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DIRECT SYSTEM AND DIRECT LIMIT OF H_v-MODULES*

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Abstract – The largest class of algebraic hyper structures satisfying the module like axioms is the H_v-module. In this paper, we consider the category of H_v-modules and prove that the direct limit always exists in this category. Direct limits are defined by a universal property, and so are unique. The most powerful tool in order to obtain a module from a given H_v- module is the quotient out procedure. To use this method we consider the fundamental equivalence relation ε^*, and then prove some of the results about the connection between the fundamental modules, direct systems and direct limits.

Keywords – H_v-ring, H_v-module, direct system, direct limit, fundamental relation, fundamental module

1. INTRODUCTION

The theory of hyper structures was introduced by Marty in 1934 during the 8th Congress of the Scandinavian Mathematicians [1]. A hyper structure is a non-empty set H, together with a function $\cdot : H \times H \to P^*(H)$ called hyper operation, where $P^*(H)$ denotes the set of all non-empty subsets of H. Marty introduced hyper groups as a generalization of groups, and then many researchers began to work on this new field of modern algebra and have since developed it. Some basic definitions and theorems about hyperstructures can be found in [2, 3]. The concept of H_v-structures constitute a generalization of well known algebraic hyper structures (hyper group, hyper ring, hyper module and so on) where the axioms are replaced by the weak ones. That is, instead of the equality on sets, one has non-empty intersections. H_v-structures were first introduced by Vougiouklis in the Fourth AHA Congress (1990) [4]. The basic definitions and results of the object can be found in [5]. This concept has been further investigated in [6-14]. We recall the following definitions from [5] for the sake of completeness.

Definition 1. 1. A *multivalued system* $(R, +, \cdot)$ is called an H_v-ring if the following axioms hold:

1. $(R,+)$ is an H_v-group i.e.,
   $(x+y)+z \cap x+(y+z) \neq \phi$, for all $x, y, z \in R$ (weak associativity),
   $a+R = R+a = R$, for all $a \in R$ (reproduction axiom).
2. $(x \cdot y) \cdot z \cap x \cdot (y \cdot z) \neq \phi$, for all $x, y, z \in R$.
3. $(\cdot)$ is *weak distributive* with respect to $(+)$, i.e., for all $x, y, z \in R$

$$x \cdot (y+z) \cap (x \cdot y + x \cdot z) \neq \phi, \ (x+y) \cdot z \cap (x \cdot z + y \cdot z) \neq \phi .$$

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Definition 1. 2. M is a (left) H_v-module over an H_v-ring R if $(M, +)$ is a weak commutative H_v-group and there exists a map $\cdot : R \times M \to P^*(M)$ denoted by $(r, m) \to rm$ such that for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$, we have

$$r_1(m_1 + m_2) \cap (r_1 m_1 + r_1 m_2) \neq \phi,$$

$$(r_1 + r_2) m_1 \cap (r_1 m_1 + r_2 m_1) \neq \phi,$$

$$(r_1 r_2) m_1 \cap r_1(r_2 m_1) \neq \phi.$$

Definition 1. 3. Let M_1 and M_2 be two H_v-modules over an H_v- ring R. A mapping $f : M_1 \to M_2$ is called a *strong homomorphism* if, for all $x, y \in M_1$ and for all $r \in R$, the following relations hold:

$$f(x + y) = f(x) + f(y),\ f(rx) = rf(x).$$

If there exists a one to one strong homomorphism from M_1 onto M_2, then M_1 and M_2 are called isomorphic.

In this paper, we assume ϑ to be the category of H_v-modules over an H_v-ring R with strong homomorphisms, and consider the direct system and direct limit of ϑ and prove some of the results in this connection.

2. DIRECT SYSTEM AND DIRECT LIMIT OF H_v- MODULES

A partially ordered set I is said to be a *directed set* if for each pair i, j in I there exists $k \in I$, such that $i \leq k$ and $j \leq k$. Let I be a directed set and ϑ the category of H_v-modules over an H_v-ring R with strong homeomorphisms. Let $(M_i)_{i \in I}$ be a family of H_v-modules indexed by I. For each pair i, j in I such that $i \leq j$, let $\phi_j^i : M_i \to M_j$ be a strong homomorphism and suppose that the following axioms are satisfied:

1. ϕ_i^i is the identity for all $i \in I$;
2. $\phi_k^i = \phi_k^j \phi_j^i$ whenever $i \leq j \leq k$.

