“Research Note”

CONFORMAL VECTOR FIELDS ON TANGENT BUNDLE OF A RIEMANNIAN MANIFOLD*

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Abstract – Let \( M \) be an \( n \)-dimensional Riemannian manifold and \( TM \) its tangent bundle. The conformal and fiber preserving vector fields on \( TM \) have well-known physical interpretations and have been studied by physicists and geometricians. Here we define a Riemannian or pseudo-Riemannian lift metric \( \tilde{g} \) on \( TM \), which is in some senses more general than other lift metrics previously defined on \( TM \), and seems to complete these works. Next we study the lift conformal vector fields on \( (TM, \tilde{g}) \) and prove among the others that, every complete lift conformal vector field on \( TM \) is homothetic, and moreover, every horizontal or vertical lift conformal vector field on \( TM \) is a Killing vector.

Keywords – Complete lift metric, Conformal, Homothetic, Killing and Fiber-preserving vector fields.

1. INTRODUCTION

Let \( M \) be an \( n \)-dimensional differential manifold with a Riemannian metric \( g \) and \( \phi \) be a transformation on \( M \). Then \( \phi \) is called a conformal (resp. projective) transformation if it preserves the angles (resp. geodesics). Let \( V \) be a vector field on \( M \) and \( \{\varphi_t\} \) be the local one-parameter group of local transformations on \( M \) generated by \( V \). Then \( V \) is called an infinitesimal conformal (resp. projective) transformation on \( M \) if each \( \varphi_t \) is a local conformal (resp. projective) transformation of \( M \). It is well known that \( V \) is an infinitesimal conformal transformation or conformal vector field on \( M \) if and only if there is a scalar function \( \rho \) on \( M \) such that \( \mathcal{L}_V g = 2\rho g \), where \( \mathcal{L}_V \) denotes Lie derivation with respect to the vector field \( V \). \( V \) is called homothetic if \( \rho \) is constant and is called an isometry or Killing vector field when \( \rho \) vanishes.

Let \( TM \) be the tangent bundle over \( M \), and \( \Phi \) be a transformation on \( TM \). Then \( \Phi \) is called a fiber preserving transformation if it preserves the fibers. Fiber preserving transformations have well known applications in Physics. Let \( X \) be a vector field on \( TM \) and \( \{\Phi_t\} \) the local one parameter group of local transformation on \( TM \) generated by \( X \). Then \( X \) is called an infinitesimal fiber preserving transformation or fiber preserving vector field on \( TM \) if each \( \Phi_t \) is a local fiber preserving transformation of \( TM \).

Let \( \tilde{g} \) be a Riemannian or pseudo-Riemannian metric on \( TM \). The conformal vector field \( X \) on \( TM \) is said to be essential if the scalar function \( \Omega \) on \( TM \) in \( \mathcal{L}_X \tilde{g} = 2\Omega \tilde{g} \) depends only on \( (y^b) \) (with

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respect to the induced coordinates \((x^i, y^i)\) on \(TM\), and is said to be \textit{inessential} if \(\Omega\) depends only on \((x^i)\). In other words, \(\Omega\) is a function on \(M\).

There are some lift metrics on \(TM\) as follows: complete lift metric or \(g_2\), diagonal lift metric or \(g_1 + g_3\), lift metric \(g_2 + g_3\) and lift metric \(g_1 + g_2\).

In this area the following results are well known:

Let \((M, g)\) be a Riemannian manifold. If we consider \(TM\) with metrics \(g_1 + g_3\) or \(g_2 + g_3\), then every infinitesimal fiber preserving conformal transformation on \(TM\) is homothetic, and induces a homothetic vector field on \(M\) [1].

Let \((M, g)\) be a complete, simply connected Riemannian manifold. If we consider \(TM\) with metric \(g_1 + g_3\), and \(TM\) admits an essential infinitesimal conformal transformation, then \(M\) is isometric to the standard sphere [2].

