REAL GROUP ALGEBRAS*

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Abstract – In this paper we initiate the study of real group algebras and investigate some of its aspects. Let \( L^p(G, \tau) \) be a group algebra of a locally compact group \( G \), \( \tau : G \rightarrow G \) be a group homeomorphism such that \( \tau^2 = \tau \circ \tau = 1 \), the identity map, and \( L^p(G, \tau) = \{ f \in L^p(G) : f \circ \tau = f \} \) \( (p \geq 1) \). In this paper, among other results, we clarify the structure of \( L^p(G, \tau) \) and identify its multipliers.

Keywords – Real Banach algebra, amenability, multiplier, derivation, group involution

1. INTRODUCTION

In 1965, Ingelstam [1] introduced the theory of real Banach algebras. The real function algebra theory was developed further by Kulkarni and Limaye [2]. In their excellent monograph, “Real function algebras”, Kulkarni and Limaye present interesting aspects of the theory of \( C(X, \tau) \). We refer to [3] for our notations.

Let \( G \) be a locally compact group. An automorphism \( \tau : G \rightarrow G \) is called a topological group involution on \( G \) if \( \tau \) is a homeomorphism and \( \tau(x) = x^{-1} \) for all \( x \in G \). For example, in group \( (\mathbb{C}, +) \), \( \tau(z) = \overline{z} \) and in \( (\mathbb{R} \setminus \{0\}, \cdot) \), \( \tau(x) = x^{-1} \) are topological group involutions. Note that we do not assume that \( \tau(xy) = \tau(y) \tau(x) \).

Let \( C_\circ(G, \tau) = \{ f \in C_\circ(G) : f \circ \tau(x) = f(x), x \in G \} \), and \( C_\circ(G, \tau) = \{ f \in C_\circ(G) : f \circ \tau(x) = f(x), x \in G \} \) it is clear that, if \( \tau \) is the identity map on \( G \), then \( C_\circ(G, \tau) = C_\circ'(G) \), \( C_\circ(G, \tau) = C_\circ'(G) \). If \( 1 \leq p \leq \infty \), we define \( f \circ \tau(x) = \overline{f(x)} \), for all \( x \in G \).

Clearly, \( L^p(G, \tau) \subseteq L^p(G) \) and if \( \tau \) is the identity map, \( L^p(G, \tau) \) consists of real functions.

2. THE STRUCTURE OF \( L^1(G, \tau) \) AND \( M(G, \tau) \)

Lemma 2. 1. Let \( G \) be a locally compact group and \( \tau \) be a topological group involution on \( G \). If \( \sigma : C_\circ(G) \rightarrow C_\circ(G) \) is defined by \( \sigma(f) = \overline{f \circ \tau} \), then (i) \( \sigma \) is an algebra involution on \( C_\circ(G) \) and \( C_\circ(G, \tau) = \{ f \in C_\circ(G) : \sigma(f) = f \} \).

(ii) \( C_\circ(G) = C_\circ(G, \tau) \oplus iC_\circ(G, \tau) \).

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Proof. (i) We must show that whenever \( f \in C_c(G) \), then \( \tilde{f} \circ \tau \in C_c(G) \). To do this, we have

\[
\text{supp}(\tilde{f} \circ \tau) = \text{cl}([\text{supp}(\tilde{f} \circ \tau)^{-1} \{0\}]' \subseteq \tau^{-1}(\text{supp} \tilde{f})).
\]

It follows that \( \text{supp}(\tilde{f} \circ \tau) \) is compact, i.e., \( (\tilde{f} \circ \tau) \in C_c(G) \). Hence, \( \text{supp}(\tilde{f} \circ \tau) \) is compact, i.e., \( \tilde{f} \circ \tau \in C_c(G) \). The rest of (i) is clear.

(ii) Clearly, \( f = \frac{f + \sigma(f)}{2} + i \left( \frac{f - \sigma(f)}{2} \right) \). Since \( \sigma^2 = i \), \( (\sigma(f) + \sigma(f)) = \frac{f + \sigma(f)}{2} \) and \( \sigma(f - \sigma(f)) = \frac{f - \sigma(f)}{2i} \). It follows that \( f = g + ih \) where \( g, h \in C_c(G, \tau) \).

Note. By the same argument one can conclude that \( C_0(G) = C_0(G, \tau) \oplus iC_0(G, \tau) \). In fact it is enough to show that \( \tilde{f} \circ \tau \in C_c(G) \) whenever \( f \in C_c(G) \). Since \( f \in C_c(G) \), for a given \( \varepsilon > 0 \), there is a compact set \( F \) in \( G \) such that \( |f(x)| < \varepsilon \) whenever \( x \in F' \). Clearly, \( \tau^{-1}(F) \) is compact, and if \( x \in \tau^{-1}(F) \), then \( \tau(x) \notin F \) i.e., \( |\tilde{f} \circ \tau(x)| < \varepsilon \). Therefore, \( \tilde{f} \circ \tau \in C_c(G) \).