Then the H_v-modules M_i and strong homomorphisms ϕ_j^i are said to be a direct system $M = (M_i, \phi_j^i)$ over the directed set I.

Let $M = (M_i, \phi_j^i)$ be a direct system in ϑ. The direct limit of this system, denoted by $\underrightarrow{lim} M_i$, is an H_v-module and a family of strong homomorphisms $\alpha_i : M_i \to \underrightarrow{lim} M_i$ with $\alpha_i = \alpha_j \phi_j^i$ whenever $i \leq j$ satisfying the following universal mapping property:

$$\begin{tikzcd}
\underrightarrow{lim} M_i \arrow[rr, dashed, "\beta"] & & X \\
& M_i \arrow[ul, "\alpha_i"'] \arrow[ur, "f_i"] \arrow[d, "\phi_j^i"] & \\
& M_j \arrow[uul, "\alpha_j"] \arrow[uur, "f_j"'] &
\end{tikzcd}$$

Fig 1. commutative diagram for direct limit for every H_v-module X and every family of strong homomorphisms $f_i : M_i \to X$ with $f_i = f_j\phi_j^i$, whenever $i \le j$, there is a unique strong homomorphism $\beta : \underrightarrow{lim}\, M_i \to X$ making the above diagram (Fig. 1) commute.

Note that the construction of the direct system and direct limit are similar as in the usual group theory and modules. (See [15, 16]). Also, already Leoreanu [17-19] and Romeo [20] wrote on direct limits of hyper structures. But in this paper, we consider the previous known facts from another point of view.

Let X be the disjoint union $\bigcup M_i$, defining an equivalent relation on X by $a_i \sim a_j$, $a_i \in M_i$, $a_j \in M_j$ if there exists an index $k \ge i, j$ with $\phi_k^i a_i = \phi_k^j a_j$. The equivalent class of a_i is denoted by $[a_i]$. Suppose that $X/\sim$ is to be the set of all equivalent classes. It is clear that $a_i \sim \phi_j^i a_j$ for $j \ge i$ in $X/\sim$. Now, for $r \in R$ and $[a_i], [a_j] \in X/\sim$ we define:
$[a_i] \oplus [a_j] = \{[x] \mid x \in a_k + a'_k$ where $a_k = \phi_k^i a_i, a'_k = \phi_k^j a_j$ for some $k \ge i, j\}$, and

$$r \circ [a_i] = \{[x] \mid x \in ra_i\}.$$

Lemma 2. 1. $\oplus$ and $\circ$ defined above are well-defined hyper operations.

Proof. Suppose $[a_i] = [b_s]$ and $[a'_j] = [b'_t]$, then by definition there exists $k_1 \ge i, s$ and $k_2 \ge j, t$ such that
$\phi_{k_1}^i a_i = \phi_{k_1}^s b_s$ and $\phi_{k_2}^j a'_j = \phi_{k_2}^t b'_t$.
Now, we have

$$\begin{aligned}
[x] \in [a_i] \oplus [a'_j] &\Leftrightarrow x \in \phi_{k_3}^i a_i + \phi_{k_3}^j a'_j \text{ for some } k_3 \ge i, j \\
&\Leftrightarrow x \in \phi_k^i a_i + \phi_k^j a'_j \text{ for some } k \ge k_1, k_2, k_3 \\
&\Leftrightarrow x \in \phi_k^{k_1} \phi_{k_1}^i a_i + \phi_k^{k_2} \phi_{k_2}^j a'_j \\
&\Leftrightarrow x \in \phi_k^{k_1} \phi_{k_1}^s b_s + \phi_k^{k_2} \phi_{k_2}^t b'_t \\
&\Leftrightarrow x \in \phi_k^s b_s + \phi_k^t b'_t \\
&\Leftrightarrow [x] \in [b_s] \oplus [b'_t].
\end{aligned}$$

So $\oplus$ is well defined. If $r \in R$ and $[a_i] = [a'_j]$, then there exists $k \ge i, j$ such that $\phi_k^i a_i = \phi_k^j a'_j$. So $r\phi_k^i a_i = r\phi_k^j a'_j$, which implies $\phi_k^i(ra_i) = \phi_k^j(ra'_j)$. Hence $[ra_i] = [ra'_j]$ and so $\circ$ is also well defined.

