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COUNTEREXAMPLES IN $a-$MINIMAL SETS*

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Abstract – Several tables have been given due to $a-$minimal sets. Our main aim in this paper is to complete these tables by employing several examples.

Keywords– $a-$minimal set, distal, enveloping semigroup, proximal relation, trasformation semigroup

PRELIMINARIES

Let X be a compact Hausdorff topological space, S be a topological discrete semigroup with identity e and $\pi : X \times S \to X$ ($\pi(x,s) = xs$ $(\forall x \in X, \forall s \in S)$) be a continuous map such that for all $x \in X$ and for all $s,t \in S$, we have $xe = x$ and $x(st) = (xs)t$, then the triple (X,S,π) or simply (X,S) is called a transformation semigroup. In a transformation semigroup (X,S) we have the following definitions:

1. For each $s \in S$, define the continuous map $\pi^s : X \to X$ by $x\pi^s = xs$ ($\forall x \in X$), then $\mathrm{E}(X,S)$ or simply $\mathrm{E}(X)$ is the closure of $\{\pi^s \mid s \in S\}$ in X^X with pointwise convergence, moreover, it is called the enveloping semigroup (or Ellis semigroup) of (X,S). $\mathrm{E}(X)$ has a semigroup structure [1]. A nonempty subset K of $\mathrm{E}(X)$ is called a right ideal if $K\,\mathrm{E}(X) \subseteq K$, and it is called a minimal right ideal if none of the right ideals of $\mathrm{E}(X)$ is a proper subset of K. The set of all minimal right ideals of $\mathrm{E}(X)$ will be denoted by $\mathrm{Min}(\mathrm{E}(X))$.
2. A nonempty subset Z of X is called invariant if $ZS \subseteq Z$. Furthermore, it is called minimal if it is closed and none of the closed invariant subsets of X is a proper subset of Z. The element $a \in X$ is called almost periodic if $a\,\mathrm{E}(X)$ is a minimal subset of X.
3. Let $a \in X$, A be a nonempty subset of X, C be a nonempty subset of $\mathrm{E}(X)$, and K be a right ideal of $\mathrm{E}(X)$, then for each $p \in \mathrm{E}(X)$, $\mathrm{L}_p : \mathrm{E}(X) \to \mathrm{E}(X)$ such that $\mathrm{L}_p(q) = pq$ ($\forall q \in \mathrm{E}(X)$) is a continuous map. The following sets are introduced:

$$\mathrm{B}(K) = \{p \in K \mid \mathrm{L}_p : K \to K \text{ is bijective}\}, \qquad \mathrm{F}(a,C) = \{p \in C \mid ap = a\},$$

$$\mathrm{S}(K) = \{p \in K \mid \mathrm{L}_p : K \to K \text{ is surjective}\}, \qquad \mathrm{F}(A,C) = \bigcap_{b \in A} \mathrm{F}(b,C),$$

$$\mathrm{I}(K) = \{p \in K \mid \mathrm{L}_p : K \to K \text{ is injective}\}, \qquad \overline{\mathrm{F}}(A,C) = \{p \in C \mid Ap = A\},$$

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$$\mathrm{J}(C)=\{p\in C\mid p^2=p\}.$$

4. Let $a\in X$, A be a nonempty subset of X , and K be a closed right ideal of $\mathrm{E}(X)$, then:

- K is called an $a-$minimal set if:

$$aK=a\,\mathrm{E}(X),$$

K does not have any proper subset like L , such that L is a closed right ideal of $\mathrm{E}(X)$ and

$$aL=a\,\mathrm{E}(X),$$

the set of all $a-$minimal sets is denoted by $\mathrm{M}(a)$ and it is nonempty;

- K is called an $A-\overline{\text{minimal}}$ set if:

$$\forall b\in A\quad bK=b\,\mathrm{E}(X),$$

K does not have any proper subset like L , such that L is a closed right ideal of $\mathrm{E}(X)$ and $bL=b\,\mathrm{E}(X)$ for all $b\in A$,
the set of all $A-\overline{\text{minimal}}$ sets is denoted by $\overline{\mathrm{M}}(A)$ and is nonempty;

- K is called an $A-\overline{\overline{\text{minimal}}}$ set if:

$$AK=A\,\mathrm{E}(X),$$

K does not have any proper subset like L , such that L is a closed right ideal of $\mathrm{E}(X)$ and

$$AL=A\,\mathrm{E}(X),$$

the set of all $A-\overline{\overline{\text{minimal}}}$ sets is denoted by $\overline{\overline{\mathrm{M}}}(A)$.

5. The following sets are introduced:

$$\overline{\mathbf{M}}(X,S)=\{A\subseteq X\mid A\neq\varnothing\wedge(\forall K\in\overline{\mathrm{M}}(A)\quad \mathrm{J}(\mathrm{F}(A,K))\neq\varnothing)\},$$

$$\overline{\overline{\mathbf{M}}}(X,S)=\{A\subseteq X\mid A\neq\varnothing\wedge\overline{\overline{\mathrm{M}}}(A)\neq\varnothing\wedge(\forall K\in\overline{\overline{\mathrm{M}}}(A)\quad \mathrm{J}(\overline{\mathrm{F}}(A,K))\neq\varnothing)\}.$$

6. Let $a\in X$ and A be a nonempty subset of X , then:

- (X,S) is called $a-$distal if $\mathrm{E}(X)\in\mathrm{M}(a)$,
- (X,S) is called $A\xrightarrow{(-)}$distal (or simply $A-$distal) if (X,S) be $b-$distal for each $b\in A$,
- (X,S) is called $A\xrightarrow{(\overline{\mathrm{M}})}$distal if $\mathrm{E}(X)\in\overline{\mathrm{M}}(A)$,
- (X,S) is called $A\xrightarrow{(\overline{\overline{\mathrm{M}}})}$distal if $\mathrm{E}(X)\in\overline{\overline{\mathrm{M}}}(A)$.

7. Let A and B be nonempty subsets of X and $\mathrm{R},\mathrm{Q}\in\{\overline{\mathrm{M}},\overline{\overline{\mathrm{M}}}\}$, then:

- B is called $A\xrightarrow{(-,-)}$almost periodic if:

$$\forall a\in A\quad\forall K\in\mathrm{M}(a)\quad\forall b\in B\quad\exists L\in\mathrm{M}(b)\quad L\subseteq K,$$

- B is called $A\xrightarrow{(\mathrm{R},-)}$almost periodic if:

$$\forall a\in A\quad\forall K\in\mathrm{M}(a)\quad\exists L\in\mathrm{R}(B)\quad L\subseteq K,$$

- B is called $A\xrightarrow{(-,\mathrm{Q})}$almost periodic if $\mathrm{Q}(A)\neq\varnothing$ and:
-

$$\forall K \in \mathrm{Q}(A) \quad \forall b \in B \quad \exists L \in \mathrm{M}(b) \quad L \subseteq K\,,$$

- B is called $A \xrightarrow{(\mathrm{R},\mathrm{Q})}$ almost periodic if $\mathrm{Q}(A) \neq \varnothing$ and:

$$\forall K \in \mathrm{Q}(A) \quad \exists L \in \mathrm{R}(B) \quad L \subseteq K\,.$$

Example 1. Let $X_1 = [-1,1]$ (with the induced topology of $\mathbf{R}$) and S_1 be the group of all homeomorphisms like $f : X_1 \to X_1$ (S_1 has the discrete topology), then in the transformation semigroup (X_1, S_1) we have:

1. If $a \in [-1,1]$, and:

$$xs_a = \begin{cases} (1+a)x + a & -1 \le x \le 0 \\ (1-a)x + a & 0 \le x \le 1 \end{cases},\quad x\eta_a = \begin{cases} x & x = 1,-1 \\ a & -1 < x < 1 \end{cases},\quad x\mu_a = \begin{cases} -1 & -1 \le x \le a \wedge x \neq 1 \\ 1 & a < x \le 1 \vee x = 1 \end{cases},$$

then $s_a \in S_1$, $\eta_a, \mu_a \in \mathrm{E}(X_1)$, $\eta_1 = \mu_1$ and $\eta_{-1} = \mu_{-1}$.

2. We have:

i. Using the connectness of $[-1,1]$, for all $s \in S_1$ we have $\{-1,1\}s = \{-1,1\}$ and $-s \in S_1$, moreover, for all $a, b \in X_1 - \{-1,1\}$ there exists $t \in S_1$ such that $at = b$,

ii. $$\overline{xS_1} = \begin{cases} \{-1,1\} & x \in \{-1,1\} \\ [-1,1] = X_1 & x \in (-1,1) \end{cases},$$

iii. $$\forall x \in X_1 \quad \mu_x\, \mathrm{E}(X_1) = \{-\mu_x, \mu_x\}\,,$$

iv. $$\forall x \in (-1,1) \quad \eta_x\, \mathrm{E}(X_1) = \{(-1)^k \eta_y \mid y \in X_1, k = 1,2\}\,.$$

3. We have:

i. Only 1 and -1 are almost periodic points of (X_1, S_1),

ii. $\{-1,1\}$ is the unique minimal subset of (X_1, S_1),

iii. for $x \in X_1$, $\{-\mu_x, \mu_x\}$ is a minimal right ideal of $\mathrm{E}(X_1)$,

iv. $\{-\eta_1, \eta_1\}$, $\{-\eta_{-1}, \eta_{-1}\}$, $\{-\eta_{-1}, \eta_{-1}\} \cup \{-\eta_1, \eta_1\}$ are the only proper subsets of $\{(-1)^k \eta_y \mid y \in X_1, k = 1,2\}$ which are right ideals of $\mathrm{E}(X_1)$.

So:

v. $\{\{-\mu_x, \mu_x\} \mid x \in X_1\}$ is a subset of $\mathrm{M}(1)(= \mathrm{M}(-1) = \overline{\mathrm{M}}(\{-1,1\}) = \overline{\overline{\mathrm{M}}}(\{-1,1\}))$,

vi. $$\forall x \in (-1,1) \quad \eta_x\, \mathrm{E}(X_1) = \{(-1)^k \eta_y \mid y \in X_1, k = 1,2\} \in \mathrm{M}(x)\,,$$

vii. $$\forall A \subseteq X_1 \quad (A \cap (-1,1) \neq \varnothing \Rightarrow \{(-1)^k \eta_y \mid y \in X_1, k = 1,2\} \in \overline{\mathrm{M}}(A))\,.$$

4. Let $K = \{(-1)^k \eta_y \mid y \in X_1, k = 1,2\}$, it is easy to see that:

i. $\mathrm{J}(K) = \{\eta_y \mid y \in X_1\}$, $\mathrm{S}(K) = \mathrm{B}(K) = \mathrm{I}(K) = K - \{-\eta_{-1}, \eta_{-1}, -\eta_1, \eta_1\}$,

ii. $$\mathrm{F}(x, K) = \begin{cases} \{\eta_x, -\eta_{-x}\} & x \in (-1,1) \\ \{\eta_y \mid y \in X_1\} & x \in \{-1,1\} \end{cases},$$

iii. $$\forall x \in X_1 \quad \forall a \in \{-1,1\} \quad \mathrm{F}(a, \{-\mu_x, \mu_x\}) = \mathrm{J}(\{-\mu_x, \mu_x\}) = \{\mu_x\}\,,$$

iv. $\forall x \in X_1 \quad \mathrm{S}(\{-\mu_x,\mu_x\}) = \mathrm{I}(\{-\mu_x,\mu_x\}) = \mathrm{B}(\{-\mu_x,\mu_x\}) = \{-\mu_x,\mu_x\}$.

