Iranian Journal of Science & Technology, Transaction A, Vol. 28, No. A2
Printed in Islamic Republic of Iran, 2004

CONTROLLING CHAOS IN 2-DIMENSIONAL SYSTEMS*

L. M. SAHA[1]**, G. H. ERJAEE[2] AND M. BUDHRAJA[3]

[1]Department of Mathematics, Zakir Husain College, University of Delhi, New Delhi – 110002, India
[2]Mathematics Department, College of Science, Shiraz University, Shiraz, I. R. of Iran
[3]Department of Mathematics, Shivaji College, University of Delhi,
New Delhi, 110027, India

Abstract – A chaos control method suggested by Erjaee has been reviewed. It has been shown that this technique can be applied in various evolutionary systems of 2-dimensional types. The method has been applied for cases of the Henon map, as well as Burger's map. The limitations of the control technique have also been discussed by considering the Standard Map and the Gumowski-Mira map. The results obtained through numerical calculations are very interesting and significant. This technique has some advantages over many other techniques of chaos control in discrete systems.

Keywords – Asymptotic stability, control parameter, chaos

1. INTRODUCTION

The subject chaos control refers to the technique of manipulation of chaotic motion exhibited in nonlinear systems. Such a technique proves to be useful, as chaos can be controlled in various ways to achieve desirable results. Recent articles on chaos control, [1-11] have revealed the fact that chaos may be of great benefit if it is applied properly to irregular and complex nonlinear systems. This new technology of nonlinear dynamics has a significant impact on various engineering devices such as high-performance circuits, communication signals, chemical reactions, etc. in changing irregular behavior into regularity.

The area of chaos control is now considered as a challenge area of research and it is emerging as an interdisciplinary field because it involves research in all areas of knowledge. Chaos control poses a substantial challenge because of extreme sensitivity and complexity of chaotic dynamics. Also, as almost all the systems showing chaotic behavior are of a nonlinear type, and there is no common way to explain such behavior, chaos control techniques are different for different chaotic systems.

Erjaee [12] has formulated a method for achieving asymptotic stability of a 2-dimensional dynamical system, and has illustrated the method by applying it to a Predator-Prey system.

In this paper, we have applied the above method to 2-dimensional Henon and Burger maps to achieve asymptotic stability. Further, we have considered the case where the above technique fails, by discussing the Standard Map and Gumowski-Mira Map.

*Received by the editor September 5, 2003 and in final revised form June 29, 2004
**Corresponding author

2. DESCRIPTION OF METHOD

We consider the dynamics of the desired map, y_{n+1}, and that of the actual one, x_{n+1}, through the following mappings:

$$x_{n+1} = F(x_n, p),$$
$$y_{n+1} = F(y_n, \overline{p}).$$

The dynamics of the neighborhood of x_{n+1} and y_{n+1} can be represented as

$$x_{n+1} = \mathbf{A_R} x_n + \mathbf{B_R} p \quad (1)$$
$$y_{n+1} = \mathbf{A_D} y_n + \mathbf{B_D} \overline{p}, \quad (2)$$

where the matrices $\mathbf{A_R}$, $\mathbf{A_D}$, $\mathbf{B_R}$, $\mathbf{B_D}$ can be obtained as follows:

$$\mathbf{A_R} = D_{x_n} F(x_n, p)$$
$$\mathbf{A_D} = D_{y_n} F(y_n, \overline{p})$$
$$\mathbf{B_R} = D_p F(x_n, p)$$
$$\mathbf{B_D} = D_{\overline{p}} F(y_n, \overline{p}).$$

Now, suppose the control input is given by

$$\overline{p} = \mathbf{C_R} x_n + \mathbf{C_M} p - \mathbf{C_D} y_n \quad (3)$$

On subtracting (1) from (2), and using (3), we obtain the error equation as

$$e_{n+1} = (\mathbf{A_R} - \mathbf{B_D}\mathbf{C_R}) e_n + (\mathbf{A_R} - \mathbf{A_D} + \mathbf{B_D}(\mathbf{C_D} - \mathbf{C_R})) y_n + (\mathbf{B_R} - \mathbf{B_D}\mathbf{C_M}) p. \quad (4)$$