Let \( M(G) \) be the Banach space of all complex regular Borel measures on \( G \). For each \( \mu \in M(G) \), we define \( \mu \circ \tau = \mu \circ \tau \), then it is clear that \( \mu \circ \tau \in M(G) \). Also by Lebesgue dominated convergence theorem one can show that for every bounded Borel measurable function \( h \) on \( G \),

\[
\int_G hd\mu = \int_G (h \circ \tau) d\mu.
\]

Proposition 2.2. Let \( M(G, \tau) = \{ \mu \in M(G) \mid \mu \circ \tau = \mu \} \). Then \( M(G, \tau) \) is a real Banach algebra with the convolution product \( \mu \ast \nu(E) = \int_G \nu(x^{-1} E) d\mu(x) = \int_G \mu(E y^{-1}) d\nu(y) \) \( (\mu, \nu \in M(G, \tau)) \) and \( M(G) = M(G, \tau) \oplus iM(G, \tau) \).

Proof. Let \( \mu, \nu \in M(G, \tau) \). Then

\[
(\mu \ast \nu) \circ \tau(E) = \int_G \nu(x^{-1} \tau(E)) d\mu(x) = \int_G \nu(\tau(\tau(x)^{-1} E)) d\mu(x).
\]

\[
= \int_G \nu((\tau(x)^{-1})) E) d\mu(x) = \int_G \nu(x^{-1} E) d\mu \circ \tau
\]

\[
= \mu \ast \nu(E)
\]

Therefore \( \mu \ast \nu \in M(G, \tau) \). The rest of the proof follows the same line as the proof of Lemma 2.1. Therefore, it is omitted.

Remark. For a real linear space \( A \), the real dual space of \( A \), that is, the space of all real-valued continuous linear functional on \( A \) will be denoted by \( A' \).

Proposition 2.3. Every real-valued continuous functional \( \phi \) on \( C_0(G, \tau) \) can be represented as \( \phi(f) = \int_G f d\mu \), where \( \mu \) is the unique measure in \( M(G, \tau) \) such that \( ||\psi|| = ||\mu|| \) and vice versa.

Proof. Let \( f \in C_0(G, \tau) \). Then \( f = g + ih \) where \( g, h \in C_0(G, \tau) \). If we define \( \psi(f) = \phi(g) + i\phi(h) \), then clearly \( \psi \in C_0(G)' \) and so by the Riesz representation theorem ([3,
There exists a unique measure \( \mu \) in \( M(G) \) such that

\[
\psi(f) = \int_G fd\mu(f \in C_0(G)) \quad \text{and} \quad \| \psi \| = \| \mu \|.
\]

It follows that \( \phi(h) = \int_C h d\mu \) for every \( h \) in \( C_0(G, \tau) \). Now, in order to prove that \( \mu \in M(G, \tau) \), we have

\[
\overline{\psi}(\sigma(f)) = \overline{\psi}(g - ih) = \overline{\phi(g) - i\phi(h)} = \psi(f).
\]

Therefore,

\[
\int_G fd\psi = \int_G \sigma(f)d\mu = \int_G \sigma(f)d\overline{\mu} = \int_G fd\mu_0 \tau
\]

(\( f \in C_0(G) \)). Thus, \( \mu = \overline{\rho_0} \tau \), i.e. \( \mu \in M(G, \tau) \). Also, similar to the proof of [6, Theorem 3.2.1] we can show that \( \| \psi \| = \| \mu \| \).

Conversely, let \( \mu \in M(G, \tau) \) and \( \phi(f) = \int_G fd\mu(f \in C_0(G, \tau)) \). If \( f \in C_0(G, \tau) \), then \( \sigma(f) = f \). Hence,

\[
\overline{\phi}(f) = \overline{\phi}(\sigma(f)) = \int_G \sigma(f)d\mu = \int_G \sigma(f)d\overline{\mu} = \int_G (f \circ \tau)d\overline{\mu}
\]

\[
= \int_G fd\overline{\mu_0} \tau = \int_G fd\mu = \phi(f).
\]

Thus \( \phi(f) \) is real.

**Theorem 2.4.** Let \( G \) be a locally compact group with the left Haar measure \( \lambda \) and \( \tau \) be a topological group involution on \( G \). Then \( \lambda \circ \tau = \lambda \).