Theorem 2. 2. $(X/\sim, \oplus)$ is an H_v-group.
Proof. Suppose $[a_i], [a'_j], [a''_t] \in X/\sim$ and $k \ge i, j, t$. Since M_k is a weak associative we have

$$\left((\phi_k^i a_i + \phi_k^j a')_j + \phi_k^t a''_t\right) \cap \left(\phi_k^i a_i + (\phi_k^j a'_i + \phi_k^t a'')_t\right) \ne \phi.$$

So

$$\left(([a_i] \oplus [a'_j]) \oplus [a''_t]\right) \cap \left([a_i] \oplus ([a'_j] \oplus [a''_t])\right) \ne \phi.$$

Therefore $(X/\sim, \oplus)$ is a weak associative. Now we prove the reproduction axiom. Suppose $[a_i], [a'_j] \in X/\sim$, then for $k \ge i, j$ we have $\phi_k^i a_i = a_k \in M_k$ and $\phi_k^j a'_j = a'_k \in M_k$.

Since $(M_k,+)$ is an H_v-group, then there exists $a''_k \in M_k$ such that $a_k \in a'_k + a''_k$. So by definition we have $[a_k] \in [a'_k] \oplus [a''_k] = [a'_j] \oplus [a''_k]$, which implies that $[a_i] \in [a'_j] \oplus X/\sim$. Thus $X/\sim = [a'_j] \oplus X/\sim$.

Theorem 2. 3. $(X/\sim, \oplus, \circ)$ is an H_v-module over R.

Proof. Suppose $r_1, r_2 \in R$ and $[a_i], [a'_j] \in X/\sim$ and $k \geq i, j$. In the H_v-module M_k, we have

$$r_1(\phi^i_k a_i + \phi^j_k a'_j) \cap (r_1\phi^i_k a_i + r_1\phi^i_k a_j) \neq \phi,$$

$$(r_1 + r_2)\phi^i_k a_i \cap (r_1\phi^i_k a_i + r_2\phi^i_k a_i) \neq \phi,$$

$$(r_1 r_2)\phi^i_k a_i \cap r_1(r_2\phi^i_k a_i) \neq \phi.$$

Therefore

$$r_1 \circ ([a_i] \oplus [a'_j]) \cap (r_1 \circ [a_i] \oplus r_1 \circ [a'_j]) \neq \phi,$$

$$(r_1 + r_2) \circ [a_i] \cap (r_1 \circ [a_i] \oplus r_2 \circ [a_i]) \neq \phi,$$

$$(r_1 r_2) \circ [a_i] \cap r_1 \circ (r_2 \circ [a_i]) \neq \phi.$$

Theorem 2. 4. Let (M_i, ϕ^i_j) be a direct system of H_v-modules indexed by I. Then the H_v-module $X/\sim$ is $\varinjlim M_i$.

Proof. Define $\alpha_i : M_i \to X/\sim$ given by $a_i \mapsto [a_i]$, and consider the following diagram (Fig. 2):

$$\begin{tikzcd}
X/\sim & \\
 & M_i \arrow[ul, "\alpha_i"'] \arrow[d, "\phi^i_j"] \\
 & M_j \arrow[uul, "\alpha_j"]
\end{tikzcd}$$