It is clear from (4) that for asymptotic stability we need

$$\mathbf{A_R} - \mathbf{A_D} + \mathbf{B_D}(\mathbf{C_D} - \mathbf{C_R}) = 0 \quad (5)$$
$$\mathbf{B_R} - \mathbf{B_D}\mathbf{C_M} = 0. \quad (6)$$

Usually we have $e_0 \neq 0$.
Therefore, for asymptotic stability, our requirement is $e_n \to 0$ as $n \to \infty$, and for this we must have

$$\mathbf{A_R} - \mathbf{B_D}\mathbf{C_R} = -I. \quad (7)$$

From equations (5), (6) and (7) we can obtain $\mathbf{C_D}$, $\mathbf{C_M}$, $\mathbf{C_R}$ and then from (3), we can obtain the control parameter $\overline{p}$.

3. APPLICATION OF CONTROL TECHNIQUE

Here below, we apply the above technique to a Henon map and Burger's map.

(a) Control in Henon Map

Henon has considered a pair of difference equations:

$$\begin{cases} f(x,y) = 1 + y - a x^2 \\ g(x,y) = b x \end{cases} \tag{8}$$

for given parameters a and b.

The above system has 2 fixed points $\vec{X}$ = (X, Y), for a > $-\frac{1}{4}(1-b)^2$ such that

$$\begin{cases} X = \dfrac{[b - 1 \pm \{(1-b)^2 + 4a\}^{1/2}]}{2a} \\ Y = bX \end{cases} \tag{9}$$

The above system exhibits a strange attractor for a = 1.4, b = 0.3.

It has been observed that for such parameter values, the fixed point (0.63135448, 0.18940634) of the Henon system is unstable and we desire to stabilize it. Also, the map is unpredictable at the neighboring point (0.531245, 0.214678).

In this case, after some calculations, we obtain

$$\mathbf{A_R} = \begin{bmatrix} -1.487486 & 1 \\ 0.3 & 0 \end{bmatrix},$$

$$\mathbf{A_D} = \begin{bmatrix} -1.767792544 & 1 \\ 0.3 & 0 \end{bmatrix}.$$

Now, setting a = $\overline{a}$ = 1.4, b = $\overline{b}$ = 0.3, we get

$$\mathbf{B_R} = \begin{bmatrix} -0.28222125 & 0 \\ 0 & 0.531245 \end{bmatrix},$$

$$\mathbf{B_D} = \begin{bmatrix} -0.398608479 & 0 \\ 0 & 0.63135448 \end{bmatrix}.$$

Using these in equations (5), (6) and (7), we get

$$\mathbf{C_R} = \begin{bmatrix} 1.22297 & -2.50873 \\ 0.475169 & 1.5839 \end{bmatrix},$$

$$\mathbf{C_D} = \begin{bmatrix} 1.92618 & -2.50873 \\ 0.475169 & 1.5839 \end{bmatrix},$$

$$\mathbf{C_M} = \begin{bmatrix} 0.708016 & 0 \\ 0 & 0.841437 \end{bmatrix}.$$

Then, from (3), after certain calculation we get

$$\overline{\mathrm{p}} = \begin{bmatrix} 0.361416 \\ 0.24489 \end{bmatrix}.$$

The dynamics of the uncontrolled and corresponding controlled map of the system (8) are respectively shown in Figs. 1(a), 1(b) (uncontrolled) and 1(a'), 1(b') (controlled).