Let \((M, g)\) be a Riemannian manifold and \(V\) a vector field on \(M\) and let \(X^C\), \(X^V\), \(X^H\) be complete, vertical and horizontal lifts of \(V\) to \(TM\) respectively. If we consider \(TM\) with metric \(g_2\), then \(X^C\) is a conformal vector field on \(TM\) if and only if \(V\) is homothetic on \(M\). Moreover, if \(V\) is a Killing vector on \(M\), then \(X^C\) and \(X^V\) are Killing vectors on \(TM\) [3].

Let \((M, g)\) be a Riemannian manifold. If we consider \(TM\) with metric \(g_1 + g_3\), then

I) \(X^C\) is a conformal vector field if and only if \(X\) is homothetic.

II) \(X^V\) is a conformal vector field if and only if \(X\) is Killing vector field with vanishing second covariant derivative in \(M\).

III) \(X^H\) is a conformal vector field if and only if \(X\) is parallel [3], [4].

In this paper we are going to replace the cited lift Riemannian or pseudo-Riemannian metrics on \(TM\) by \(\hat{g} = a g_1 + b g_2 + c g_3\), that is a combination of diagonal lift and complete lift metrics, where \(a\), \(b\) and \(c\) are certain positive real numbers. More precisely, we prove the following Theorems.

**Theorem 1.** Let \(M\) be a connected n-dimensional Riemannian manifold and let \(TM\) be its tangent bundle with metric \(\hat{g}\). Then every complete lift conformal vector field on \(TM\) is homothetic, and moreover, every horizontal or vertical lift conformal vector field on \(TM\) is a Killing vector.

**Theorem 2.** Let \(M\) be a connected n-dimensional Riemannian manifold and \(TM\) be its tangent bundle with metric \(\hat{g}\). Then every inessential fiber preserving conformal vector field on \(TM\) is homothetic.

### 2. PRELIMINARIES

Let \((M, g)\) be a real \(n\)-dimensional Riemannian manifold and \((U, x)\) a local chart on \(M\), where the induced coordinates of the point \(p \in U\) are denoted by its image on \(IR^n\), \(x(p)\) or briefly \((x')\). Using the induced coordinates \((x')\) on \(M\), we have the local field of frames \(\{\frac{\partial}{\partial x^i}\}\) on \(T_p M\). Let \(\nabla\) be a Riemannian connection on \(M\) with coefficients \(\Gamma^k_{ij}\), where the indices \(a, b, c, h, i, j, k, m, \ldots\) run over the range 1, 2, ..., \(n\). The Riemannian curvature tensor is defined by

\[
K(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z, \forall X, Y, Z \in X(M).
\]

Locally we have

\[
K^m_{ij} = \partial_i \Gamma^m_{jk} - \partial_j \Gamma^m_{ik} + \Gamma^m_{ih} \Gamma^h_{jk} - \Gamma^m_{jk} \Gamma^h_{ik},
\]

where \(\partial_i = \frac{\partial}{\partial x^i}\) and \(K(\partial_i, \partial_j, \partial_k) = K^m_{ij} \partial_m\).
3. NON-LINEAR CONNECTION

Let $TM$ be the tangent bundle of $M$ and $\pi$ the natural projection from $TM$ to $M$. Consider $\pi^*: TTM \mapsto TM$ and let us put

$$\text{ker} \pi^* = \{ z \in TTM \mid \pi^*(z) = 0 \}, \forall v \in TM.$$ 

Then the vertical vector bundle on $M$ is defined by $\text{VT}_{TM} \subset TTM \pi^*$ and let us put

$$\{ (v, z) \mid v \in \text{VT}_{TM}, z \in TTM \pi^* \}, \forall \in .$$

A non-linear connection or a horizontal distribution on $TM$ is a complementary distribution $HTM$ for $\text{VT}_{TM}$ on $TTM$. The non-linear nomination arise from the fact that $HTM$ is spanned by a basis which is completely determined by non-linear functions. These functions are called coefficients of non-linear connection and will be noted in the sequel by $N_j^i$. It is clear that $HTM$ is a horizontal vector bundle.