Fig. 2. Commutative diagram

So $\alpha_j(\phi^i_j a_i) = [\phi^i_j a_i] = [a_i] = \alpha_i(a_i)$. Therefore $\alpha_j \phi^i_j = \alpha_i$, hence the diagram is commutative. Now let M be an H_v-module and $\{f_i | f_i : M_i \to M\}$ a family of strong homomorphisms with $f_i = f_j \phi^i_j$. Now, we define $\beta : X/\sim \to M$ by $[a_i] \mapsto f_i a_i$. We show that β is a strong homomorphism and so the universal mapping property holds. First we show that β is well defined. Suppose $[a_i] = [b_j]$, then there exists $k \geq i, j$ such that $\phi^i_k a_i = \phi^j_k b_j$. Then $f_k \phi^i_k a_i = f_k \phi^j_k b_j$ and so $f_i a_i = f_j b_j$. Therefore β is well defined. Now let $[a_i], [b_j] \in X/\sim$ and $r \in R$, then

$\beta([a_i] \oplus [b_j]) = \{\beta([x]) \mid x \in a_k + a'_k$ where $a_k = \phi^i_k a_i$, $a'_k = \phi^j_k b_j$ for some $k \geq i, j\}$

$= \{f_k x \mid x \in a_k + a'_k$ where $a_k = \phi^i_k a_i$, $a'_k = \phi^j_k b_j$ for some $k \geq i, j\}$

$= f_k(a_k + a'_k)$ where $a_k = \phi_k^i a_i, a'_k = \phi_k^j b_j$

$$= f_k a_k + f_k a'_k$$

$$= f_k \phi_k^i a_i + f_k \phi_k^j b_j$$

$$= f_i a_i + f_j b_j$$

$$= \beta([a_i]) + \beta([b_j]).$$

And

$$\beta(r \circ [a_i]) = \beta([ra_i]) = f_i(ra_i) = rf_i(a_i) = r\beta([a_i]).$$

Therefore β is a strong homomorphism and $\beta\alpha_i = f_i$.

3. FUNDAMENTAL RELATION ε^* AND DIRECT SYSTEMS

Every H_v-structure "hides" a corresponding structure. This structure is obtained from the H_v-structure by quotienting out the fundamental relation β^*, γ^* or ε^*. Therefore if H is an H_v-group (H_v-ring, H_v-module), then H/β^* is a group (H/γ^* is a ring, H/ε^* is a module, respectively). The above corresponding structures are the fundamental ones.

Consider the (left) H_v-module M over an H_v-ring R. According to [5], let U denote the set of all expressions consisting of finite hyper operations applied on finite subsets of R and M. A relation ε_M can be defined on M, whose transitive closure is the fundamental relation ε_M^*. The relation ε_M is defined as follows: for all $x, y \in M$, $x\varepsilon_M y$ if and only if $\{x, y\} \subseteq u$ for some $u \in U$. Suppose $\gamma^*(r)$ is the equivalence class containing $r \in R$ and $\varepsilon_M^*(x)$ is the equivalence class containing $x \in M$. The sum $\oplus$ and the external product $\circ$ using the γ^* classes in R on M/ε_M^* are defined as follows: for $x, y \in M$ and for all $r \in R$,

$$\varepsilon_M^*(x) \oplus \varepsilon_M^*(y) = \varepsilon_M^*(c) \text{ for all } c \in \varepsilon_M^*(x) + \varepsilon_M^*(y),$$

$$\gamma^*(x) \circ \varepsilon_M^*(x) = \varepsilon_M^*(d) \text{ for all } d \in \gamma^*(r) \cdot \varepsilon_M^*(x).$$

The fundamental relation ε_M^* on M over R is the smallest equivalence relation such that M/ε_M^* is a module over the ring R/γ^*, (see [5]). If $\phi : M \to M/\varepsilon_M^*$ is the canonical map, then the kernel of ϕ is called the *core* of M and is denoted by ω_M. Therefore $\omega_M = \{x \in M | \phi(x) = o\}$, where o is the unit element of the group $(M/\varepsilon_M^*, \oplus)$. One can prove that the unit element of the group $(M/\varepsilon_M^*, \oplus)$ is equal to ω_M, i.e., $\omega_M \oplus \varepsilon_M^*(x) = \varepsilon_M^*(x) \oplus \omega_M = \varepsilon_M^*(x)$, for all $x \in M$.