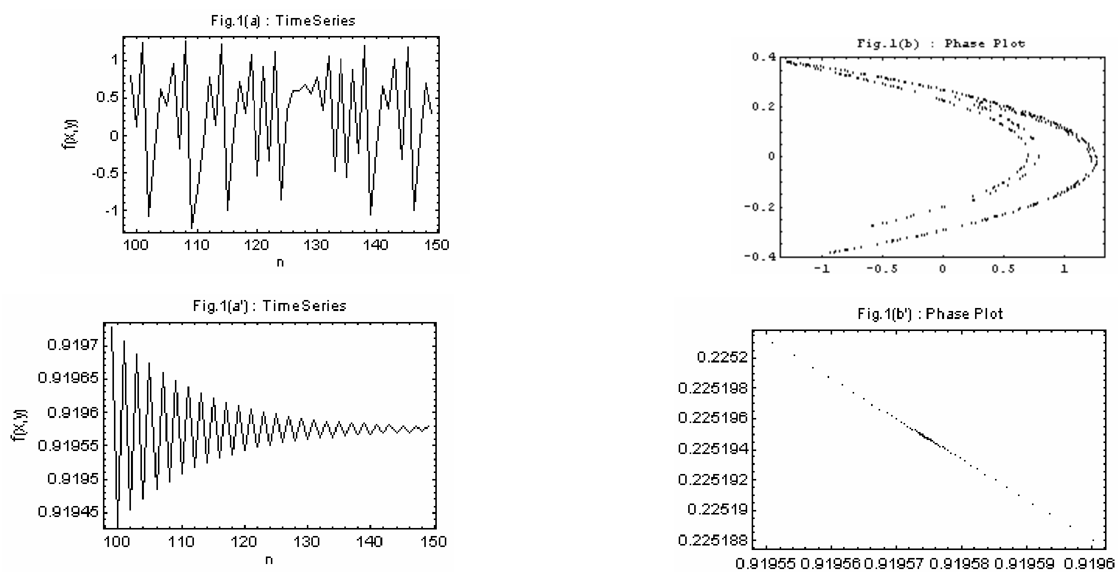


Fig. 1. The uncontrolled and controlled Henon map

Figure 1(a), 1(b) represent time series and Phase Plot for uncontrolled system and Fig. 1 (a'), 1 (b') are the corresponding plots for controlled system.

(b) Control in Burger's Map

Burger's map, [13], is defined by the following pair of equations:

$$\begin{cases} f(x,y) = (1-a)x - y^2 \\ g(x,y) = (1+b)y + xy \end{cases} \tag{10}$$

where a and b are non-zero parameters.

This map is the discrete form of differential equations appearing in hydrodynamics.

There are two fixed points ($-b$, $\pm\sqrt{ab}$), which are stable for $b < 0.5$

For $a = 0.9$, $b = 0.856$, the motion about the fixed point (-0.856, 0.87772433) is unstable and unpredictable.

For example, if we consider the dynamics starting from point (–1, 0.7), which is a neighboring point of the above fixed point, the ensuing motion is unpredictable, as is evident from the time series and the Phase-Plot in Fig 2(a), 2(b) respectively.

Proceeding, as in the case of the Henon map, we obtain

$$\mathbf{C_R} = \begin{bmatrix} 1.28505 & -1.63551 \\ 0.797517 & 2.11456 \end{bmatrix},$$

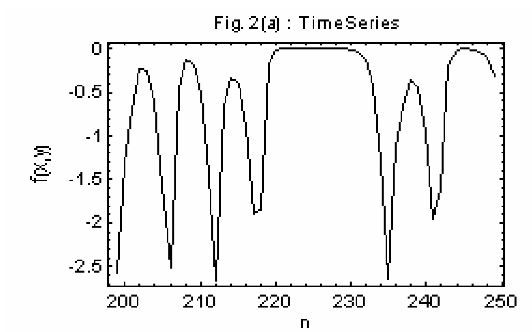
$$\mathbf{C_D} = \begin{bmatrix} 1.28505 & -2.05076 \\ 1 & 2.27862 \end{bmatrix},$$

$$\mathbf{C_M} = \begin{bmatrix} 1.16822 & 0 \\ 0 & 0.797517 \end{bmatrix}.$$

Then, by substituting $\mathbf{C_R}, \mathbf{C_D}, \mathbf{C_M}$ in (3), we get

$$\overline{\mathrm{p}} = \begin{bmatrix} 1.5215 \\ 0.221349 \end{bmatrix}$$

The dynamics of system (10) corresponding to a = 1.5215, b = 0.221349 have been shown in Figs. 2(a'), 2(b').