5. $\overline{\overline{\mathsf{M}}}(X_1,S_1) = \overline{\mathsf{M}}(X_1,S_1) = \{A \subseteq X_1 \mid A \neq \varnothing \wedge \mathrm{card}(A \cap (-1,1)) \leq 1\}$.

(Caution: for each $p \in {X_1}^{X_1}$ by $-p$ we mean $x(-p) = -(xp)$ (for all $x \in X_1$).)

Proof.

1. We have the following cases:

- $a = -1,1$: For each $n \in \mathbf{N}$, define:

$$xf_n = \begin{cases} (2n-1)x + (2n-2)a & -1 \leq ax \leq -1 + \dfrac{1}{n} \\ \dfrac{x}{2n-1} + \dfrac{(2n-2)a}{2n-1} & -1 + \dfrac{1}{n} \leq ax \leq 1 \end{cases}.$$

For each $n \in \mathbf{N}$, $f_n \in S_1$, and $\lim\limits_{n \to +\infty} f_n = \eta_a$, thus $\mu_a = \eta_a \in \mathrm{E}(X_1)$.

- $-1 < a < 1$: Choose $m \in \mathbf{N}$ such that $\left\{a - \dfrac{1}{n+m} \mid n \in \mathbf{N}\right\} \cup \left\{a + \dfrac{1}{n+m} \mid n \in \mathbf{N}\right\}$ is a subset of $(-1,1)$. For each $n > m$ define:

$$xf_n = \begin{cases} (na+n-1)x + na + n - 2 & -1 \leq x \leq -1 + \dfrac{1}{n} \\ \dfrac{x}{n-1} + a & -1 + \dfrac{1}{n} \leq x \leq 1 - \dfrac{1}{n} \\ (-na+n-1)x + na - n + 2 & 1 - \dfrac{1}{n} \leq x \leq 1 \end{cases},$$

$$xg_n = \begin{cases} \dfrac{x}{n(a+1)} + \dfrac{1-n(a+1)}{n(a+1)} & -1 \leq x \leq a \\ x(2n-2) + \dfrac{(2an+1)(1-n)}{n} & a \leq x \leq a + \dfrac{1}{n} \\ \dfrac{-x}{n(a-1)+1} + \dfrac{n(a-1)+2}{n(a-1)+1} & a + \dfrac{1}{n} \leq x \leq 1 \end{cases}.$$

For each $n > m$, $f_n, g_n \in S_1$, and $\lim\limits_{n>m} f_n = \eta_a$, $\lim\limits_{n>m} g_n = \mu_a$ thus $\eta_a, \mu_a \in \mathrm{E}(X_1)$.

2.

iii. Let $a \in X_1$. Each $s \in S_1$ is monotone, thus if s is increasing we have $\mu_a s = \mu_a$ and if s is decreasing, then $\mu_a s = -\mu_a$, so $\mu_a S_1 = \{-\mu_a, \mu_a\}$ and $\mu_a\, \mathrm{E}(X_1) = \overline{\mu_a S_1} = \{-\mu_a, \mu_a\}$.

iv. Let $-1 < a < 1$. Each $s \in S_1$ is monotone, thus if s is increasing we have $\eta_a s = \eta_{as}$ and if s is decreasing, then $\eta_a s = -\eta_{-as}$. On the other hand, for each $-1 < b < 1$ there exists an increasing $s \in S_1$ such that $as = b$, thus $\eta_a S_1 = \{(-1)^k \eta_y \mid y \in (-1,1), k = 1,2\}$ and $\{(-1)^k \eta_y \mid y \in [-1,1], k = 1,2\} \subseteq \overline{\eta_a S_1} = \eta_a\, \mathrm{E}(X_1)$. Now let $\{\eta_{y_\alpha}\}_{\alpha \in \Gamma}$ be a convergent net in $\eta_a\, \mathrm{E}(X_1)$. By compactness of $[-1,1]$ there exists $y \in [-1,1]$ and a subnet of $\{y_\alpha\}_{\alpha \in \Gamma}$ like $\{y_{\alpha_\beta}\}_{\beta \in \Omega}$ such that $\lim\limits_{\beta \in \Omega} y_{\alpha_\beta} = y$, thus $\lim\limits_{\beta \in \Omega} \eta_{y_{\alpha_\beta}} = \eta_y$, so $\eta_a\, \mathrm{E}(X_1) = \overline{\eta_a S_1} \subseteq \{(-1)^k \eta_y \mid y \in [-1,1], k = 1,2\}$, which completes the proof.

3.

i. Use (i) and (ii) in item (2).

ii. Use (i) in item (2) and (i).

iii. Use (iii) in item (2).

iv. By a similar argument as in (iv) in item (2), for $a \in \{-1,1\}$ we have $\eta_a \, \mathrm{E}(X_1) = \{-\eta_a, \eta_a\}$. Now use (iv) in item (2).

v. By (i) the elements -1 and 1 are almost periodic, thus $\mathrm{M}(1) = \mathrm{M}(-1) = \overline{\mathrm{M}}(\{-1,1\}) = \overline{\overline{\mathrm{M}}}(\{-1,1\}) = \mathrm{Min}(\mathrm{E}(X_1))$,on the other hand by (iii) we have

$$\{\{-\mu_x, \mu_x\} \mid x \in X_1\} \subseteq \mathrm{Min}(\mathrm{E}(X_1)) .$$

vi. If $-1 < a < 1$ by (ii) in item (2), $a\,\mathrm{E}(X_1) = X_1$, on the other hand by (iv) in item (2) we have $a\eta_a \, \mathrm{E}(X_1) = a\{(-1)^k \eta_y \mid y \in X_1, k = 1,2\} = X_1$. Moreover, for each $b \in \{-1,1\}$ we have $a\{-\eta_b, \eta_b\} = \{-1,1\} \neq X_1$, which completes the proof by (iv).

vii. Use (vi).

4. For each $-1 < a < 1$ we have $\mathrm{J}(\mathrm{F}(a, \{(-1)^k \eta_y \mid y \in X_1, k = 1,2\})) = \{\eta_a\}$. Thus if $\mathrm{card}(A \cap (-1,1)) \leq 1$ (and $A \neq \varnothing$), then by (vii) and (v) in item (3), we have $A \in \overline{\mathbf{M}}(X_1, S_1)$, and if $\mathrm{card}(A \cap (-1,1)) \geq 2$ we have $\mathrm{J}(\mathrm{F}(A, \{(-1)^k \eta_y \mid y \in X_1, k = 1,2\})) = \varnothing$, but by (vii) in item (3) $\{(-1)^k \eta_y \mid y \in X_1, k = 1,2\} \in \overline{\mathrm{M}}(A)$, thus $A \notin \overline{\mathbf{M}}(X_1, S_1)$. Therefore $\overline{\mathbf{M}}(X_1, S_1) = \{A \subseteq X_1 \mid A \neq \varnothing \wedge \mathrm{card}(A \cap (-1,1)) \leq 1\}$. Now by a similar method described for (vi) in item (3), for each subset A of X_1 such that $A \cap (-1,1) \neq \varnothing$ we have $\{(-1)^k \eta_y \mid y \in X_1, k = 1,2\} \in \overline{\overline{\mathrm{M}}}(A)$, moreover if $\mathrm{card}(A \cap (-1,1)) \geq 2$, then

$$\mathrm{J}(\overline{\mathrm{F}}(A, \{(-1)^k \eta_y \mid y \in X_1, k = 1,2\})) = \mathrm{J}(\mathrm{F}(A, \{(-1)^k \eta_y \mid y \in X_1, k = 1,2\})) = \varnothing ,$$

which leads us to the desired result.

Example 2. Let X_2 be an infinite fort space with the particular point b (i.e., X_2 is infinite, $b \in X_2$, and X_2 is occupied with topology $\{U \subseteq X_2 \mid b \notin U \vee \mathrm{card}(X_2 - U) < \aleph_0\}$) $\xi : X_2 \to X_2$ is a one to one map such that for each $x \in X_2$ and $n \in \mathbf{N}$, $x\xi^n = x$ if and only if $x = b$, and let $S_2 = \{\xi^n \mid n \geq 0\}$ (S_2 has the discrete topology), then in the transformation semi group (X_2, S_2) we have:

1.

i. $\mathrm{E}(X_2) = S_2 \cup \{b\}$,

ii. $\overline{bS_2} = \{b\}$, b is the unique almost periodic point of (X_2, S_2), and $\{b\}$ is the unique minimal right ideal of $\mathrm{E}(X_2)$,

iii. if L is a right ideal of $\mathrm{E}(X_2)$ and $L \neq \{b\}$, then there exists $n \geq 0$ such that $L = \xi^n \, \mathrm{E}(X_2)$.

2. If A is a nonempty subset of X_2, then:

i. $\mathrm{M}(b) = \{\{b\}\}$,

ii. $\overline{\mathrm{M}}(A) = \{\mathrm{E}(X_2)\}$ (i.e., (X_2, S_2) is $A \frac{(\overline{\mathrm{M}})}{}$ distal) if and only if $A \neq \{b\}$,

iii. (X_2, S_2) is $A-$distal if and only if $b \notin A$,

iv. $\mathrm{J}(\mathrm{E}(X_2)) = \{b, \mathrm{id}_{X_2}\}$, $\mathrm{I}(\mathrm{E}(X_2)) = S_2$, $\mathrm{S}(\mathrm{E}(X_2)) = \mathrm{B}(\mathrm{E}(X_2)) = \{\mathrm{id}_{X_2}\}$,

v. $\overline{\mathbf{M}}(X_2, S_2) = \{A \subseteq X_2 \mid A \neq \varnothing\}$.

Proof. First note that $\xi : X_2 \to X_2$ is continuous. For this aim let U be an open subset of X_2 if $b \notin U$, then $b \notin \xi^{-1}(U)$ and $\xi^{-1}(U)$ is an open subset of X_2, also if $X_2 - U$ is finite since ξ is

$1-1$, so $X_2-\xi^{-1}(U)$ is finite too and $\xi^{-1}(U)$ is an open subset of X_2, thus $\xi: X_2 \to X_2$ is continuous.