Proposition 3. 1. Let (M_i, ϕ_j^i) be a direct system of H_v-modules over an H_v-ring R indexed by a directed set I. Then $(M_i/\varepsilon_{M_i}^*, \phi_j^{*i})$ is a direct system of modules over the ring R/γ^*, where

$$\phi_j^{*i} : M_i/\varepsilon_{M_i}^* \to M_j/\varepsilon_{M_j}^*$$
$$\varepsilon_{M_i}^*(a_i) \mapsto \varepsilon_{M_j}^*(\phi_j^i a_i).$$

Proof. By Lemma 3.1 of [7], $(M_i/\varepsilon_{M_i}^*, \phi_j^{*i})$ is a family of R/γ^*-modules and R/γ^*-homomorphisms. It is clear that ϕ_i^{*i} is the identity for all $i \in I$. Now, for $i \le j \le k$, we have

$$\left(\phi_k^j\phi_j^i\right)^*\left(\varepsilon_{M_i}^*(a_i)\right)=\phi_k^{*i}\left(\varepsilon_{M_i}^*(a_i)\right)=\varepsilon_{M_k}^*\left(\phi_k^i a_i\right)=\varepsilon_{M_k}^*\left(\left(\phi_k^j\phi_j^i\right)a_i\right)$$
$$=\varepsilon_{M_k}^*\left(\phi_k^j\left(\phi_j^i a_i\right)\right)=\phi_k^{*j}\left(\varepsilon_{M_j}^*\left(\phi_j^i a_i\right)\right)=\phi_k^{*j}\phi_j^{*i}\left(\varepsilon_{M_i}^*(a_i)\right)$$

Therefore $\left(\phi_k^j\phi_j^i\right)^*=\phi_k^{*i}=\phi_k^{*j}\phi_j^{*i}$.

Propostion 3. 2. Let $[a_i],[b_j]\in X/_\sim$, as described in Section 2, we define $[a_i]\theta[b_j]$ if there exists $k\geq i,j$ such that $\phi_k^i a_i\varepsilon_{M_k}\phi_k^j b_j$, then $\theta=\varepsilon_{X/_\sim}$.

Proof. Let $[a_i]\varepsilon_{X/_\sim}[b_j]$, then there exist $r_1,\cdots,r_n\in R$ and $t_1,\cdots,t_m\in\bigcup M_i$ such that $\{[a_i],[b_j]\}\subseteq r_1[t_1]+\cdots+r_m[t_m]$. Suppose $t_1\in M_{i_1},\cdots,t_m\in M_{i_m}$ and $k\geq i_1,i_2,\cdots,i_m,i,j$. Then

$$r_1[t_1]+\cdots+r_m[t_m]=r_1\left[\phi_k^{i_1}t_1\right]+\cdots+r_m\left[\phi_k^{i_m}t_m\right]$$
$$=\left\{[x]\;\middle|\;x\in\phi_k^{i_1}r_1t_1+\cdots+\phi_k^{i_m}r_mt_m\right\},$$

and so $\left\{\left[\phi_k^i a_i\right],\left[\phi_k^j b_j\right]\right\}\subseteq\left\{[x]\;\middle|\;x\in\phi_k^{i_1}r_1t_1+\cdots+\phi_k^{i_m}r_mt_m\right\}$. Now, for some $n\geq k$, we obtain $\left\{\phi_n^i a_i,\phi_n^j b_j\right\}\subseteq\phi_n^{i_1}r_1t_1+\cdots+\phi_n^{i_m}r_mt_m$ which implies that $\phi_n^i a_i\varepsilon_{M_n}\phi_n^j b_j$. Hence $[a_i]\theta[b_j]$.

Conversely, if $[a_i]\theta[b_j]$, then there exists $k\geq i,j$ such that $\phi_k^i a_i\varepsilon_{M_k}\phi_k^j b_j$ and so there exist $t_1,\cdots,t_m\in M_k\subset\bigcup M_i$ and $r_1,\cdots,r_m\in R$ such that

$$\left\{\phi_k^i a_i,\phi_k^j b_j\right\}\subseteq r_1t_1+\cdots+r_mt_m.$$

So $\left\{\left[\phi_k^i a_i\right],\left[\phi_k^j b_j\right]\right\}\subseteq r_1[t_1]+\cdots+r_m[t_m]$ which implies $[a_i]\varepsilon_{X/_\sim}[b_j]$. Thus $\phi=\varepsilon_{X/_\sim}$.