**Proof.** It is easy to show that \( \lambda \circ \tau \) is a positive measure on \( G \). Also if \( B \) is a Borel set, then \( \lambda \circ \tau(xB) = \lambda(\tau(xB)) = \lambda(\tau(x)\tau(B)) = \lambda(\tau(B))(x \in G) \). Therefore, \( \lambda \circ \tau \) is left invariant. So, there is a positive number \( c \) such that \( \lambda \circ \tau(B) = c\lambda(B) \) for every Borel set \( B \). If \( U \) is an open set, then \( \lambda \circ \tau(\tau(U)) = c\lambda(U) \), i.e., \( \lambda(U) = c\lambda(\tau(U)) \) which is equal to \( c^2\lambda(U) \). Therefore, for every open set \( U \) we have \( \lambda(U) = c^2\lambda(U) \). So, \( c = 1 \). Hence, \( \lambda \circ \tau = \lambda \).

For a locally compact group \( G \) and the Haar measure \( \lambda \) we defined

\[
L^p(G) = \{ f \in L^p(G) \mid f \circ \tau = f \} (1 \leq p \leq \infty).
\]

Clearly \( L^p(G) \subseteq L^p(G, \tau) \) is a real algebra and \( L^p(G, \tau) = L^p(G, \tau) \cap L^p(G) \).

**Theorem 2.5.** (a) For \( 1 \leq p \leq \infty \), \( L^p(G, \tau) \) is a real Banach space, and \( L^2(G, \tau) \) is a real Hilbert space with an inner product,

\[
< f, g > = \int_G \overline{f}d\lambda.
\]

(b) For each \( f, g \in L^1(G, \tau) \), max \( \{ \| f \|_p, \| g \|_p \} \leq \| f + ig \|_p \leq \| f \|_p + \| g \|_p \).

(c) \( L^1(G, \tau) = L^\infty(G, \tau) \).

(d) \( L^1(G, \tau) \) has a bounded approximate identity of norm 1.

**Proof.** (a). Clearly, \( L^p(G, \tau) \) is a real subspace of \( L^p(G) \). Let \( f, g \in L^p(G, \tau) \) then \( f \ast g \in L^p(G) \), [4]. We will show that \( f \ast g \in L^p(G, \tau) \). In order to do this, by (2.4) and (1) we have

\[
(f \ast g)(\tau(x)) = \int_G f(y)g(y^{-1}\tau(x))d\lambda(y).
\]
for every \( x \in G \), hence \( f \ast g \in L^p(G, \tau) \). We now prove that \( L^p(G, \tau) \) is complete. Let \( \{ f_n \}_{n=1}^\infty \) be a Cauchy sequence in \( L^p(G, \tau) \). Since \( L^p(G) \) is complete, there exists \( f \in L^p(G) \) such that \( \lim_{n \to \infty} \| f_n - f \|_p = 0 \). Now, there exists a subsequence of \( \{ f_n \}_{n=1}^\infty \) as \( \{ f_{n_k} \}_{k=1}^\infty \) such that \( \lim_{k \to \infty} f_{n_k}(x) = f(x) \), \( \lambda \)-almost everywhere, and so \( f(\tau(x)) = \lim_{k \to \infty} f_{n_k}(\tau(x)) = \lim_{k \to \infty} f_{n_k}(x) = f(x) \), \( \lambda \)-almost everywhere. Therefore, \( f \in L^p(G, \tau) \). Hence \( L^p(G, \tau) \) is a real Banach algebra and not a complex algebra.

If \( \langle f, g \rangle := \int_G f \overline{g} d\lambda \) for every \( f, g \in L^1(G, \tau) \), then \( \langle f, g \rangle = \langle f, g \rangle \). Therefore \( L^1(G, \tau) \) is a real Hilbert space.

(b) For \( f, g \in L^1(G, \tau) \) we have \( \| f \|_p \leq \frac{1}{2} (\| f + ig \|_p + \| f - ig \|_p) = \| f + ig \|_p \). Similarly, \( \| g \|_p \leq \| f + ig \|_p \).

(c) We know that \( L^1(G)^* \cong L^\infty(G) \). Let \( f \in L^1(G) \). So \( f = g + ih \) where \( g, h \in L^1(G, \tau) \). Now, we define \( \psi(f) = \phi(f) + i\phi(g) \) where \( \phi \in L^1(G, \tau)^* \). It is clear that \( \psi \in L^1(G)^* \) and therefore, there exists a unique \( p \in L^\infty(G) \) such that \( \psi(f) = \int_G f \, p \, d\lambda(f \in L^1(G)) \).