1.

i. We claim that $\lim_{n\in\mathbf{N}} \xi^n = b$. Let $a \in X_2$ and U be an open neighborhood of b, then X_2-U is finite, by the hypothesis on ξ the set $\{m\in\mathbf{N} \mid a\xi^m \in (X_2-U)\}$ is finite and for each $k > \max\{m\in\mathbf{N} \mid a\xi^m \in (X_2-U)\}$, $a\xi^k \in U$, thus $\lim_{n\in\mathbf{N}} a\xi^n = b$ and $\lim_{n\in\mathbf{N}} \xi^n = b$, so $S_2 \cup \{b\} \subseteq \mathrm{E}(X_2)$. Moreover if $\vartheta \in \mathrm{E}(X_2)$, then there exists a net $\{\xi^{n_\alpha}\}_{\alpha\in\Gamma}$ such that $\lim_{\alpha\in\Gamma} \xi^{n_\alpha} = \vartheta$, if $\vartheta \neq b$ there exists $a \neq b$ such that $a\vartheta \neq b$ and $\lim_{\alpha\in\Gamma} a\xi^{n_\alpha} = a\vartheta$. The set $\{a\vartheta\}$ is an open neighborhood of $a\vartheta$, thus there exists $\beta \in \Gamma$ such that for each $\alpha \geq \beta$ we have $a\xi^{n_\alpha} \in \{a\vartheta\}$ and $a\xi^{n_\alpha} = a\vartheta$, thus for each $\alpha \geq \beta$ we have $n_\alpha = n_\beta$ (by our hypothesis on ξ and $a\vartheta \neq b$) thus $\vartheta = \xi^{n_\beta}$, so $\mathrm{E}(X_2) \subseteq S_2 \cup \{b\}$. Therefore $\mathrm{E}(X_2) = S_2 \cup \{b\}$.

ii. $\overline{bS_2} = \{b\}$ thus b is almost periodic. If $a \neq b$ then $b \in \overline{aS_2}$ (by (i)) and $a \notin \overline{bS_2} = \{b\}$ thus a is not almost periodic. Moreover, if I is a right ideal of $\mathrm{E}(X_2)$, then $\{b\} = (S_2 \cup \{b\})b = \mathrm{E}(X_2)b \supseteq Ib$, thus $\{b\} = Ib$ is a subset of I. Moreover $\{b\} = b(S_2 \cup \{b\}) = b\,\mathrm{E}(X_2)$, thus $\{b\}$ is a minimal right ideal of $\mathrm{E}(X_2)$, so $\{b\}$ is the unique minimal right ideal of $\mathrm{E}(X_2)$.

iii. Let $n = \min\{m \mid \xi^m \in L\}$, thus by (i) $L \subseteq \xi^n\,\mathrm{E}(X_2) = \{\xi^m \mid m \geq n\} \cup \{b\}$. On the other hand, let $m > n$ thus $\xi^m = \xi^n \xi^{m-n} \in L$ (since $\xi^n \in L$ and L is a right ideal of $\mathrm{E}(X_2)$) therefore $\{\xi^m \mid m \geq n\} \subseteq L$, thus $\{\xi^m \mid m \geq n\} \cup \{b\} \subseteq L$ (by the argument in (ii)). Therefore $L = \xi^n\,\mathrm{E}(X_2)$.

2. Let A be a nonempty subset of X_2.

i. Use (ii) in item (1).

ii. If $\overline{\overline{\mathrm{M}}}(A) = \{\mathrm{E}(X_2)\}$, then by (i) $A \neq \{b\}$. If $A \neq \{b\}$ choose $a \in A - \{b\}$ and $K \in \overline{\mathrm{M}}(A)$ so $a\,\mathrm{E}(X_2) = aK$, by (i) in item (1) there exists $n \geq 0$ such that $a\xi^n = a$ and $\xi^n \in K$. By our hypothesis on ξ we have $n = 0$ so $\mathrm{id}_{X_2} \in K$ and $K = \mathrm{E}(X_2)$.

iii, v. Use (ii).

Example 3. Let $X_3 = \left\{\frac{1}{n} \mid n \in \mathbf{N}\right\} \cup \{0\}$ (with the induced topology of $\mathbf{R}$), $\xi: X_3 \to X_3$ by $x\xi = \dfrac{x}{x+1}$ ($x \in X_3$) and let $S_3 = \{\xi^n \mid n \geq 0\}$ (S_3 has the discrete topology), then in the transformation semigroup (X_3, S_3), $\overline{\overline{\mathbf{M}}}(X_3,S_3) = \overline{\mathbf{M}}(X_3,S_3) = \{A \subseteq X_3 \mid A \neq \varnothing\}$, and for each nonempty subset A of X_3 we have $\overline{\mathrm{M}}(A) = \overline{\overline{\mathrm{M}}}(A)$ (this example is a special case of Example 2).

Proof. By Example 2 we have $\overline{\mathbf{M}}(X_3,S_3) = \{A \subseteq X_3 \mid A \neq \varnothing\}$. Now let $\{0\} \neq A \in \overline{\mathbf{M}}(X_3,S_3)$ and K be a closed right ideal of $\mathrm{E}(X_3)$ such that $AK = A\,\mathrm{E}(X_3)$. As $A \neq \{0\}$ we have $K \neq \{0\}$, thus by (iii) of item (1) in Example 2 there exists $n \geq 0$ such that $K = \xi^n\,\mathrm{E}(X_3)$ we have:

$$\min\left\{m \mid \frac{1}{m} \in A\,\mathrm{E}(X_3)\right\} + n = \min\left\{m \mid \frac{1}{m} \in A\xi^n\,\mathrm{E}(X_3)\right\}$$

$$= \min\left\{m \mid \frac{1}{m} \in AK\right\} = \min\left\{m \mid \frac{1}{m} \in A\,\mathrm{E}(X_3)\right\}$$

thus $n = 0$, $K = \mathrm{E}(X_3)$ and $\overline{\overline{\mathrm{M}}}(A) = \{\mathrm{E}(X_3)\}$. So $\overline{\overline{\mathrm{M}}}(A) = \{\mathrm{E}(X_3)\}$ if and only if $A \neq \{0\}$ and $\overline{\overline{\mathbf{M}}}(X_3,S_3) = \{A \subseteq X_3 \mid A \neq \varnothing\}$. Using (ii) in item (2) of Example 2 will complete the proof.

Example 4. Let $X_4 = \left\{ \frac{1}{n} \mid n \in \mathbf{N} \right\} \cup \{0\}$ (with the induced topology of $\mathbf{R}$). Define $\xi : X_4 \to X_4$ by:

$$x\xi = \begin{cases} \frac{x}{1-x} & x \in X_4 - \{1\} \\ 0 & x = 1 \end{cases}.$$

Take $S_4 = \{\xi^n \mid n \geq 0\}$ (S_4 with the discrete topology), then in the transformation semigroup (X_4, S_4) we have:

1.

i. $\mathrm{E}(X_4) = S_4 \cup \{0\}$,

ii. 0 is the unique almost periodic point of the transformation semigroup (X_4, S_4) and $\{0\}$ is the unique minimal right ideal of $\mathrm{E}(X_4)$,

iii. $\mathrm{M}(0) = \{\{0\}\}$.

2. If A is a nonempty subset of X_4, then:

i. $$\overline{\mathrm{M}}(A) = \begin{cases} \{\mathrm{E}(X_4)\} & A \neq \{0\} \quad ((X_4, S_4) \text{ is } A\text{-}\overline{(\mathrm{M})}\text{ distal}) \\ \{\{0\}\} & A = \{0\} \end{cases},$$

ii. $$\overline{\overline{\mathrm{M}}}(A) = \begin{cases} \varnothing & \mathrm{card}(A) = \aleph_0 \\ \{\{0\}\} & A = \{0\} \\ \{\mathrm{E}(X_4)\} & \text{otherwise} \quad ((X_4, S_4) \text{ is } A\text{-}\overline{\overline{(\mathrm{M})}}\text{ distal}) \end{cases},$$

iii. (X_4, S_4) is $A-$distal if and only if $0 \notin A$.

Proof.

1.

i. We have $\lim_{n \in \mathbf{N}} \xi^n = 0$, so $S_4 \cup \{0\} \subseteq \mathrm{E}(X_4)$. Moreover, for $\vartheta \in \mathrm{E}(X_4)$, there exists a net $\{\xi^{n_\alpha}\}_{\alpha \in \Gamma}$ such that $\lim_{\alpha \in \Gamma} \xi^{n_\alpha} = \vartheta$ and if $\vartheta \neq 0$, there exists $n \in \mathbf{N}$ such that $\frac{1}{n}\vartheta \neq 0$ and $\lim_{\alpha \in \Gamma} \frac{1}{n}\xi^{n_\alpha} = \frac{1}{n}\vartheta$, thus there exists $\beta \in \Gamma$ such that for each $\alpha \geq \beta$ we have $\frac{1}{n}\xi^{n_\alpha} = \frac{1}{n}\vartheta$, therefore for $\alpha \geq \beta$ we have $\frac{1}{n}\xi^{n_\alpha} = \frac{1}{n}\xi^{n_\beta} = \frac{1}{n}\vartheta \neq 0$ thus $\frac{1}{n - n_\alpha} = \frac{1}{n - n_\beta}$, i.e., $n_\alpha = n_\beta$ (for all $\alpha \geq \beta$), thus $\vartheta = \xi^{n_\beta}$ and $\mathrm{E}(X_4) \subseteq S_4 \cup \{0\}$. Therefore $\mathrm{E}(X_4) = S_4 \cup \{0\}$.

ii. $0\,\mathrm{E}(X_4) = \{0\}$, therefore 0 is almost periodic. For each $n \in \mathbf{N}$ we have $0 \in \frac{1}{n}\mathrm{E}(X_4) (= \frac{1}{n}S_4 = \left\{\frac{1}{m} \mid m \leq n\right\} \cup \{0\} = \left\{\frac{1}{m} \mid m \leq n\right\} \cup \{0\})$, but $\frac{1}{n} \notin 0\,\mathrm{E}(X_4) (= \{0\})$, therefore $\frac{1}{n}$ is not almost periodic. Moreover, if I is a right ideal of $\mathrm{E}(X_4)$, then $\{0\} = I0 \subseteq I\,\mathrm{E}(X_4) \subseteq I$, thus $\{0\}$ is the unique minimal right ideal of $\mathrm{E}(X_4)$.

iii. Use (ii).

2.

i. If $A \neq \{0\}$ choose $n \in \mathbf{N}$ such that $\frac{1}{n} \in A$ and $K \in \overline{\mathrm{M}}(A)$. As $\frac{1}{n}K = \frac{1}{n}\mathrm{E}(X_4)$ there exists $m \geq 0$ such that $\xi^m \in K$ and $\frac{1}{n}\xi^m = \frac{1}{n}$, therefore $m = 0$, $\mathrm{id}_{X_4} \in K$ and $K = \mathrm{E}(X_4)$. Using (iii) in item (1) will complete the proof.

ii. If $\mathrm{card}(A) < \aleph_0$ and $A \neq \{0\}$, let $m = \max\left\{n \mid \frac{1}{n} \in A\right\}$ and let K be a closed right ideal of $\mathrm{E}(X_4)$ such that $AK = A\mathrm{E}(X_4)$, (thus $A \subseteq AK$) so there exist $q \in \mathbf{N}$ and $p \in \mathbf{N} \cup \{0\}$ such that $\frac{1}{q} \in A$, $\xi^p \in K$ and $\frac{1}{m} = \frac{1}{q}\xi^p$ by $m \geq q$ we get $p = 0$, $\mathrm{id}_{X_4} \in K$ and $K = \mathrm{E}(X_4)$, thus $\overline{\overline{\mathrm{M}}}(A) \neq \varnothing$. If $\mathrm{card}(A) = \aleph_0$, then for each $n \in \mathbf{N}$ we have $A\xi^n\,\mathrm{E}(X_4) = A\mathrm{E}(X_4) = X_4$, and if K be a closed right ideal of $\mathrm{E}(X_4)$ such that $AK = A\mathrm{E}(X_4)$ we have $K \neq \{0\}$. Let $m = \min\{n \mid \xi^n \in K\}$ then $\xi^{m+1}\,\mathrm{E}(X_4)$ is a proper subset of K and a closed right ideal of $\mathrm{E}(X_4)$, moreover $A\xi^{m+1}\,\mathrm{E}(X_4) = A\mathrm{E}(X_4)$.

iii. Use (i).

