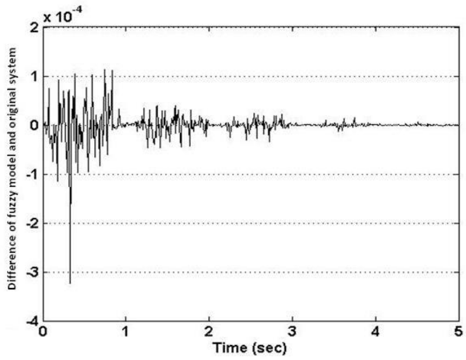

Fig. 8. Modeling error for Rossler system

When membership functions are determined or tuned based on numerical data, the uncertainties in the numerical data translates into uncertainties in the membership functions as FOU which has depicted in Figs. 6 and 7. In these Figs., infinite number of type-1 fuzzy model can be defined based on available FOU. So, the proposed IT2 T-S fuzzy model is a collection of type-1 T-S fuzzy models. For example, we simulated the difference between fuzzy model and original system of Rossler system for one value of initial condition. As it is evident in Fig. 8, the modeling error or difference between fuzzy model and original system becomes less than 1×10^{-4} after about 1 second which means that fuzzy model tends toward the original system.

The main advantage of this method of modeling is simplicity. In this method, if there are the lower and upper bounds of scheduling variable as an interval, an IT2 T-S fuzzy model can be defined. The lower and upper bounds of scheduling variable can be determined in chaos systems. So, this method can be extended for every chaotic and hyper chaotic system with any uncertain parameter and variation in initial conditions.

As it is evident from the Figs., the fuzzy models can be used instead of original systems for several chaotic applications. However, there is no proof for robustness of proposed method; we have just surveyed the effect of variability in initial conditions in modeling. For more information about robustness and future works, the research work by Jianbin et al. (2010) and Zhang et al. (2012) can be cited to improve this work.

4. Conclusion

To realize a design procedure based on fuzzy model, chaotic systems must first be represented by T-S fuzzy models. Examining many famous chaotic systems, we saw that nonlinear terms in chaotic systems are composed of just one variable or more and we focused on nonlinear terms of the chaotic systems. In this paper, we introduced a new fuzzy modeling method based on sector nonlinearity approach for chaotic systems based on variations in initial condition using the interval type-2 Takagi–Sugeno (IT2 T-S) fuzzy model. This model is covered by the lower and upper membership functions of the IT2 fuzzy sets. The proposed method was employed for the Genesio-Tesi and Rossler systems mentioned above. Result of numerical simulations on the famous Genesio-Tesi and Rossler system showed the validity of the approach. This modeling can be extended to a wider variety of chaotic systems and hyperchaotic systems. The main advantage of the proposed method is simplicity in mathematical computations and modeling.

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