Hence we have,

\[
\overline{\psi}(\sigma(f)) = \overline{\psi}(g - ih) = \overline{\phi(g) - i\phi(h)} = \psi(f).
\]

This implies that

\[
\int_G f \, p d\lambda = \int_G \sigma(f) \, p d\lambda = \int_G (f \tau) \, p d\lambda = \int_G f \, \bar{p} \, d\alpha = \int_G f \, \bar{p} \, d\lambda \quad (f \in L^1(G)).
\]

Therefore, \( p \circ \tau = \bar{p} \), i.e., \( p \in L^\infty(G, \tau) \). Also, we have \( \phi(f) = \int_G f \, p \, d\lambda \) for every \( f \in L^1(G, \tau) \) and by \( (*) \) \( \phi(f) \) is real. Conversely, if \( \phi : L^1(G, \tau) \to \mathbb{R} \) is defined by \( \phi(f) = \int_G f \) where \( p \in L^\infty(G, \tau) \) and \( f \) is an arbitrary function, then \( \phi \in L^1(G, \tau)^* \) and the proof is complete.

(d) Let \( U \) be any compact neighborhood of \( e \) and \( \{ U_a \} \) be the collection of all compact neighborhoods of \( e \) in \( U \), which is directed by a set inclusion (\( \alpha \leq \beta \) if and only if \( U_\alpha \supseteq U_\beta \)). If we define \( f_a = \frac{\chi_{U_a}}{\lambda(U_a)} \) and \( g_a = \frac{\chi_{U_a} \circ \tau}{\lambda(U_a)} \), then, since \( \tau \) is a homeomorphism, \( \{ f_a \} \) and \( \{ g_a \} \) are bounded approximate identities of norm one for \( L^1(G) \). If we define \( e_a = \frac{f_a + g_a}{2} \), then \( \{ e_a \} \) is a bounded approximate identity of norm one for \( L^1(G) \), and also for \( L^1(G, \tau) \) since \( e_a \in L^1(G, \tau) \).

**Lemma 2.6.** For \( 1 \leq p \leq \infty \), the linear space \( C_c(G, \tau) \) is a dense subspace of \( L^p(G, \tau) \).
Proof. Suppose that \( f \in L^p(G, \tau) \), since \( C_c(G) \) is a dense subspace of \( L^p(G) \), there exists a sequence \( \{ f_n \}_{n=1}^\infty \) in \( C_c(G) \) such that \( \lim_{n \to \infty} \| f_n - f \|_p = 0 \). Let \( g_n = \frac{f_n + f}{2} \). Then \( g_n \in C_c(G, \tau) \) and \( \lim_{n \to \infty} \| g_n - (f + f) \|_p = \lim_{n \to \infty} \| g_n - f \|_p = 0 \).

**Theorem 2.7.** For \( \mu \in M(G, \tau) \) and \( \psi \in L^1(G, \tau) \), let \( T_\mu \psi = \mu * \psi \). Each \( T_\mu \) is a bounded operator on the real Hilbert space \( L^2(G, \tau) \), and the mapping \( \mu \to T_\mu \) is a faithful \( * \)-representation of \( M(G, \tau) \). Note that \( M(G, \tau) \) is a \( * \)-Banach algebra.

Proof. The linearity of \( T_\mu \) on \( L^2(G, \tau) \) is obvious, and the boundedness of \( T_\mu \), with \( \| T_\mu \| \leq \| \mu \| \), follows from [3,(20.12.ii)]. For \( \psi \in L^1(G, \tau) \cap L^2(G, \tau) \), we have
\[
(\mu * \nu) * \psi = \mu * (\nu * \psi)
\]
[2, (19.2.iv)]. Thus \( T_{\mu \nu} \psi = T_\mu (T_\nu \psi) \) for all \( \psi \in L^1(G, \tau) \cap L^2(G, \tau) \). Since \( C_c(G, \tau) \subseteq L^1(G, \tau) \cap L^2(G, \tau) \), by Lemma (2.7), \( L^1(G, \tau) \cap L^2(G, \tau) \) is dense in \( L^2(G, \tau) \). It follows that \( T_{\mu \nu} = T_\mu T_\nu \). To show that \( T_\mu \neq 0 \) if \( \mu \neq 0 \), consider an \( f \in C_c(G, \tau) \) such that \( \int_G f^* d\mu \neq 0 \). Since \( \mu * f(e) = \int_G f^* d\mu \neq 0 \) and \( \mu * f \) is continuous; thus \( T_\mu f \) is not a zero element of \( L^2(G, \tau) \). Note that \( f^* \) is the involution of \( f \).

### 3. AMENABILITY AND WEAK AMENABILITY OF REAL GROUP ALGEBRAS

In this section, we show that amenability of \( L^1(G, \tau) \) and \( L^1(G) \) are equivalent. We shall use some notions of [1].

**Definition 3.1.** A Banach algebra \( A \) over \( F \) is called amenable if for every Banach \( A \)-module \( X \) over \( F \), \( H^1(A, X^*') = \{ 0 \} \).