Example 5. Let $X_5 = \left\{\frac{1}{n} \mid n \in \mathbf{N}\right\} \cup \{0\}$ (with the induced topology of $\mathbf{R}$), for each $n \in \mathbf{N}$, define the following maps:

$$x\rho_n = \begin{cases} 0 & x \neq \frac{1}{n} \\ x & x = \frac{1}{n} \end{cases}, \quad x\vartheta_n = \begin{cases} x & x \neq \frac{1}{n} \\ 0 & x = \frac{1}{n} \end{cases}, \quad x\psi = \begin{cases} x & x \neq 1, \frac{1}{2} \\ 1 & x = \frac{1}{2} \\ \frac{1}{2} & x = 1 \end{cases}.$$

Let S_5 be the semigroup generated by $\{\vartheta_n \mid n \in \mathbf{N}\} \cup \{\psi, \mathrm{id}_{X_5}\}$ (S_5 with the discrete topology), then in the transformation semigroup (X_5, S_5) we have:

1. $\mathrm{E}(X_5) = \{p\psi^i \mid (\exists A \subseteq X_5 \quad (p|_A = \mathrm{id}_A \wedge p|_{X_5 - A} = 0 \wedge 0p = 0)), i = 1,2\}$.

2. 0 is the unique almost periodic point of the transformation semigroup (X_5, S_5) and $\{0\}$ is the unique minimal right ideal of $\mathrm{E}(X_5)$.

3. $$\mathrm{M}(x) = \begin{cases} \{\{0\}\} & x = 0 \\ \{\{\rho_n, \rho_n\psi, 0\}\} & x = \frac{1}{n}, n = 1,2 \\ \{\{\rho_n, 0\}\} & x = \frac{1}{n}, n \geq 3 \end{cases}.$$

4. $$\overline{\overline{\mathrm{M}}}(\left\{1, \frac{1}{2}\right\}) = \mathrm{M}(1) \cup \mathrm{M}(\frac{1}{2}) = \{\{\rho_n, \rho_n\psi, 0\} \mid n = 1,2\}.$$

5. $\forall A \subseteq X_5 \quad (0 < \mathrm{card}(A) < \aleph_0 \Rightarrow \overline{\overline{\mathrm{M}}}(A) = \left\{\left\{\rho_n \psi^i \mid \frac{1}{n} \in A, i = 1,2\right\} \cup \{0\}\right\})$.

Proof. Let $n \in \mathbf{N}$, for $k > n$ define $\eta_k = \vartheta_1 \cdots \vartheta_{n-1}\vartheta_{n+1} \cdots \vartheta_k$, then we have $\rho_n = \lim_{k>n} \eta_k$, thus $\rho_n \in \mathrm{E}(X_5)$.

1. Use the fact that for each $p : X_5 \to X_5$, there exists a finite subset A of X_5 such that $p|_{X_5 - A} = \mathrm{id}_{X_5 - A}$ and $p|_A = 0$ if and only if there exists $k_1, ..., k_n \in \mathbf{N}$ such that $p = \vartheta_{k_1} \cdots \vartheta_{k_n}$. Moreover $\psi^2 = \mathrm{id}_{X_5}$. For $n \geq 3$ we have $\psi\vartheta_n = \vartheta_n\psi$ and for $n, m \in \mathbf{N}$ we have $\vartheta_n\vartheta_m = \vartheta_m\vartheta_n$, $\vartheta_1\psi = \psi\vartheta_2$ and $\vartheta_2\psi = \psi\vartheta_1$.

2. $\{0\} = 0\,\mathrm{E}(X_5)$ thus 0 is almost periodic, for each $n \in \mathbf{N}$ we have $0 = \frac{1}{n}\vartheta_n \in \frac{1}{n}\mathrm{E}(X_5)$, but $\frac{1}{n} \notin \{0\} = 0\,\mathrm{E}(X_5)$ thus $\frac{1}{n}$ is not quite periodic. Moreover for each right ideal I of $\mathrm{E}(X_5)$ we have $0\,\mathrm{E}(X_5) = \{0\} = I0 \subseteq I\,\mathrm{E}(X_5) \subseteq I$, thus $\{0\}$ is the unique minimal right ideal of $\mathrm{E}(X_5)$.

3. Using (2) we have $\mathrm{M}(0) = \{\{0\}\}$. For each $n \in \mathbf{N}$ we have $\rho_n\,\mathrm{E}(X_5) = \rho_n\psi\,\mathrm{E}(X_5) = \{\rho_n, \rho_n\psi, 0\}$, thus $\{\rho_n, \rho_n\psi, 0\}$ is a closed right ideal of $\mathrm{E}(X_5)$, moreover $\frac{1}{n}\mathrm{E}(X_5) = \left\{\frac{1}{n}, \frac{1}{n}\psi, 0\right\} = \frac{1}{n}\{\rho_n, \rho_n\psi, 0\}$. On the other hand, $\{0\}$ is the only proper subset of $\{\rho_n, \rho_n\psi, 0\}$ such that it is a right ideal of $\mathrm{E}(X_5)$, but $\frac{1}{n}\{0\} = \{0\} \neq \left\{\frac{1}{n}, \frac{1}{n}\psi, 0\right\} = \{0\}\{\rho_n, \rho_n\psi, 0\}$ thus $\{\rho_n, \rho_n\psi, 0\} \in \mathrm{M}(\frac{1}{n})$. Conversely, if $K \in \mathrm{M}(\frac{1}{n})$, then $\frac{1}{n}\mathrm{E}(X_5) = \frac{1}{n}K$ thus there exists $p \in K$ such that $\frac{1}{n}p = \frac{1}{n}$, therefore $\frac{1}{n}p\rho_n = \frac{1}{n}\rho_n$. It is easy to verify that $p\rho_n = \rho_n$. But $p\rho_n \in K$ thus $\rho_n \in K$, so $\{\rho_n, \rho_n\psi, 0\} = \rho_n\,\mathrm{E}(X_5) \subseteq K$. As $\{\rho_n, \rho_n\psi, 0\}, K \in \mathrm{M}(\frac{1}{n})$ we get $K = \{\rho_n, \rho_n\psi, 0\}$ and $\mathrm{M}(\frac{1}{n}) = \{\{\rho_n, \rho_n\psi, 0\}\}$. Now if $n \geq 3$, then $\rho_n = \rho_n\psi$ so $\mathrm{M}(\frac{1}{n}) = \{\{\rho_n, 0\}\}$.

4. By the argument in (3) we have $\mathrm{M}(1) = \{\{\rho_1, \rho_1\psi, 0\}\}$ and $\mathrm{M}(\frac{1}{2}) = \{\{\rho_2, \rho_2\psi, 0\}\}$, moreover $\left\{1, \frac{1}{2}\right\}\{\rho_2, \rho_2\psi, 0\} = \left\{0, \frac{1}{2}, 1\right\} = \left\{1, \frac{1}{2}\right\}\mathrm{E}(X_5) = \left\{1, \frac{1}{2}\right\}\{\rho_1, \rho_1\psi, 0\}$ which shows $\mathrm{M}(1) \cup \mathrm{M}(\frac{1}{2}) \subseteq \overline{\overline{\mathrm{M}}}(\left\{1, \frac{1}{2}\right\})$. On the other hand, if $K \in \overline{\overline{\mathrm{M}}}(\left\{1, \frac{1}{2}\right\})$, then $\left\{1, \frac{1}{2}\right\}K = \left\{1, \frac{1}{2}\right\}\mathrm{E}(X_5) = \left\{0, \frac{1}{2}, 1\right\}$ thus there exists $p \in K$ such that $1p = 1$ or $\frac{1}{2}p = 1$ thus $1p\rho_1 = 1\rho_1 = 1$ or $\frac{1}{2}p\psi\rho_2 = 1\psi\rho_2 = \frac{1}{2} = \frac{1}{2}\rho_2$, therefore $p\rho_1 = \rho_1$ or $p\psi\rho_2 = \rho_2$, since $p \in K$ we have $\rho_1 \in K$ or $\rho_2 \in K$, thus $\{\rho_1, \rho_1\psi, 0\} = \rho_1\,\mathrm{E}(X_5) \subseteq K$ or $\{\rho_2, \rho_2\psi, 0\} = \rho_2\,\mathrm{E}(X_5) \subseteq K$. Since $\{\rho_1, \rho_1\psi, 0\} \in \overline{\overline{\mathrm{M}}}(\left\{1, \frac{1}{2}\right\})$ and $\{\rho_2, \rho_2\psi, 0\} \in \overline{\overline{\mathrm{M}}}(\left\{1, \frac{1}{2}\right\})$, we have $K = \{\rho_1, \rho_1\psi, 0\}$ or $K = \{\rho_2, \rho_2\psi, 0\}$. Therefore $\overline{\overline{\mathrm{M}}}(\left\{1, \frac{1}{2}\right\}) = \{\{\rho_1, \rho_1\psi, 0\}, \{\rho_2, \rho_2\psi, 0\}\} = \mathrm{M}(1) \cup \mathrm{M}(\frac{1}{2})$.

5. Let A be a nonempty finite subset of X_5. Then for each $m \in \mathbf{N}$ with $\frac{1}{m} \in A$ we have (by (3)) $\frac{1}{m}\mathrm{E}(X_5) = \frac{1}{m}\{\rho_m, \rho_m\psi, 0\} \subseteq \frac{1}{m}(\left\{\rho_n\psi^i \mid \frac{1}{n} \in A, i = 1,2\right\} \cup \{0\}) \subseteq \frac{1}{m}\mathrm{E}(X_5)$, thus for each $a \in A$ we have $a\,\mathrm{E}(X_5) = a(\left\{\rho_n\psi^i \mid \frac{1}{n} \in A, i = 1,2\right\} \cup \{0\})$. Since A is finite, $\left\{\rho_n\psi^i \mid \frac{1}{n} \in A, i = 1,2\right\} \cup \{0\}$ is finite and closed, hence

$$\begin{aligned}(\left\{\rho_n\psi^i \mid \frac{1}{n}\in A, i=1,2\right\}\cup\{0\})\,\mathrm{E}(X_5) &= \bigcup\left\{\{\rho_n,\rho_n\psi,0\}\,\mathrm{E}(X_5)\mid \frac{1}{n}\in A\right\}\\ &\subseteq \bigcup\left\{\{\rho_n,\rho_n\psi,0\}\mid \frac{1}{n}\in A\right\}\\ &= \left\{\rho_n\psi^i \mid \frac{1}{n}\in A, i=1,2\right\}\cup\{0\}\end{aligned}$$

thus $\left\{\rho_n\psi^i \mid \frac{1}{n}\in A, i=1,2\right\}\cup\{0\}$ is a closed right ideal of $\mathrm{E}(X_5)$. Let K be a closed right ideal of $\mathrm{E}(X_5)$ such that $aK = a\,\mathrm{E}(X_5)$ for each $a\in A$, thus there exists $L\in\overline{\mathrm{M}}(A)$ such that $L\subseteq K$. By (3) we have $\left\{\rho_n\psi^i \mid \frac{1}{n}\in A, i=1,2\right\}\cup\{0\}\subseteq K$, therefore

$$\left\{\rho_n\psi^i \mid \frac{1}{n}\in A, i=1,2\right\}\cup\{0\} = K\in\overline{\mathrm{M}}(A) \text{ and}$$

$$\overline{\mathrm{M}}(A) = \left\{\left\{\rho_n\psi^i \mid \frac{1}{n}\in A, i=1,2\right\}\cup\{0\}\right\}.$$

Example 6. Let $X_6 = \left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}\cup\{0\}$ (with the induced topology of $\mathbf{R}$), define $\tau: X_6\to X_6$ by

$$x\tau = \begin{cases} x & x\neq \frac{1}{3},\frac{1}{4}\\ \frac{1}{3} & x=\frac{1}{4}\\ \frac{1}{4} & x=\frac{1}{3}\end{cases}.$$

With the same assumptions as in Example 5 let S_6 be the semigroup generated by $\{\vartheta_n \mid n\in\mathbf{N}\}\cup\{\psi\tau,\mathrm{id}_{X_6}\}$ (S_6 with the discrete topology), and let S_6' be the semigoup generated by $\{\vartheta_n \mid n\in\mathbf{N}\}\cup\{\psi,\tau,\mathrm{id}_{X_6}\}$ (S_6' with the discrete topology), then in the transformation semigroups (X_6,S_6) and (X_6,S_6') we have

1. $\mathrm{E}(X_6,S_6)\subseteq \mathrm{E}(X_6,S_6') = \{p\tau^i \mid p\in\mathrm{E}(X_5,S_5), i=1,2\}$.