Theorem 3. 3. Let $\left(M_i,\phi_j^i\right)$ be a direct system of H_v-modules over an H_v-ring R indexed by a directed set I, and let ε^* be the fundamental relation of $\underrightarrow{lim}M_i$. Then

$$\underrightarrow{lim}\left(M_i/\varepsilon_{M_i}^*\right)\underset{R/\gamma^*}{\cong}\left(\underrightarrow{lim}\;M_i\right)/\varepsilon^*.$$

Proof. We define $f:\underrightarrow{lim}\left(M_i/\varepsilon_{M_i}^*\right)\to\left(\underrightarrow{lim}\;M_i\right)/\varepsilon^*$ by $\left[\varepsilon_{M_i}^*(a_i)\right]\mapsto\varepsilon^*\left([a_i]\right)$. Then

$$\varepsilon^*\left([a_i]\right)=\varepsilon^*\left([b_j]\right)\Leftrightarrow\phi_k^i\,a_i\,\varepsilon_{M_k}^*\,\phi_k^j\,b_j\ \text{ for some } k\geq i,j$$
$$\Leftrightarrow\varepsilon_{M_k}^*\left(\phi_k^i a_i\right)=\varepsilon_{M_k}^*\left(\phi_k^j b_j\right)$$
$$\Leftrightarrow\phi_k^{*i}\varepsilon_{M_i}^*(a_i)=\phi_k^{*j}\varepsilon_{M_j}^*(b_j)$$
$$\Leftrightarrow\left[\varepsilon_{M_i}^*(a_i)\right]=\left[\varepsilon_{M_j}^*(b_j)\right].$$

Therefore f is well defined and one to one. Now, we have
$f\left(\left[\varepsilon_{M_i}^*(a_i)\right]\oplus\left[\varepsilon_{M_j}^*(b_j)\right]\right)=f\left(\left[\phi_k^{*i}\,\varepsilon_{M_i}^*(a_i)\oplus\phi_k^{*j}\,\varepsilon_{M_j}^*\,b_j\right]\right)$ for some $k\geq i,j$

$$= f\left(\left[\varepsilon^*_{M_k}\left(\phi^i_k a_i\right) \oplus \varepsilon^*_{M_k}\left(\phi^j_k b_j\right)\right]\right)$$
$$= f\left(\left[\varepsilon^*_{M_k}\left(\phi^i_k a_i + \phi^j_k b_j\right)\right]\right)$$

$$= f\left(\left[\varepsilon^*_{M_k}(x)\right]\right) \text{ where } x \in \phi^i_k a_i + \phi^j_k b_j$$

$$= \varepsilon^*([x]) \text{ where } x \in \phi^i_k a_i + \phi^j_k b_j .$$

On the other hand,

$$f\left(\left[\varepsilon^*_{M_i}(a_i)\right]\right) \oplus f\left(\left[\varepsilon^*_{M_j}(b_j)\right]\right) = \varepsilon^*([a_i]) \oplus \varepsilon^*([b_j])$$

$$= \varepsilon^*([x]) \text{ where } [x] \in [a_i] + [b_j]$$

$$= \varepsilon^*([x]) \text{ where } x \in \phi^i_k a_i + \phi^j_k b_j, \text{ for } k \geq i, j .$$

Thus

$$f\left(\left[\varepsilon^*_{M_i}(a_i)\right] \oplus \left[\varepsilon^*_{M_j}(b_j)\right]\right) = f\left(\left[\varepsilon^*_{M_i}(a_i)\right]\right) \oplus f\left(\left[\varepsilon^*_{M_j}(b_j)\right]\right) .$$