Let \( A \) be a Banach algebra over \( F \), and \( X \) be a Banach \( A \)-module over \( F \). If \( F = R \), we say that \( X \) is a real Banach \( A \)-module for the real Banach algebra \( A \). If \( F = C \), we say \( X \) is a Banach \( A \)-module for the Banach algebra \( A \).

**Definition 3.2.** Let \( X \) be a real Banach space. Then \( BL_k(X, C) \), consists of all complex-valued continuous real-linear functional on \( X \), which is a real Banach space, denoted by \( X^* \) and called the complex dual of \( X \).

If \( A \) is a real Banach algebra and \( X \) is a real Banach \( A \)-module, then \( X^* \) with the natural module action is also a real Banach \( A \)-module.

Note that in this case \( X^* \) is isomorphic to \( X^* \times X^* \).

**Lemma 3.3.** Let \( G \) be a locally compact group and let \( \tau \) be a topological involution on \( G \). Suppose \( X \) is a real Banach \( L^1(G, \tau) \)-module. Then \( H^1(L^1(G, \tau), X^*) = \{ 0 \} \) if and only if \( H^1(L^1(G, \tau), X^*) = \{ 0 \} \).

**Proof.** It is easy to see that \( Z^1(L^1(G, \tau), X^*) = Z^1(L^1(G, \tau), X^*) \oplus iZ^1(L^1(G, \tau), X^*) \). Now, let \( H^1(L^1(G, \tau), X^*) = \{ 0 \} \) and let \( D \in Z^1(L^1(G, \tau), X^*) \). There exist elements \( a \) and \( b \) in \( X^* \) such that \( D = \delta_a + i\delta_b \). If \( c = a + ib \), then \( c \in X^* \) and \( d = \delta_c \). Hence \( H^1(L^1(G, \tau), X^*) = \{ 0 \} \).
Conversely, we assume that \( H^1(L^1(G, \tau), X') = \{0\} \) and let \( D \in Z^1(L^1(G, \tau), X^*) \). By the assumption \( D \in B^1(L^1(G, \tau), X') \). Clearly, \( B^1(L^1(G, \tau), X') = B^1(L^1(G, \tau), X^*) \oplus iB^1(L^1(G, \tau), X^*) \).

Hence there exist unique elements \( D_1, D_2 \) in \( B^1(L^1(G, \tau), X^*) \) such that \( D = D_1 + iD_2 \). On the other hand, \( D = D + i0 \) where \( D, 0 \in Z^1(L^1(G, Z), X^*) \). Therefore, we have \( D_1 = D \) and \( D_2 = 0 \). Hence \( D \in B^1(L^1(G, \tau), X^*) \) and so \( H^1(L^1(G, \tau), X^*) = \{0\} \).

Lemma 3.4. Let \((X, ||\cdot||)\) be a real Banach space and \(X \times X\) be the (complex) linear space under the standard operations of addition and scalar multiplication. If we equip \(X \times X\) by the norm \(||\cdot,\cdot||\), which satisfies the inequalities

\[
\max\{||x||,||y||\} \leq C_1 ||x||
\]

and

\[
||x,y|| \leq C_2 \max\{||x||,||y||\},
\]

for constants \(C_1\) and \(C_2\), then

(i) \((X \times X, ||\cdot,\cdot||)\) is a Banach space

(ii) The map \(\eta : X \to X \times X\), defined by \(\eta(x) = (x,0)\), is a real-linear continuous mapping.

(iii) The map \(\psi : X' \to (X \times X)'\), defined by \(\psi(\lambda)(x,y) = \lambda(x) + i\lambda(y)\), is a real-linear continuous mapping onto the real Banach space \((X \times X)'\).

Proof. (i) and (ii) are clear. (iii) \(\psi\) is a well-defined real-linear mapping. For each \(\lambda \in X'\) we have

\[
||\psi(\lambda)|| = \sup\{|\psi(x,y)| : (x,y) \in X, ||x,y|| \leq 1\}
\]

\[
\leq \sup\{|\lambda(x)| + |\lambda(y)| : x \in X, ||x|| \leq C_1, y \in X, ||y|| \leq C_1, x,y \in X\}
\]

\[
\leq 2C_1 ||\lambda||.
\]

Hence \(\psi\) is continuous. On the other hand, for each \(\lambda \in X'\) we have

\[
||\psi(\lambda)|| = \sup\{|\psi(\lambda)(x,0) : x \in X, ||(x,0)|| \leq 1\}
\]

\[
\geq \{|\lambda(x) : x \in X, ||x|| \leq 1\} = C_2^{-1} ||\lambda||.
\]

Hence \(\psi\) is one-to-one. To show that \(\psi\) is onto, let \(\Lambda \in (X \times X)'\). Then \(\Lambda \circ \eta \in X'\) and \(\psi(\Lambda \circ \eta) = \Lambda\).