2. In the transformation semigroup (X_6,S_6) we have

i. 0 is the unique, almost periodic point of the transformation semigroup (X_6,S_6), and $\{0\}$ is the unique minimal right ideal of $\mathrm{E}(X_6)$,

ii. $$\mathrm{M}(x) = \begin{cases} \{\{0\}\} & x=0\\ \{\{\rho_n,\rho_n\psi,0\}\} & x=\frac{1}{n}, n=1,2\\ \{\{\rho_n,\rho_n\tau,0\}\} & x=\frac{1}{n}, n=3,4\\ \{\{\rho_n,0\}\} & x=\frac{1}{n}, n\geq 5\end{cases},$$

iii. if A is a nonempty finite subset of X_6, then

iv. $$\overline{\mathrm{M}}(A)=\left\{\left\{\rho_n\psi^i\tau^j \mid \frac{1}{n}\in A, i,j\in\{1,2\}\right\}\cup\{0\}\right\},$$

v. $$\forall n\in\{1,3\}\quad \overline{\overline{\mathrm{M}}}(\left\{\frac{1}{n},\frac{1}{n+1}\right\})=\mathrm{M}(\frac{1}{n})\cup\mathrm{M}(\frac{1}{n+1}),$$

vi. $$\forall m\in\{1,2\}\quad \forall n\in\{3,4\}\quad \overline{\overline{\mathrm{M}}}(\left\{\frac{1}{n},\frac{1}{m}\right\})=\{\{\rho_n,\rho_n\tau,\rho_m,\rho_m\psi,0\}\},$$

vii. $$\overline{\overline{\mathrm{M}}}(\left\{\frac{1}{n}\mid 1\le n\le 4\right\})=\{\{\rho_n,\rho_n\tau,\rho_m,\rho_m\psi,0\}\mid m\in\{1,2\},n\in\{3,4\}\}.$$

Proof. Use a similar method described in Example 5.

Example 7. Let $X_7=\left\{\frac{1}{n}+\frac{1}{m}\mid n,m\in\mathbf{N},m>n(n-1)\right\}\cup\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}\cup\{0\}$ (with the induced topology of $\mathbf{R}$), consider the following maps on X_7:

$$x\vartheta=\begin{cases}\frac{1}{n}+\frac{1}{m+1} & n,m\in\mathbf{N},m>n(n-1),x=\frac{1}{n}+\frac{1}{m},\\ x & \text{otherwise}\end{cases}$$

$$x\psi=\begin{cases}\frac{1}{n} & n,m\in\mathbf{N},m>n(n-1),x=\frac{1}{n}+\frac{1}{m}\\ x & \text{otherwise}\end{cases}$$

and let $S_7=\{\vartheta^n\mid n\ge 0\}$ (S_7 with the discrete topology), then in the transformation semigroup (X_7,S_7) we have

1. $$\mathrm{E}(X_7)=S_7\cup\{\psi\}.$$

2. $$\overline{xS_7}=\begin{cases}\{x\} & x\in\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}\cup\{0\}\\ \left\{\frac{1}{n}+\frac{1}{k}\mid k\in\mathbf{N},k\ge m\right\}\cup\left\{\frac{1}{n}\right\} & n,m\in\mathbf{N},m>n(n-1),x=\frac{1}{n}+\frac{1}{m}\end{cases}.$$

3. $\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}\cup\{0\}$ is the set of almost all periodic points of (X_7,S_7) and $\{\psi\}$ is the unique minimal right ideal of $\mathrm{E}(X_7)$.

4. If A is a nonempty subset of X_7, then:

i. $$\overline{\mathrm{M}}(A)=\begin{cases}\{\{\psi\}\} & A\subseteq\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}\cup\{0\}\\ \mathrm{E}(X_7) & A\not\subset\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}\cup\{0\}\quad ((X_7,S_7)\text{ is }A\text{-}\overline{(\mathrm{M})}\text{ distal})\end{cases}$$

ii. $$(A\subseteq\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}\cup\{0\})\Leftrightarrow(\overline{\overline{\mathrm{M}}}(A)=\{\{\psi\}\}),$$

iii. $A\cap(\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}\cup\{0\})=\varnothing$ if and only if (X_7,S_7) is $A-$distal.

Proof.

1. Since $\lim\limits_{n\in\mathbf{N}}\vartheta^n=\psi$, $S_7\cup\{\psi\}\subseteq \mathrm{E}(X_7)$. On the other hand, let $p\in\mathrm{E}(X_7)$ and $\{\vartheta^{n_\alpha}\}_{\alpha\in\Gamma}$ be a net such that $\lim\limits_{\alpha\in\Gamma}\vartheta^{n_\alpha}=p$. We have

- $$\forall x\in\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}\cup\{0\}\quad xp=\lim_{\alpha\in\Gamma}x\vartheta^{n_\alpha}=\lim_{\alpha\in\Gamma}x=x,$$

- for all $m,n\in\mathbf{N}$ such that $m>n(n-1)$ we have

$$(\forall\alpha\in\Gamma\quad(\frac{1}{n}+\frac{1}{m})\vartheta^{n_\alpha}\in\left\{\frac{1}{n}+\frac{1}{k}\mid k\in\mathbf{N},k\geq m\right\})$$

$$\Rightarrow(\forall\alpha\in\Gamma\quad(\frac{1}{n}+\frac{1}{m})\vartheta^{n_\alpha}\in\left\{\frac{1}{n}+\frac{1}{k}\mid k\in\mathbf{N},k\geq m\right\}\cup\left\{\frac{1}{n}\right\})$$

$$\Rightarrow\lim_{\alpha\in\Gamma}(\frac{1}{n}+\frac{1}{m})\vartheta^{n_\alpha}\in\left\{\frac{1}{n}+\frac{1}{k}\mid k\in\mathbf{N},k\geq m\right\}\cup\left\{\frac{1}{n}\right\}$$

$$\Rightarrow(\frac{1}{n}+\frac{1}{m})p\in\left\{\frac{1}{n}+\frac{1}{k}\mid k\in\mathbf{N},k\geq m\right\}\cup\left\{\frac{1}{n}\right\}.$$

Whenever $p\neq\psi$ there exist $m,n,k\in\mathbf{N}$ such that $k\geq m>n(n-1)$ and $\lim\limits_{\alpha\in\Gamma}(\frac{1}{n}+\frac{1}{m})\vartheta^{n_\alpha}=\frac{1}{n}+\frac{1}{k}$, further,

$$\lim_{\alpha\in\Gamma}(\frac{1}{n}+\frac{1}{m})\vartheta^{n_\alpha}=\frac{1}{n}+\frac{1}{k}$$

$$\Rightarrow(\exists\beta\in\Gamma\quad\forall\alpha\in\Gamma\quad(\alpha\geq\beta\Rightarrow(\frac{1}{n}+\frac{1}{m})\vartheta^{n_\alpha}=\frac{1}{n}+\frac{1}{k}))$$

$$\Rightarrow(\exists\beta\in\Gamma\quad\forall\alpha\in\Gamma\quad(\alpha\geq\beta\Rightarrow n_\alpha=n_\beta))$$

$$\Rightarrow(\exists\beta\in\Gamma\quad\vartheta^{n_\beta}=\lim_{\alpha\in\Gamma}\vartheta^{n_\alpha}=p)$$

$$\Rightarrow p\in S_7$$

thus $\mathrm{E}(X_7)\subseteq S_7\cup\{\psi\}$.

4. i. Let A be a nonempty subset of X_7 such that $A\not\subset\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}\cup\{0\}$ and $K\in\overline{\mathrm{M}}(A)$. Let $x\in A-(\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}\cup\{0\})$, thus $xK=x\,\mathrm{E}(X_7)$, therefore, $\mathrm{J}(\mathrm{F}(x,K))\neq\varnothing$. Using $\mathrm{J}(\mathrm{E}(X_7))=\{\mathrm{id}_{X_7},\psi\}$, we have $\mathrm{id}_{X_7}\in K$ and $K=\mathrm{E}(X_7)$. Therefore $\overline{\mathrm{M}}(A)=\{\mathrm{E}(X_7)\}$.

Example 8. Let $X_8 = \left\{\frac{1}{n}+\frac{1}{m} \mid n,m\in\mathbf{N}, m>n(n-1)\right\} \cup \left\{\frac{1}{n} \mid n\in\mathbf{N}\right\} \cup \{0\}$ (with the induced topology of $\mathbf{R}$) and let S_8 be the group of all homeomorphisms on X_8 (S_8 with the discrete topology), then in the transformation semigroup (X_8, S_8) we have

1. 0 is the unique almost periodic point of the transformation semigroup (X_8, S_8), $\{0\}$ is the unique minimal right ideal of $\mathrm{E}(X_8)$ and $\mathrm{M}(0)=\{\{0\}\}$.

2. $$\forall n\in\mathbf{N} \quad (n\geq 2 \Rightarrow \overline{\frac{1}{n}S_8} = \left\{\frac{1}{m} \mid m\in\mathbf{N}, m\geq 2\right\} \cup \{0\}).$$

3. $$\forall x\in X_8 - (\left\{\frac{1}{n} \mid m\in\mathbf{N}, m\geq 2\right\} \cup \{0\}) \quad \overline{xS_8} = X_8.$$

4. If A is a nonempty subset of X_8, then (X_8, S_8) is not $A-$distal.

Proof. For each $m,n\in\mathbf{N}$ define

$$x\eta_{m,n} = \begin{cases} \frac{1}{n} & x=\frac{1}{m} \\ \frac{1}{m} & x=\frac{1}{n} \\ \frac{1}{m}+\frac{1}{k-n(n-1)+m(m-1)} & k\in\mathbf{N}, k>n(n-1), x=\frac{1}{n}+\frac{1}{k} \\ \frac{1}{n}+\frac{1}{k-m(m-1)+n(n-1)} & k\in\mathbf{N}, k>m(m-1), x=\frac{1}{m}+\frac{1}{k} \\ x & \text{otherwise} \end{cases},$$

$$x\eta_n = \begin{cases} 0 & x=\frac{1}{n} \\ 0 & k\in\mathbf{N}, k>n(n-1), x=\frac{1}{n}+\frac{1}{k}, \\ x & \text{otherwise} \end{cases}$$

$\eta_{m,n}\in S_8$; and for each $n\in\mathbf{N}$, $\lim_{m\in\mathbf{N}}\eta_{m,n}=\eta_n$. Now use a similar method described for the previous examples.