Also, we have

$$f\left(\gamma^*(r) \circ \left[\varepsilon^*_{M_i}(a_i)\right]\right) = f\left(\left[\gamma^*(r) \circ \varepsilon^*_{M_i}(a_i)\right]\right)$$

$$= f\left(\left[\varepsilon^*_{M_i}(x)\right]\right) \text{ for } x \in ra_i$$

$$= \varepsilon^*([x]) \text{ for } x \in ra_i ,$$

and

$$\gamma^*(r) \circ f\left(\left[\varepsilon^*_{M_i}(a_i)\right]\right) = \gamma^*(r) \circ \varepsilon^*([a_i])$$

$$= \varepsilon^*(r[a_i])$$

$$= \varepsilon^*([x]) \text{ for } [x] \in r[a_i]$$

$$= \varepsilon^*([x]) \text{ for } x \in ra_i .$$

So $f\left(\gamma^*(r) \circ \left[\varepsilon^*_{M_i}(a_i)\right]\right) = \gamma^*(r) \circ f\left(\left[\varepsilon^*_{M_i}(a_i)\right]\right)$.

Proposition 3. 4. If $a_i \in \omega_{M_i}$ then $\left[\varepsilon^*_{M_i}(a_i)\right]$ is the zero element in $\underrightarrow{lim}\left(M_i / \varepsilon^*_{M_i}\right)$ and $[a_i] \in \omega_{\underrightarrow{lim} M_i}$.

Proof. Suppose that $\left[\varepsilon^*_{M_i}(a_i)\right] \oplus \left[\varepsilon^*_{M_j}(b_j)\right] = [X]$ where $X = \phi^{*i}_k \varepsilon^*_{M_i}(a_i) \oplus \phi^{*j}_k \varepsilon^*_{M_j}(b_j)$ for $k \geq i, j$. So,

$$X = \phi^{*i}_k\left(\omega_{M_i}\right) \oplus \phi^{*j}_k \varepsilon^*_{M_j}(b_j) = \omega_{M_k} \oplus \phi^{*j}_k \varepsilon^*_{M_j}(b_j) = \phi^{*j}_k \varepsilon^*_{M_j}(b_j).$$

Using Theorem 3.3, we obtain $\varepsilon^*([a_i])$ is zero element in $\underrightarrow{lim}\, M_i / \varepsilon^*$, and so $[a_i] \in \omega_{\underrightarrow{lim} M_i}$.

Lemma 3. 5. (Corollary 3.7, [7]). Let M be an H_v-module over an H_v-ring R. Then ω_M is an H_v- submodule of M.

Corollary 3. 6. Let (M_i, ϕ_j^i) be a direct system of H_v- modules over an H_v-ring R indexed by a directed set *I*. Then $\left(\omega_{M_i}, \phi_j^i\big|_{\omega_{M_i}}\right)$is a direct system and $\underrightarrow{lim}\,\omega_{M_i} / \varepsilon^*$ is a zero module.

Proof. Using Theorem 3.3, $\underrightarrow{lim}\,\omega_{M_i} / \varepsilon^* \cong \underrightarrow{lim}_i\left(\omega_{M_i} / \varepsilon^*_{M_i}\right) = 0.$

Lemma 3. 7. $\omega_{\underrightarrow{lim} M_i} = \left\{[a_i] \;\middle|\; a_i \in \bigcup M_i, \phi_k^i a_i \in \omega_{M_K}\right.$, for some $\left. k \geq i\right\}.$

Proof. It is clear and omitted.

Theorem 3. 8. Let $Dir(I)$be the category of all direct systems of H_v-modules and strong homomorphisms over directed set I, $_R m$ be the category of left H_v-modules and strong homomorphisms over an H_v-ring *R*. Suppose

$$\underrightarrow{lim} : Dir(I) \mapsto {}_R m ,$$

$$\{F_i, \phi_j^i\} \mapsto \{[a] \mid a \in \bigcup F_i\}$$

and when $t : \{F_i, \phi_j^i\} \rightarrow \{C_i, \phi_j'^i\}$ is a natural transformation, $\underrightarrow{lim}\, t = \vec{t} : \underrightarrow{lim}\, F_i \rightarrow \underrightarrow{lim}\, C_i$, where $a = a_i \in F_i$, and $\vec{t}([a]) = [t_i a_i]$ where $t_i : F_i \rightarrow C_i$, Then $\underrightarrow{lim}$ is an exact functor.