Theorem 3.5 Let \(G\) be a locally compact group and \(\tau\) be a topological involution on \(G\). Then \(L^1(G, \tau)\) is amenable if and only if \(L^1(G)\) is amenable.

Proof. Let \(L^1(G, \tau)\) be amenable, \(X\) be a Banach \(L^1(G)\)-module and \(\Delta \in Z^1(L^1(G), X^*)\). If \(X_\delta\) represents \(X\) as a real Banach space then it is a real Banach \(L^1(G, \tau)\)-module under the module actions defined by
Now we define the map \( D : L^1(G, \tau) \to X^*_R \) by \( Df = \text{Re} \Delta(f + i0) \). Clearly \( D \) is a real-linear mapping and since for each \( f \in L^1(G, \tau) \)
\[
\| Df \| \leq \sup \{ |\Delta(f + i0)(x)| : x \in X, \| x \| \leq 1 \} \leq \| \Delta \| \| f \| ,
\]
\( D \) is continuous. On the other hand, for each \( f, g \in L^1(G, \tau) \),
\[
(Df).g = \text{Re}(\Delta(f + i0)(g + i0)), f.(Dg) = \text{Re}((f + i0) \Delta(g + i0)).
\]
Hence \( D(fg) = (Df).g + f.(Dg) \) and so \( D \in Z^1(L^1(G, \tau), X^*_R) \). The amenability of \( L^1(G, \tau) \) implies that there exists \( u \in X^*_R \) such that \( D = \delta_u \). Now we define \( \lambda : X \to \mathbb{C} \) by \( \lambda(x) = u(x) - iu(ix) \). Clearly \( \lambda \in X^* \) and for \( f \in L^1(G, \tau), x \in X \) we have
\[
(\lambda.(f + i0))(x) = u(fx) - iu(fix), ((f + i0)\lambda)(x) = u(xf) - iu(ixf).
\]
We can show that \( \Delta(f + ig)(x) = (\delta_u(f + ig))(x) \) for every \( f, g \in L^1(G, \tau) \) and \( x \in X \). Hence \( \Delta = \delta_u \) and so \( \Delta \) is an inner derivation, i.e. \( H^1(L^1(G), X^*) = \{0\} \) Thus \( L^1(G) \) is amenable.

Conversely, let \( L^1(G) \) be amenable and let \( X \) be a real Banach \( L^1(G, \tau) \)-module. By Lemma 3.3 it is enough to show that \( H^1(L^1(G), X^*) = \{0\} \). Let \( D : L^1(G, \tau) \to X^* \) be a continuous real derivation. By Lemma 3.4, \( X \times X \) is a Banach space under the norm \( \| (x, y) \| = \max \{ \| x \|, \| y \| \} \).

The map \( \psi : X^* \to (X \times X)^* \), defined by \( \psi(\lambda)(x, y) = \lambda(x) + i\lambda(y)(x, y \in X, \lambda \in X^*) \) is a continuous real-linear mapping which is one-one and onto. The space \( X \times X \) is a Banach \( L^1(G) \)-module under the familiar module actions. Now we define the map \( \Delta : L^1(G) \to (X \times X)^* \) by \( \Delta(f + ig) = \psi(Df) + i\psi(Dg) \). Clearly \( \Delta \) is a complex linear mapping and for \( f, g \in L^1(G) \),
\[
\| \Delta(f + ig) \| \leq \| \psi \| \| D \| \| f \|_1 + \psi \| D \| \| g \| \leq 2 \| \psi \| \| D \| \max \{ \| f \|_1, \| g \|_1 \}
\]
\[
\leq 2 \| \psi \| \| D \| \| f + ig \|_1.
\]
Hence \( \Delta \) is continuous. Considering the module actions on \( X \times X \) we can show that
\[
\psi((Df).g) = \psi(Df)(g + i0)
\]
and
\[
\psi(f.(Dg)) = (f + i0)\psi(Dg)
\]
Since \( D \) is a \( X^* \)-derivation on \( L^1(G, \tau) \), by using the above equation we have
\[
\Delta((f_1 + ig_1)(f_2 + ig_2)) = (\Delta(f_1 + ig_1)(f_2 + ig_2) + (f_1 + ig_1)(\Delta(f_2 + ig_2)).
\]
Therefore, \( \Delta \in Z^1(L^1(G), (X \times X)^*) \) and so there exists \( \Lambda \in (X \times X)^* \) such that \( \Delta = \delta_\Lambda \). Since \( \psi \) is onto and one-to-one there exists a unique \( \lambda \in X^* \) such that \( \Lambda = \psi(\lambda) \).