Since the following examples will not be used to complete the tables, we have omitted their proofs.

Example 9. Let $X_9 = (\bigcup_{n\in\mathbf{N}}\left[\frac{1}{2n},\frac{1}{2n-1}\right]) \cup \{0\}$ (with the induced topology of $\mathbf{R}$), let S_9 be the group of all homeomorphisms on X_9 (S_9 with the discrete topology), for each $m,n\in\mathbf{N}$ define

$$xp_{n,m}=\begin{cases}\frac{1}{2m} & x=\frac{1}{2n-1}\\ \frac{1}{2m-1} & \frac{1}{2n}\le x<\frac{1}{2n-1}\\ 0 & \text{otherwise}\end{cases},\quad xq_{n,m}=\begin{cases}\frac{1}{2m-1} & x=\frac{1}{2n-1}\\ \frac{1}{2m} & \frac{1}{2n}\le x<\frac{1}{2n-1}\\ 0 & \text{otherwise}\end{cases},$$

then in the transformation semigroup (X_9,S_9) we have

1. 0 is the unique almost periodic point of the transformation semigroup (X_9,S_9), $\{0\}$ is the unique minimal right ideal of $\mathrm{E}(X_9)$ and $\mathrm{M}(0)=\{\{0\}\}$.
2. For each $n\in\mathbf{N}$ we have:

   i. $$\frac{1}{n}S_9=\left\{\frac{1}{m}\mid m\in\mathbf{N}\right\}\wedge\overline{\frac{1}{n}S_9}=\left\{\frac{1}{m}\mid m\in\mathbf{N}\right\}\cup\{0\},$$

   ii. $$p_{n,n}\,\mathrm{E}(X_9)=q_{n,n}\,\mathrm{E}(X_9)=\{p_{n,m}\mid m\in\mathbf{N}\}\cup\{q_{n,m}\mid m\in\mathbf{N}\}\cup\{0\},$$

   iii. $$p_{n,n}\,\mathrm{E}(X_9)p_{n,n}=\{p_{n,n},q_{n,n},0\},$$

   iv. $$p_{n,n}\,\mathrm{E}(X_9)\in\mathrm{M}(\frac{1}{2n})\cap\mathrm{M}(\frac{1}{2n-1}),\ \mathrm{J}(p_{n,n}\,\mathrm{E}(X_9))=\{0,q_{n,n}\},$$

   v. $$\mathrm{F}(\frac{1}{2n},p_{n,n}\,\mathrm{E}(X_9))=\mathrm{F}(\frac{1}{2n-1},p_{n,n}\,\mathrm{E}(X_9))=\{q_{n,n}\},$$

   vi. $$\mathrm{S}(p_{n,n}\,\mathrm{E}(X_9))=\mathrm{B}(p_{n,n}\,\mathrm{E}(X_9))=\mathrm{I}(p_{n,n}\,\mathrm{E}(X_9))=\{p_{n,n},q_{n,n}\}.$$

3. Let $A=\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}$, $K=\{p_{n,m}\mid n,m\in\mathbf{N}\}\cup\{q_{n,m}\mid n,m\in\mathbf{N}\}\cup\{0\}$, and for each $i,j\in\mathbf{N}$ define $\delta_{ij}=0$ if $i\neq j$ and $\delta_{ij}=1$, thus we have

   i. $$\forall m,n,k\in\mathbf{N}\quad p_{n,m}p_{k,r}=\delta_{mk}p_{n,r},$$

   ii. $$K\in\overline{\mathrm{M}}(A),\ \mathrm{S}(K)=\mathrm{B}(K)=\mathrm{I}(K)=\mathrm{F}(A,K)=\overline{\mathrm{F}}(A,K)=\varnothing.$$

Example 10. Let $n\in\mathbf{N}$:

$$X_{10}=\left\{(x,y)\in\mathbf{R}^2\mid |y|=1-4(x-m)^2,m-\frac{1}{2}\le x\le m+\frac{1}{2},m\in\{-n,-n+1,\ldots,n-1,n\}\right\}$$

(with the induced topology of $\mathbf{R}^2$) and let S_{10} be the group of all homeomorphisms like $f:X_{10}\to X_{10}$ such that $\left\{(-n-\frac{1}{2},0),(n+\frac{1}{2},0)\right\}f=\left\{(-n-\frac{1}{2},0),(n+\frac{1}{2},0)\right\}$ (S_{10} with the discrete topology), then in the transformation semigroup (X_{10},S_{10}) we have

1. $$\forall m\in\{-n,-n+1,\ldots,n,n+1\}\quad (m-\frac{1}{2},0)S_{10}=\left\{(m-\frac{1}{2},0),(\frac{1}{2}-m,0)\right\}.$$

2. $\left\{(m-\frac{1}{2},0)\mid m\in\{-n,-n+1,\ldots,n,n+1\}\right\}$ is the set of all almost periodic points in the transformation semigroup (X_{10},S_{10}).

3. (X_{10}, S_{10}) has $2n+2$ almost periodic point and $n+1$ minimal subset, moreover each of its minimal subsets has 2 elements.
4. For each $m \in \{-n, -n+1, ..., n\}$ and $(x, y) \in X_{10}$ if $m - \frac{1}{2} < x < m + \frac{1}{2}$, then

$$\overline{(x, y)S_{10}} = \left\{(u, v) \in X_{10} \mid (m - \frac{1}{2} \le u \le m + \frac{1}{2}) \vee (-m - \frac{1}{2} \le u \le -m + \frac{1}{2})\right\}.$$

1. Let $\{(x_m, y_m)\}_{-n \le m \le n}$ and $\{(x'_m, y'_m)\}_{-n \le m \le n}$ be finite sequences in X_{10}, such that for each $-n \le m \le n$ we have $m - \frac{1}{2} \le x_m \le m + \frac{1}{2}$, $m - \frac{1}{2} \le x'_m \le m + \frac{1}{2}$, $y_m \ge 0$, and $y'_m \le 0$, choose $p \in X_{10}{}^{X_{10}}$ such that for each $-n \le m \le n$, $\{(x_m, y_m), (x'_m, y'_m)\}p \subseteq \left\{(m - \frac{1}{2}, 0), (m + \frac{1}{2}, 0)\right\}$ and

$$(x, y)p = \begin{cases} (m - \frac{1}{2}, 0) & (m - \frac{1}{2} \le x < x_m \wedge y \ge 0) \vee (m - \frac{1}{2} \le x < x'_m \wedge y \le 0) \vee x = m - \frac{1}{2} \\ (m + \frac{1}{2}, 0) & (x_m < x \le m + \frac{1}{2} \wedge y \ge 0) \vee (x'_m < x \le m + \frac{1}{2} \wedge y \le 0) \vee x = m + \frac{1}{2} \end{cases}$$

then $p \in \mathrm{E}(X_{10})$, $\{-p, p\}$ is a minimal right ideal of $\mathrm{E}(X_{10})$. In addition, each minimal right ideal of $\mathrm{E}(X_{10})$ has a similar structure, so $\mathrm{card}(\mathrm{Min}(\mathrm{E}(X_{10}))) = c (= \mathrm{card}(\mathbf{R}))$.
2. For each nonempty subset A of X_{10}, (X_{10}, S_{10}) is not $A-$distal.
3. (X_{10}, S_{10}) is not point transitive.

Example 11. Let $n \in \mathbf{N}$:

$$X_{11} = \{(x, y) \in \mathbf{R}^2 \mid |y| = 2((x - m) - (x - m)^2), m \le x \le m+1, m \in \{-n, -n+1, ..., n-1\}\}$$

(with the induced topology of $\mathbf{R}^2$) and let S_{11} be the group of all homeomorphisms like $f : X_{11} \to X_{11}$ such that $\{(-n, 0), (n, 0)\}f = \{(-n, 0), (n, 0)\}$ (S_{11} with the discrete topology), then in the transformation semigroup (X_{11}, S_{11}) we have

1. $\forall m \in \{-n, -n+1, ..., n\} \quad (m, 0)S_{11} = \overline{(m, 0)S_{11}} = \{(m, 0), (-m, 0)\}$.

2. $\{(m, 0) \mid m \in \{-n, -n+1, ..., n\}\}$ is the set of nearly all periodic points in the transformation semigroup (X_{11}, S_{11})
3. (X_{11}, S_{11}) has $2n+1$ almost periodic point and $n+1$ minimal subset (n minimal subset with two elements and a singleton minimal subset).
4. For each $m \in \{-n, -n+1, ..., n-1\}$ and $(x, y) \in X_{11}$ if $m < x < m+1$, then

$$\overline{(x, y)S_{11}} = \{(u, v) \in X_{11} \mid (m \le u \le m+1) \vee (-m-1 \le u \le -m)\}.$$

5. Let $\{(x_m, y_m)\}_{-n \le m \le n-1}$ and $\{(x'_m, y'_m)\}_{-n \le m \le n-1}$ be finite sequences in X_{11}, such that for each $-n \le m \le n-1$ we have $m \le x_m \le m+1$, $m \le x'_m \le m+1$, $y_m \ge 0$, and $y'_m \le 0$. Choose $p \in X_{11}{}^{X_{11}}$ such that for each $-n \le m \le n-1$, $\{(x_m, y_m), (x'_m, y'_m)\}p \subseteq \{(m, 0), (m+1, 0)\}$ and

$$(x, y)p = \begin{cases} (m, 0) & (m \le x < x_m \wedge y \ge 0) \vee (m \le x < x'_m \wedge y \le 0) \vee x = m \\ (m+1, 0) & (x_m < x \le m+1 \wedge y \ge 0) \vee (x'_m < x \le m+1 \wedge y \le 0) \vee x = m+1 \end{cases}$$

then $p \in \mathrm{E}(X_{11})$, $\{-p, p\}$ is a minimal right ideal of $\mathrm{E}(X_{11})$. In addition, each minimal right ideal of $\mathrm{E}(X_{11})$ has a similar structure (therefore has two elements), so $\mathrm{card}(\mathrm{Min}(\mathrm{E}(X_{10}))) = c(= \mathrm{card}(\mathbf{R}))$.

1. For each nonempty subset A of X_{11}, (X_{11}, S_{11}) is not $A-$distal.
2. (X_{11}, S_{11}) is not point transitive.

Example 12. In Example 11, let $X_{12} = \dfrac{X_{11} \cap ([0,n]\times[-1,1])}{\{(0,0),(n,0)\}}$ (with the quotient topology), for each $(x,y) \in X_{11} \cap ([0,n]\times[-1,1])$ the image of (x,y) under the quotient map is denoted by $[x,y]_\pi$, and let S_{12} be the group of all homeomorphisms like $f: X_{12} \to X_{12}$ (S_{12} with the discrete topology), then in the transformation semigroup (X_{12}, S_{12}) we have

1. $\forall m \in \{1,\dots,n-1\} \quad [m,0]_\pi S_{12} = \overline{[m,0]_\pi S_{12}} = \{[k,0]_\pi \mid k \in \{1,\dots,n\}\}$.
2. $\forall [x,y]_\pi \in X_{12} - \{[k,0]_\pi \mid k \in \{1,\dots,n\}\}$

$$([x,y]_\pi S_{12} = X_{12} - \{[k,0]_\pi \mid k \in \{1,\dots,n\}\} \wedge \overline{[x,y]_\pi S_{12}} = X_{12}).$$

3. $\{[k,0]_\pi \mid k \in \{1,\dots,n\}\}$ is the set of all almost periodic points, the unique minimal subset and the unique proper closed invariant subset of (X_{12}, S_{12}).