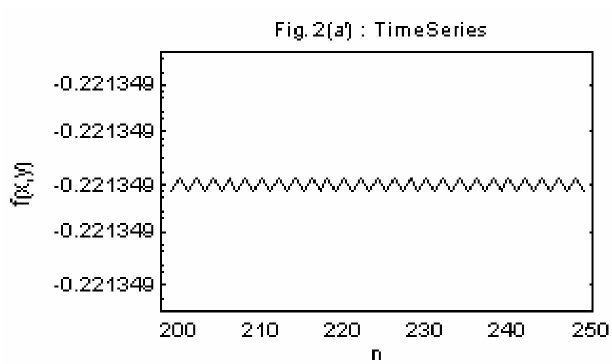
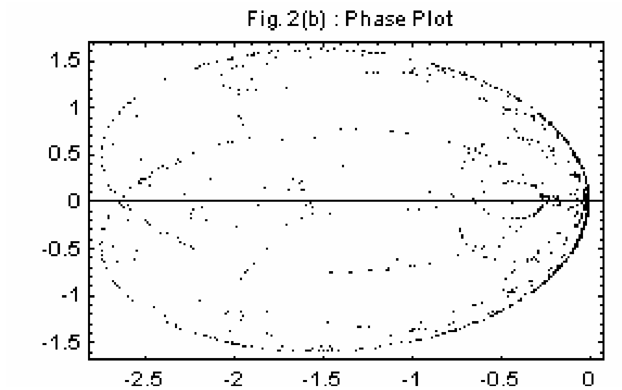


Fig. 2. The uncontrolled and controlled Burger's Map



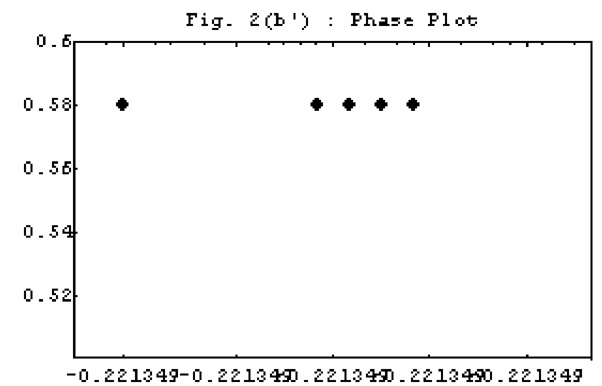


Fig. 2. (Continued). The uncontrolled and controlled Burger's Map

Figures 2(a), 2(b) represent Time Series and Phase Plot for uncontrolled system and Figs. 2(a′), 2(b′) are the corresponding plots for controlled system.

4. LIMITATIONS OF THE ABOVE METHOD

Suppose we desire to solve the system

$$\mathbf{A}\,\mathbf{X} = \mathbf{B},$$

where **A** and **B,** respectively, m × n and m × r are known matrices and **X** is an n × r unknown matrix to be determined.

Writing **X** as $\mathbf{X} \equiv (\mathbf{X_1}\ \mathbf{X_2}\ \ldots\ \mathbf{X_r})$ and **B** as $\mathbf{B} \equiv (\mathbf{B_1}\ \mathbf{B_2}\ \ldots\ \mathbf{B_r})$, where $\mathbf{X_i}$ (i = 1, 2, …, r) are n × 1 columns and $\mathbf{B_i}$ (i = 1, 2, …, r) are m × 1 columns.

We observe the system **A X = B** is consistent if and only if

$$\text{rank}\,(\mathbf{A}) = \text{rank}\,(\mathbf{A}\ \ \mathbf{B_i}),\ \forall\ i = 1, 2, \ldots, r \tag{11}$$

We rewrite equations (5), (6), (7) as:

$$\mathbf{B_D}\,(\mathbf{C_D} - \mathbf{C_R}) = \mathbf{A_D} - \mathbf{A_R} \tag{12}$$

$$\mathbf{B_D}\,\mathbf{C_M} = \mathbf{B_R} \tag{13}$$

$$\mathbf{B_D}\,\mathbf{C_R} = \mathbf{A_R} + \mathbf{I} \tag{14}$$

These equations (12), (13), (14) are consistent if and only if each of these satisfy condition(s) (11).

Now, we consider the following examples where the conditions (11) fail to hold.

I. The Standard Map

The Standard Map is described by the following pair of equations:

$$\begin{cases} f(x,\, y) = x + y + k\sin x \\ g(x,\, y) = y + k\sin x \end{cases}, \tag{15}$$

where k is a non-zero parameter.

The fixed points of this system are $(\pm n\pi,\ 0)$.

For k = 1, the fixed point (0, 0) is unstable, and the motion about it is unpredictable. In this case, the matrix $\mathbf{B_D}$ is

$$\mathbf{B_D} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Clearly, rank ($\mathbf{B_D}$) = 0. Now, consider equation (13), $\mathbf{B_D}\,\mathbf{C_M} = \mathbf{B_R}$.

Rank ($\mathbf{B_D} : \mathbf{B_R}$) ≠ 0, whatever reference point we may take in the neighborhood of (0, 0).