Now we notice that \( \psi(f, \eta) = (f + i0)\psi(\eta) \) and \( \psi(\eta, f) = \psi(\eta)(f + i0) \) for every \( f \in L^1(G, \tau) \) and \( \eta \in X^* \). Hence
\[
\psi(Df) = \psi(Df) + i\psi(D0) = \Delta(f + i0) = \delta_\Lambda(f + i0)
\]
(f + i0).\Lambda - \Lambda.(f + i0) = (f + i0).\psi(\lambda) - \psi(\lambda).(f + i0) \\
= \psi(f, \lambda) - \psi(\lambda, f) = \psi(\delta_{\lambda}(f)).$

Since \psi is one-to-one, it implies that \( D(f) = \delta_{\lambda}(f) \) for each \( f \in L^1(G, \tau) \) and \( D = \delta_{\lambda} \). This completes the proof.

**Theorem 3.6.** Let \( G \) be a locally compact group and let \( \tau \) be a topological involution on \( G \). Then \( L^1(G, \tau) \) is weakly amenable if and only if \( L^1(G) \) is weakly amenable.

**Proof.** Let \( L^1(G, \tau) \) be a weakly amenable real Banach algebra. We show that for each \( \Delta \in Z^1(L^1(G), L^1(G)^*) \) there exists \( \Lambda \in L^1(G)^* \) such that \( \Delta = \delta_{\lambda} \). Let \( \eta : L^1(G, \tau) \to L^1(G) \) be defined by \( \eta(f) = f + i0 \) and \( \psi : L^1(G, \tau)^* \to L^1(G)^* \) be defined by \( \psi(\lambda)(f + ig) = \lambda(f) + i\lambda(g) \). By Lemma 3.4, \( \eta \) and \( \psi \) are continuous real-linear mapping. Also, \( \psi \) is a one-to-one and onto mapping from the real Banach space \( L^1(G, \tau)^* \) onto \( L^1(G)^* \) as a real Banach space. By the open Mapping Theorem for real Banach spaces, \( \psi^{-1} : L^1(G)^* \times L^1(G)^* \to L^1(G, \tau)^* \) is a real-linear continuous mapping. Now if we define \( D = \psi^{-1} o \Delta o \eta \), then it is easy to see that \( D \) is a real-linear continuous mapping. To show that \( D \) is an \( L^1(G, \tau)^* \)-derivation on \( L^1(G, \tau) \) we see that

\[
\psi(D(fg)) = (\psi o D)(fg) = \Delta(fg + i0) = \Delta((f + i0).(g + i0)) \\
= (\Delta o \eta)(f). (g + i0) + (f + i0). (\Delta o \eta)(g) \\
= \psi(DF).(g + i0) + (f + i0). \psi(Dg).
\]

On the other hand, \( \psi(\mu)(f + i0) = \psi(\mu, f) \) and \( (f + i0). \psi(\mu) = \psi(a, \mu) \) for \( f \in L^1(G) \) and \( \mu \in L^1(G, \tau)^* \). Hence, for each \( f, g \in L^1(G, \tau) \) we have

\[
\psi(D(fg)) = \psi(DF.g) + \psi(f.Dg) = \psi(DF.g + f.Dg).
\]

Since \( \psi \) is one-one, we conclude that \( D \) is an \( L^1(G, \tau)^* \)-derivation, i.e. \( D \in Z^1(L^1(G, \tau), L^1(G, \tau)^*) \). By Lemma 3.3, the weak amenability of \( L^1(G, \tau) \) implies that there exists \( \lambda \in L^1(G)^* \) such that \( D = \delta_{\lambda} \). By definition of \( D \) and the above equalities it implies that \( \Delta = \delta_{\lambda} \), and so \( L^1(G) \) is weakly amenable.

Conversely, let \( L^1(G) \) be weakly amenable and \( D \in Z^1(L^1(G), L^1(G, \tau)^*) \). By Lemma 3.4 the map \( \psi : L^1(G, \tau)^* \to L^1(G)^* \times L^1(G)^* \), defined by \( \psi(\lambda)(f + ig) = \lambda(f) + i\lambda(g) \), is a real-linear continuous one-to-one mapping onto \( L^1(G)^* \times L^1(G)^* \), as a real Banach space.

Now we define the map \( \Delta : L^1(G) \to L^1(G)^* \times L^1(G)^* \) by \( \Delta(f + ig) = \psi(DF) + i \psi(Dg) \). Similar to the proof of Theorem 3.4 we can show that \( \Delta \) is a continuous derivation. Hence there exists \( \Lambda \in L^1(G)^* \times L^1(G)^* \) such that \( \Lambda = \delta_{\lambda} \). Since \( \psi \) is one-to-one and onto, there exists a unique \( \lambda \in L^1(G, \tau)^* \) such that \( \Lambda = \psi(\lambda) \). It can be shown that \( D = \delta_{\lambda} \) and so \( L^1(G, \tau) \) is weakly amenable by Lemma 3.3.