Completion 13. We have the following table (see [3], Conclusion 16, Table 1), where T1 is the affiliation: The mark "√" indicates that if (X,S) is a transformation semigroup, $a \in X$ and $K \in \mathrm{M}(a)$, then $\Gamma \subseteq \Omega$. The mark "+" indicates that a transformation semigroup (X,S) exists, $a \in X$ and $K \in \mathrm{M}(a)$, such that $\Gamma \subseteq \Omega$. The mark "–" indicates that a transformation semigroup (X,S), exists, $a \in X$ and $K \in \mathrm{M}(a)$, such that $\Gamma \not\subset \Omega$. The mark "±" indicates "+" and "–".

Table 1. In the corresponding case T1 is valid

	1st. column ↓		3rd. column ↓		5th. column ↓
$\Gamma \backslash \Omega$	$\mathrm{F}(a,K)$	$\mathrm{B}(K)$	$\mathrm{S}(K)$	$\mathrm{I}(K)$	K
$\mathrm{F}(a,K)$	√	√	√	√	√
$\mathrm{B}(K)$	±	√	√	√	√
$\mathrm{S}(K)$	±	+	√	+	√
$\mathrm{I}(K)$	±	±	±	√	√
K	±	±	±	±	√

Proof. In Example 2 let $(X,S) = (X_2, S_2)$, $a = b$ and $K = \{b\}$, then $K \in \mathrm{M}(a)$, and $\mathrm{F}(a,K) = \mathrm{B}(K) = \mathrm{S}(K) = \mathrm{I}(K) = K$, which follows "+" in all cells in Table 1. Now in order to obtain "–" items we have

- 1st. column: In Example 1 let $(X,S) = (X_1, S_1)$, $a = 1$ and $K = \{-\mu_1, \mu_1\}$, then $K \in \mathrm{M}(a)$ and $\mathrm{B}(K) = \mathrm{S}(K) = \mathrm{I}(K) = K = \{-\mu_1, \mu_1\} \not\subset \{\mu_1\} = \mathrm{F}(a,K)$.
- 2nd., 3rd. and 4th. columns: In Example 2 let $(X,S) = (X_2, S_2)$, $a \in X_2 - \{b\}$ and $K = \mathrm{E}(X_2)$, then $K \in \mathrm{M}(a)$ and:

i. $$K = \mathrm{E}(X_2) = S_2 \cup \{b\} \not\subset S_2 = \mathrm{I}(K),$$

ii. $$K = \mathrm{E}(X_2) = S_2 \cup \{b\} \not\subset \{\mathrm{id}_{X_2}\} = \mathrm{S}(K) = \mathrm{B}(K),$$

iii. $$\mathrm{I}(K) = S_2 \not\subset \{\mathrm{id}_{X_2}\} = \mathrm{S}(K) = \mathrm{B}(K).$$

Completion 14. We have the following table (see [2], Corollary 5, Table 2), where T2 is the affiliation: The mark "√" indicates that if (X,S) is a transformation semigroup, A is a nonempty subset of X, and $K \in \overline{\mathrm{M}}(A)$, then $\Gamma \subseteq \Omega$. The mark "+" indicates that a transformation semi group (X,S) exists, a nonempty subset A of X and $K \in \overline{\mathrm{M}}(A)$, such that $\Gamma \subseteq \Omega$. The mark "–" indicates that a transformation semigroup (X,S) exists, a nonempty subset A of X and $K \in \overline{\mathrm{M}}(A)$, such that $\Gamma \not\subset \Omega$. The mark "±" indicates "+" and "–".

Table 2. In the corresponding case T2 is valid

	1st. column ↓		3rd. column ↓		5th. column ↓		7th. column ↓
Γ \ Ω	$\mathrm{F}(A,K)$	$\overline{\mathrm{F}}(A,K)\cap\mathrm{B}(K)$	$\overline{\mathrm{F}}(A,K)\cap\mathrm{S}(K)$	$\overline{\mathrm{F}}(A,K)$	$\mathrm{B}(K)$	$\mathrm{S}(K)$	$\mathrm{I}(K)$
$\mathrm{F}(A,K)$	√	√	√	√	√	√	√
$\overline{\mathrm{F}}(A,K)\cap\mathrm{B}(K)$	+	√	√	√	√	√	√
$\overline{\mathrm{F}}(A,K)\cap\mathrm{S}(K)$	+	+	√	√	+	√	+
$\overline{\mathrm{F}}(A,K)$	+	+	+	√	+	+	+
$\mathrm{B}(K)$	±	±	±	±	√	√	√
$\mathrm{S}(K)$	±	±	±	±	+	√	+
$\mathrm{I}(K)$	±	±	±	±	±	±	√

Proof. Note that in the transformation semigroup (X,S) for $a \in X$, we have $\mathrm{M}(a) = \overline{\mathrm{M}}(\{a\})$. So using the above note and Table 1 (Completion 13) we are able to obtain "+" items and complete the 7th., 6th. and 5th. columns in Table 2. In order to complete the remainder in Example 1, let $(X,S) = (X_1,S_1)$, $A = \{-1,1\}$ and $K = \{-\mu_1,\mu_1\}$, then $K \in \overline{\mathrm{M}}(A)(=\overline{\mathrm{M}}(A))$ and

$$\begin{aligned}\overline{\mathrm{F}}(A,K)\cap\mathrm{S}(K) &= \overline{\mathrm{F}}(A,K)\cap\mathrm{B}(K) = \overline{\mathrm{F}}(A,K)\\ &= \mathrm{B}(K) = \mathrm{S}(K) = \mathrm{I}(K) = K = \{-\mu_1,\mu_1\}\\ &\not\subset \{\mu_1\} = \mathrm{F}(A,K).\end{aligned}$$

Completion 15. We have the following table (see [2], Corollary 5, Table 2), where T3 is the affiliation: The mark "√" indicates that if (X,S) is a transformation semigroup, A is a nonempty subset of X, and $K \in \overline{\mathrm{M}}(A)$, then $\Gamma \subseteq \Omega$. The mark "+" indicates that a transformation semigroup (X,S) exists, a nonempty subset A of X and $K \in \overline{\mathrm{M}}(A)$, such that $\Gamma \subseteq \Omega$. The mark "–" indicates that a transformation semigroup (X,S) exists, a nonempty subset A of X and $K \in \overline{\mathrm{M}}(A)$, such that $\Gamma \not\subset \Omega$. The mark "±" indicates "+" and "–".

Table 3. In the corresponding case T3 is valid

	1st. column ↓		3rd. column ↓		5th. column ↓
Γ \ Ω	$\mathrm{F}(A,K)$	$\overline{\mathrm{F}}(A,K)$	$\mathrm{B}(K)$	$\mathrm{S}(K)$	$\mathrm{I}(K)$
$\mathrm{F}(A,K)$	√	√	√	√	√
$\overline{\mathrm{F}}(A,K)$	±	√	√	√	√
$\mathrm{B}(K)$	±	±	√	√	√
$\mathrm{S}(K)$	±	±	+	√	+
$\mathrm{I}(K)$	±	±	±	±	√

Proof. Use the argument in Completion 14 (Table 2) to conclude.

Completion 16. We have the following table (see [2], Corollary 5, Table 3), where T4 is the affiliation: The mark "√" indicates that if (X,S) is a transformation semigroup, A is a nonempty subset of X, $I \in \overline{\mathrm{M}}(A)$, $J \in \overline{\mathrm{M}}(A)$ (existence of such a J is not necessary) and K is a right ideal of $\mathrm{E}(X)$, then $\mathrm{J}(\Gamma) \subseteq \mathrm{J}(\Omega)$. The mark "+" indicates that a transformation semigroup (X,S) exists, a nonempty subset A of X, $I \in \overline{\mathrm{M}}(A)$ (if it is mentioned in that item), $J \in \overline{\mathrm{M}}(A)$ (if it is mentioned in that item) and a right ideal K of $\mathrm{E}(X)$ (if it is mentioned in that item) such that $\mathrm{J}(\Gamma) \subseteq \mathrm{J}(\Omega)$. The mark "–" indicates that a transformation semigroup (X,S) exists, a nonempty subset A of X, $I \in \overline{\mathrm{M}}(A)$ (if it is mentioned in that item), $J \in \overline{\mathrm{M}}(A)$ (if it is mentioned in that item) and a right ideal K of $\mathrm{E}(X)$ (if it is mentioned in that item) such that $\mathrm{J}(\Gamma) \not\subset \mathrm{J}(\Omega)$. The mark "±" indicates "+" and "–".

Table 4. In the corresponding case T4 is valid

C	Γ \ Ω	$\mathrm{F}(A,C)\cap\mathrm{B}(C)$, $\mathrm{F}(A,C)\cap\mathrm{S}(C)$, $\overline{\mathrm{F}}(A,C)\cap\mathrm{B}(C)$, $\overline{\mathrm{F}}(A,C)\cap\mathrm{S}(C)$	$\mathrm{F}(A,C)$, $\overline{\mathrm{F}}(A,C)$	$\mathrm{B}(C)$, $\mathrm{S}(C)$, $\mathrm{I}(C)$
$\mathrm{F}(A,C)\cap\mathrm{B}(C)$, $\mathrm{F}(A,C)\cap\mathrm{S}(C)$, $\overline{\mathrm{F}}(A,C)\cap\mathrm{B}(C)$, $\overline{\mathrm{F}}(A,C)\cap\mathrm{S}(C)$	K,I,J	√	√	√
$\mathrm{F}(A,C)$, $\overline{\mathrm{F}}(A,C)$	K	±	√	±
	I,J	√	√	√
$\mathrm{B}(C)$, $\mathrm{S}(C)$, $\mathrm{I}(C)$	K,I,J	±	±	√

Proof. In Example 1 let $(X,S)=(X_1,S_1)$ and $K=\{(-1)^k\eta_x \mid x\in X_1, k=1,2\}(\in \mathrm{M}(0))$, then $\mathrm{J}(\mathrm{F}(\{-1,1\},K))=\{\eta_x \mid x\in X_1\}\not\subset\{\eta_x \mid x\in X_1-\{-1,1\}\}=\mathrm{J}(\mathrm{F}(\{-1,1\},K)\cap\mathrm{B}(K))=\mathrm{J}(\mathrm{B}(K))$ and $\mathrm{J}(\mathrm{B}(K))=\{\eta_x \mid x\in X_1-\{-1,1\}\}\not\subset\{\eta_0\}=\mathrm{J}(\mathrm{F}(0,K)\cap\mathrm{B}(K))=\mathrm{J}(\mathrm{F}(0,K))$.