Proof. It is easy to see that $\underrightarrow{lim}$ is a functor. We prove that if $\{A_i, \phi_j^i\} \rightarrow \{B_i, \phi_j'^i\} \rightarrow \{C_i, \phi_j''^i\}$ is a sequence of morphisms of direct systems over I, such that $\omega_{A_i} \xrightarrow{n_i} A_i \xrightarrow{t_i} B_i \xrightarrow{s_i} C_i \xrightarrow{u_i} \omega_{C_i}$ $(*)$ is exact for each $i \in I$. Then $\omega_{\underrightarrow{lim} A_i} \xrightarrow{\vec{n}} \underrightarrow{lim}\, A_i \xrightarrow{\vec{t}} \underrightarrow{lim}\, B_i \xrightarrow{\vec{s}} \underrightarrow{lim}\, C_i \xrightarrow{\vec{u}} \omega_{\underrightarrow{lim} C_i}$ is an exact sequence of H_v-modules.

i) $ker\,\vec{t} = Im\,\vec{n}$.

Suppose $[a] \in \ker \vec{t}$, then $\vec{t}[a] \in \omega_{\underrightarrow{lim} B_i}$. Let $a = a_i \in A_i$, then $\vec{t}[a] = [t_i a_i] \in \omega_{\underrightarrow{lim} B_i}$, which implies $\phi_k'^i(t_i a_i) \in \omega_{B_k}$ for some $k \geq i$, and so $\phi_k'^i(t_i a_i) = t_k \phi_k^i a_i \in \omega_{B_k}$. The sequence $(*)$ is exact for every i, so $\phi_k^i a_i \in Im\, n_k$, and thus $\phi_k^i a_i \in \omega_{A_k}$, so

$$[a_i] = [\phi_k^i a_i] \in \underrightarrow{lim}\, \omega_{A_i} = Im\,\vec{n}.$$

Conversely, if $[a] \in \underrightarrow{lim}\, \omega_{A_i}$, then $a_i \in \omega_{A_i}$ and so $t_i a_i \in \omega_{B_i}$ which implies $\vec{t}[a_i] = [t_i a_i] \in \omega_{\underrightarrow{lim} B_i}$. Therefore $[a_i] \in Ker\,\vec{t}$.

ii) $Ker\,\vec{s} = Im\,\vec{t}$.

Suppose $[x] = [b_i] \in Ker\,\vec{s}$ then $\vec{s}[b_i] = [s_i b_i] \in \omega_{\underrightarrow{lim} C_i}$. Therefore $\phi_k''^i s_i b_i \in \omega_{C_k}$ for some $k \geq 1$. We have $\phi_k''^i s_i b_i = s_k \phi_k'^i b_i \in \omega_{C_k}$. Since $(*)$ is exact for every i, we conclude that: $\phi_k'^i b_i \in Ker\, s_k = Im\, t_k$ and so there exists $a_k \in A_k$ such that $t_k a_k = \phi_k'^i b_i$.

Thus $\vec{t}\left[a_k\right]=\left[t_k a_k\right]=\left[\phi_k^i b_i\right]=\left[b_i\right]$, and $\left[b_i\right]\in Im\,\vec{t}$.

Conversely, $\vec{s}\circ\vec{t}\left[a_i\right]=\vec{s}\left(\left[t_i a_i\right]\right)=\left[s_i\circ t_i a_i\right]$. Since $(*)$ is exact, $s_i\circ t_i a_i\in\omega_{C_i}$, so $\left[s_i\circ t_i a_i\right]\in\omega_{\underrightarrow{lim}\,C_i}$. Thus $\vec{s}\circ\vec{t}\left[a_i\right]\in\omega_{\underrightarrow{lim}\,C_i}$ and $Im\,\vec{t}\subseteq Ker\,\vec{s}$.

iii) $Ker\,\vec{u}=Im\,\vec{s}$.

The proof is similar to (ii).

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