**Corollary 3.7.** Let \( G \) be a locally compact group and \( \tau \) be a topological involution on \( G \). Then

(i) \( L^1(G, \tau) \) is amenable if and only if \( G \) is amenable.

(ii) \( L^1(G, \tau) \) is weakly amenable.
**Proof.** By Theorem 3.5 and 3.6 the amenability and weak amenability of $L^1(G, \tau)$ and $L^1(G)$ are equivalent. Since $L^1(G)$ is amenable if and only if $G$ is amenable [5], (i) follows. Since $L^1(G)$ is weakly amenable [6], we conclude that $L^1(G, \tau)$ is also weakly amenable.

4. MULTIPLIERS

In this section we characterize the multipliers of $L^1(G, \tau)$. A bounded real linear operator $T$ on $L^1(G, \tau)$ is called a left (right) multiplier if $T(f \ast g) = (Tf) \ast g = f \ast (Tg)$, $f, g \in L^1(G, \tau)$).

**Definition 4.1.** Let $\delta_x$ be the point mass at $x \in G$. We define $m_x = \frac{\delta_x + \delta_{\tau(x)}}{2}$ and $R_x(f) = m_x \ast f (f \in L^1(G, \tau))$.

It is clear that $m_x \ast \tau = m_x$ (since $\tau^2 = \tau$) and $\|m_x\| = 1$. Therefore, $m_x \in M(G, \tau)$.

**Lemma 4.2.** Let $\mu$ be a measure in $M(G)$ such that $f \ast \mu \in L^1(G, \tau)$ for every $f \in L^1(G, \tau)$. Then $\mu \in M(G, \tau)$.

**Proof.** We have $f \ast \mu(x) = \int_G f(xy^{-1}) d\mu((y) (x \in G)$. Therefore,
\[
f \ast (\overline{\mu} \ast \tau)(x) = \int_G f(xy^{-1}) d(\overline{\mu} \ast \tau)(y) = \int_G f(x(\tau(y^{-1}))) d\overline{\mu}(y) = \int_G f(\tau(x)y^{-1}) d\overline{\mu}(y) = \int_G f(\tau(x)y^{-1}) d\overline{\mu}(y) = f \ast \mu(\tau(x)) = f \ast \mu(x).
\]
So $f \ast (\overline{\mu} \ast \tau) = f \ast \mu$ for every $f$ in $L^1(G, \tau)$. Since $L^1(G, \tau)$ has a bounded approximate identity, we have $\overline{\mu} \ast \tau = \mu$. Hence $\mu \in M(G, \tau)$.

**Theorem 4.3.** Let $T$ be a left multiplier on $L^1(G, \tau)$. Then there exists a unique $\mu \in M(G, \tau)$ such that $Tf = f \ast \mu (f \in L^1(G, \tau))$ and $\|\mu\| = \|T\|$.

**Proof.** We define $T_0 : L^1(G) \rightarrow L^1(G)$ by $T_0(f) = T(g) + iT(h)$ where $f = g + ih$. We have
\[
T_0(f_1 \ast f_2) = T_0((g_1 + ih_1) \ast (g_2 + ih_2)
= T_0(g_1 \ast g_2 + ih_1 \ast g_2 + ig_1 \ast h_2 - h_1 \ast h_2)
= T_0(g_1 \ast g_2 - h_1 \ast h_2) + iT(h_1 \ast g_2 + g_1 \ast h_2)
= g_1 \ast Tg_2 - h_1 \ast Th_2 + ih_1 \ast Tg_2 + ig_1 \ast Th_2
= (g_1 + ih_1) \ast (Tg_2 + iTh_2)
= f_1 \ast T_0 f_2.
\]
Hence, $T_0$ is a left multiplier on $L^1(G)$. Therefore, by [7] there exists a unique $\mu \in M(G, \tau)$ such that $T_0 h = h \ast \mu$ for every $h$ in $L^1(G)$ and $\| T_0 \| = |\mu| (G)$. Consequently, $Tf = f \ast \mu$ for every $L^1(G, \tau)$ and $\| T \| \leq \| T_0 \| = |\mu| (G)$. Now, since $f \ast \mu \in L^1(G, \tau)$ for all $f \in L^1(G, \tau)$, $\mu \in M(G, \tau)$ by Lemma 4.2. But by the proof of [7, Theorem 1] we have $\mu = \omega^* - \lim_a T_0 e_a = \omega^* - \lim_a T e_a$. Now, since $\| T e_a \| \leq \| T \|$, 

$$|\mu| (G) \leq \| T \| . \tag{26}$$

Therefore, $\| T \| = \| \mu \|$

REFERENCES