Completion 17. We have the following table (see [2], Theorem 17, Table 5), where T5 is the affiliation: The mark "√" indicates that if (X,S) is a transformation semigroup, A and B be nonempty subsets of X such that B is $A^{\underline{\alpha}}$ almost periodic, then B is $A^{\underline{\beta}}$ almost periodic. The mark "(√)" indicates that if (X,S) is a transformation semigroup, A and B are nonempty subsets of X such that A is $A^{\underline{(\overline{\overline{M}},\overline{M})}}$ almost periodic, B is $B^{\underline{(\overline{\overline{M}},\overline{M})}}$ almost periodic, B is $A^{\underline{\alpha}}$ almost periodic, then B is $A^{\underline{\beta}}$ almost periodic. The mark "+" indicates that a transformation semigroup (X,S) exists, nonempty subsets A and B of X, such that A is $A^{\underline{(\overline{\overline{M}},\overline{M})}}$ almost periodic, B is $B^{\underline{(\overline{\overline{M}},\overline{M})}}$ almost periodic, B is $A^{\underline{\alpha}}$ almost periodic and B is $A^{\underline{\beta}}$ almost periodic. The mark "–" indicates that there exists a transformation semigroup (X,S), nonempty subsets A and B of X, such that A is $A^{\underline{(\overline{\overline{M}},\overline{M})}}$ almost periodic, B is $B^{\underline{(\overline{\overline{M}},\overline{M})}}$ almost periodic, B is $A^{\underline{\alpha}}$ almost periodic and B is not $A^{\underline{\beta}}$ almost periodic. The mark "(–)" indicates that a transformation (X,S) exists, nonempty subsets A and B of X, such that B is $A^{\underline{\alpha}}$ almost periodic and B is not $A^{\underline{\beta}}$ almost periodic. The mark "±"indicates "+" and "–". The mark "(±)" indicates "+" and "(–)".

Table 5. In the corresponding case T5 is valid

	1st. column ↓	2nd. column ↓	3rd. column ↓	4th. column ↓	5th. column ↓	6th. column ↓
α \ β	$(-,-)$, $(\overline{M},-)$	$(-,\overline{M})$, $(\overline{M},\overline{M})$	$(-,\overline{\overline{M}})$, $(\overline{M},\overline{\overline{M}})$	$(\overline{\overline{M}},-)$	$(\overline{\overline{M}},\overline{M})$	$(\overline{\overline{M}},\overline{\overline{M}})$
$(-,-)$, $(\overline{M},-)$	√	√	+	(√) (±)	(√) (±)	(±)
$(-,\overline{M})$, $(\overline{M},\overline{M})$	±	√	±	±	(√) (±)	±
$(-,\overline{\overline{M}})$, $(\overline{M},\overline{\overline{M}})$	±	(√) +	√	±	(√) (±)	(√) (±)
$(\overline{\overline{M}},-)$	±	±	±	√	√	+
$(\overline{\overline{M}},\overline{M})$	±	±	±	±	√	±
$(\overline{\overline{M}},\overline{\overline{M}})$	±	±	±	±	(√) (±)	√

Proof.

- 1st. and 3rd. columns:

In Example 7 let $(X,S)=(X_7,S_7)$ and $x\in X_7-(\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}\cup\{0\})$, then $M(0)=\{\{\psi\}\}$, $M(x)=\overline{M}(\{0,x\})=\overline{\overline{M}}(\{0,x\})=\{E(X_7)\}$ and:

1. $\{0,x\}$ is $\{0,x\}^{\underline{(\overline{M},M)}}$ almost periodic,
2. $\{x\}$ is $\{0,x\}^{\underline{(-,\overline{\overline{M}})}}$ almost periodic,
3. $\{x\}$ is $\{0,x\}^{\underline{(\overline{M},\overline{\overline{M}})}}$ almost periodic,
4. $\{x\}$ is $\{0,x\}^{\underline{(\overline{\overline{M}},\overline{\overline{M}})}}$ almost periodic,
5. $\{x\}$ is $\{0,x\}^{\underline{(\overline{\overline{M}},\overline{M})}}$ almost periodic,
6. $\{x\}$ is not $\{0,x\}^{\underline{(-,-)}}$ almost periodic.

In Example 5 let $(X,S)=(X_5,S_5)$, then $\mathrm{M}(1)=\{\{\rho_1,\rho_1\psi,0\}\}$, $\mathrm{M}(\frac{1}{2})=\{\{\rho_2,\rho_2\psi,0\}\}$, $\overline{\overline{\mathrm{M}}}(\left\{\frac{1}{2},1\right\})=\{\{\rho_1,\rho_1\psi,0\},\{\rho_2,\rho_2\psi,0\}\}$, and:

1. $\left\{\frac{1}{2},1\right\}$ is $\left\{\frac{1}{2},1\right\}\xrightarrow{(\overline{\overline{\mathrm{M}}},\overline{\mathrm{M}})}$ almost periodic,
2. $\left\{\frac{1}{2},1\right\}$ is $\left\{\frac{1}{2},1\right\}\xrightarrow{(\overline{\overline{\mathrm{M}}},-)}$ almost periodic,
3. $\left\{\frac{1}{2},1\right\}$ is $\left\{\frac{1}{2},1\right\}\xrightarrow{(\overline{\overline{\mathrm{M}}},\overline{\overline{\mathrm{M}}})}$ almost periodic,
4. $\left\{\frac{1}{2},1\right\}$ is $\left\{\frac{1}{2},1\right\}\xrightarrow{(\overline{\mathrm{M}},\overline{\mathrm{M}})}$ almost periodic,
5. $\left\{\frac{1}{2},1\right\}$ is not $\left\{\frac{1}{2},1\right\}\xrightarrow{(-,-)}$ almost periodic,
6. $\left\{\frac{1}{2},1\right\}$ is not $\left\{\frac{1}{2},1\right\}\xrightarrow{(-,\overline{\overline{\mathrm{M}}})}$ almost periodic.

- 2nd. column: In Example 5 let $(X,S)=(X_5,S_5)$, then $\overline{\mathrm{M}}(\left\{\frac{1}{2},1\right\})=\{\{\rho_1,\rho_2,\rho_1\psi,\rho_2\psi,0\}\}$ (other items have been described in the proof of 1st. and 3rd. columns). Also we have

1. $\left\{\frac{1}{2},1\right\}$ is $\{1\}\xrightarrow{(\overline{\overline{\mathrm{M}}},-)}$ almost periodic,
2. $\left\{\frac{1}{2},1\right\}$ is $\{1\}\xrightarrow{(\overline{\overline{\mathrm{M}}},\overline{\mathrm{M}})}$ almost periodic,
3. $\left\{\frac{1}{2},1\right\}$ is $\{1\}\xrightarrow{(\overline{\overline{\mathrm{M}}},\overline{\overline{\mathrm{M}}})}$ almost periodic,
4. $\left\{\frac{1}{2},1\right\}$ is not $\{1\}\xrightarrow{(-,\overline{\mathrm{M}})}$ almost periodic.

- 4th. column:

In Example 4 let $(X,S)=(X_4,S_4)$, then for each $k\in\mathbf{N}$ we have $\mathrm{M}(\frac{1}{k})=\overline{\mathrm{M}}(\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\})=\{\mathrm{E}(X_4)\}$, $\overline{\overline{\mathrm{M}}}(\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\})=\varnothing$, and

1. $\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}$ is $\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}\xrightarrow{(-,-)}$ almost periodic,
2. $\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}$ is not $\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}\xrightarrow{(\overline{\overline{\mathrm{M}}},\overline{\mathrm{M}})}$ almost periodic,
3. $\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}$ is not $\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}\xrightarrow{(\overline{\overline{\mathrm{M}}},-)}$ almost periodic.

In Example 7 let $(X,S)=(X_7,S_7)$ and $x\in X_7-(\left\{\frac{1}{n}\mid n\in\mathbf{N}\right\}\cup\{0\})$, then $\{x\}$ is not $\{0,x\}\xrightarrow{(\overline{\overline{\mathrm{M}}},-)}$ almost periodic. Considering the proof of the 1st. column completes the proof.

- 5th. and 6th. columns:

In Example 4 let $(X,S)=(X_4,S_4)$, then (consider the proof of the 4th. column):

1. $\left\{\frac{1}{n} \mid n \in \mathbf{N}\right\}$ is $\{1\}^{\underline{(-,-)}}$ almost periodic,
2. $\left\{\frac{1}{n} \mid n \in \mathbf{N}\right\}$ is $\{1\}^{\underline{(-,\overline{M})}}$ almost periodic,
3. $\left\{\frac{1}{n} \mid n \in \mathbf{N}\right\}$ is $\{1\}^{\underline{(-,\overline{\overline{M}})}}$ almost periodic,
4. $\left\{\frac{1}{n} \mid n \in \mathbf{N}\right\}$ is not $\{1\}^{\underline{(\overline{\overline{M}},\overline{M})}}$ almost periodic,
5. $\left\{\frac{1}{n} \mid n \in \mathbf{N}\right\}$ is not $\{1\}^{\underline{(\overline{\overline{M}},\overline{\overline{M}})}}$ almost periodic.

In Example 5 let $(X,S)=(X_5,S_5)$, then (consider the proof of the 1st. column):

1. $\{1\}$ is $\left\{\frac{1}{2},1\right\}^{\underline{(\overline{\overline{M}},\overline{M})}}$ almost periodic,
2. $\{1\}$ is not $\left\{\frac{1}{2},1\right\}^{\underline{(\overline{\overline{M}},\overline{\overline{M}})}}$ almost periodic.

Completion 18. We have the following table (see [2], Theorem 20, Table 6), where T6 is the affiliation: The mark "√" indicates that if (X,S) is a transformation semigroup and A is a nonempty subset of X such that (X,S) is $A^{\underline{\alpha}}$ distal, then (X,S) is $A^{\underline{\beta}}$ distal. The mark "+" indicates that there exists a transformation semigroup (X,S) and a nonempty subset A of X, such that (X,S) is $A^{\underline{\alpha}}$ distal and $A^{\underline{\beta}}$ distal. The mark "–" indicates that there exists a transformation semigroup (X,S) and a nonempty subset A of X, such that (X,S) is $A^{\underline{\alpha}}$ distal but it is not $A^{\underline{\beta}}$ distal. The mark "±" indicates "+" and "–".

Table 6. In the corresponding case

α \ β	$(-)$	$(\overline{M})$	$(\overline{\overline{M}})$
$(-)$	√	√	±
$(\overline{M})$	±	√	±
$(\overline{\overline{M}})$	±	√	√

Proof.

- In Example 4 let $(X,S)=(X_4,S_4)$, then (consider the proof of the 4th. column in Table 5 (Completion 17)):

1. (X_4,S_4) is $\left\{\frac{1}{n} \mid n \in \mathbf{N}\right\}^{\underline{(-)}}$ distal,
2. (X_4,S_4) is $\left\{\frac{1}{n} \mid n \in \mathbf{N}\right\}^{\underline{(\overline{M})}}$ distal,
3. (X_4,S_4) is not $\left\{\frac{1}{n} \mid n \in \mathbf{N}\right\}^{\underline{(\overline{\overline{M}})}}$ distal.

- In Example 7 let $(X,S)=(X_7,S_7)$, then (consider the proof of the 1st. column in Table 5 (Completion 17)):

1. (X_7,S_7) is not $\{0,x\}^{\underline{(-)}}$ distal,

2. (X_7, S_7) is $\{0, x\} \frac{(\overline{\mathrm{M}})}{=}$ distal,
3. (X_7, S_7) is $\{0, x\} \frac{(\mathrm{M})}{}$ distal.

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