∴ Equation (13) is inconsistent.

i. e. we can't obtain $\mathbf{C_M}$.

Similarly, we can show that equations (12), (14) are inconsistent. Hence, the technique fails to stabilize (0, 0).

We can similarly show that the method fails to stabilize any of the fixed points $(\pm n\pi,\ 0)$ of the above system.

II. The Gumowski-Mira Map

The Gumowski-Mira map is described by the following pair of equations:

$$\begin{cases} f(x,\; y) = y + a(1 - b\,y^2)\,y + \mu\,x + \dfrac{2(1-\mu)\,x^2}{1 + x^2} \\ g(x,\, y) = -x + \mu\; f(x,\, y) + \dfrac{2(1-\mu)[\,f(x,\, y)]^2}{1 + [\,f(x,\, y)]^2} \end{cases}, \tag{16}$$

where a, b are constants and μ is a parameter.

For a = 0.008, b = 0.05, μ = – 0.8, the fixed point (0, 0) is unstable.

In this case, the matrices $\mathbf{A_D}$, $\mathbf{B_D}$ are:

$$\mathbf{A_D} = \begin{pmatrix} -0.8 & 1.008 \\ -0.36 & -0.8064 \end{pmatrix}$$

$$\mathbf{B_D} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The point (–0.9, 0), a neighboring point of (0, 0), has been taken as a reference point.

$$\mathbf{B_R} = \begin{pmatrix} -1.7950 \\ 1.35012 \end{pmatrix}$$

and $$\mathbf{A_R} = \begin{pmatrix} -2.77796 & 1.008 \\ 0.0960121 & -0.397695 \end{pmatrix}$$

Clearly, rank ($\mathbf{B_D} : \mathbf{B_R}$) ≠ 0, whereas rank ($\mathbf{B_D}$) = 0.

∴ Equation (13) is inconsistent.

Hence, the technique fails in the case of Gumowski-Mira map as well.

5. CONCLUSION

We observe that the asymptotic stability method can be applied successfully to achieve control in many 2, 3 or higher dimensional systems. But the technique has some limitations and can not be applied to all chaotic systems. So, no chaos control technique is universal. Different techniques are required to achieve stability in different non-linear systems.

Acknowledgements- This work is supported by a grant from the University Grant Commission, India, to Dr. L. M. Saha.

REFERENCES

1. Auerbach, D., Grebogi, C., Ott E. & Yorke, J. A. (1992). Controlling chaos in high dimensional systems, *Phys. Rev. Lett.,* 69, 3479-3482.
2. Braiman, Y., Lindner, J. F. & Ditto, W. L. (1995). Taming spatiotemporal chaos with disorder, *Nature*, 378, 465-467.
3. Carroll, T. L. & Pecora, L. M. (1993). Cascading synchronized chaotic systems, *Physica D*, 67, 126-140,
4. Chen, G. (1994). Optimal control of chaotic systems, *Int'l J. of Bifur. Chaos*, 4, 461-463.
5. Garfinkel, A., Spano, M. L., Ditto, W. L. & Weiss, J. N. (1992). Controlling cardiac chaos, Science, 257, 1230-1235.
6. Kocarev, L. & Parlitz, U. (1995). General approach for chaotic synchronization with application to communication, *Phys. Rev. Lett.*, 74, 5028-5031.
7. Ott, E., Grebogi, C. & Yorke, J. A. (1990). Controlling chaos, *Phys. Rev. Lett*., 64, 1196-1199.
8. Pan, S., & Yin, F. (1997). Using chaos to control chaotic systems, *Phys. Lett. A*, 231, 173.
9. Pecora, L. M. & Carroll, T. L. (1990). Synchronization in chaotic systems, *Phys. Rev. Lett*., 64, 821-823.
10. Pyragas, K. (1992). Continuous control of chaos by self-controlling feedback, *Phys. Lett. A*, 170, 421-428.
11. Shinbrot, T., Grebogi, C., Ott, E. & Yorke, J. A. (1993). Using small perturbations to control chaos, *Nature*, 363, 411-417.
12. Erjaee, G. H. (2002). On the asymptotic stability of a dynamical system, *IJST, Transaction A*, 26A1, 131-135.
13. Korsch, H. J. & Jodl, H. J. (1999). *Chaos*: A program collection for the PC. Springer Verlag.