By definition, we have decomposition $TTM = \text{VT}_{TM} \oplus HTM$ [5].

Using the induced coordinates $(x^i, y^j)$ on $TM$, where $x^i$ and $y^j$ are called respectively position and direction of a point on $TM$, we have the local field of frames $\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \}$ on $TTM$. Let $\{ dx^i, dy^j \}$ be the dual basis of $\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \}$. It is well known that we can choose a local field of frames $\{ X_i, \frac{\partial}{\partial y^j} \}$ adapted to the above decomposition, i.e. $X_i \in X(HTM)$ and $\frac{\partial}{\partial y^j} \in X(VTM)$ are sections of horizontal and vertical sub-bundle on $HTM$ and $VTM$, defined by $X_i = \frac{\partial}{\partial x^i} - N_j^i \frac{\partial}{\partial y^j}$, where $N_j^i(x, y)$ are functions on $TM$ and have the following coordinate transformation rule in local coordinates $(x^i, y^j)$ and $(x^i, y^j)'$ on $TM$.

$$N_j^i = \frac{\partial x^h}{\partial x^i} \frac{\partial y^j}{\partial y^h} N_j^h + \frac{\partial^2 x^h}{\partial x^i \partial x^j} y^h.$$ 

To see a relation between linear and non-linear connections let $\Gamma_j^{k i}$ be the coefficients of the Riemannian connection of $(M, g)$. Then it is easy to check that $y^a \Gamma_a^{k i}$ satisfies the above relation and thus can be regarded as coefficients of the non-linear connection on $TM$ in the sequel.

Let us put $X_h = \frac{\partial}{\partial x^i} - y^a \Gamma_a^{k i} \frac{\partial}{\partial y^h}$ and $X_\pi = \frac{\partial}{\partial y^j}$. Then $\{ X_h, X_\pi \}$ is the adapted local field of frames of $TM$ and let $\{ dx^h, dy^j \}$ be the dual basis of $\{ X_h, X_\pi \}$, where $\delta y^b = dy^b + y^a \Gamma_a^{b i} dx^i$ and the indices $i, j, h, \ldots$ and $i, j, h, \ldots$ run over the range 1, 2, ...$n$.

4. THE RIEMANNIAN OR PSEUDO-RIEMANNIAN METRIC $\tilde{g}$ ON TANGENT BUNDLE

Let $(M, g)$ be a Riemannian manifold. The Riemannian metric $g$ has components $g_{ij}$, which are functions of variables $x^j$ on $M$, and by means of the above dual basis it is well known that [3];

$g_1 := g_{ij} dx^i dx^j, g_2 := 2g_{ij} dx^i dy^j$ and $g_3 := g_{ij} dy^i dy^j$ are all bilinear differential forms defined globally on $TM$.

The tensor field:

$$\tilde{g} = a g_1 + b g_2 + c g_3,$$

on $TM$ where $a$, $b$ and $c$ are certain positive real numbers, has components

$$\begin{pmatrix}
ag_{ij} & bg_{ij} \\
bg_{ij} & cg_{ij}
\end{pmatrix}.$$
with respect to the dual basis of the adapted frame of $TM$. From linear algebra we have $\det \hat{g} = (ac - b^2)^n \det g$. Therefore $\hat{g}$ is nonsingular if $ac - b^2 \neq 0$ and positive definite if $ac - b^2 > 0$ and define, respectively, pseudo-Riemannian or Riemannian lift metrics on $T(M)$.

5. LIE DERIVATIVE

Let $M$ be an $n$-dimensional Riemannian manifold, $V$ a vector field on $M$, and $\{\phi_t\}$ any local group of local transformations of $M$ generated by $V$. Take any tensor field $S$ on $M$, and denote by $\phi_t^*(S)$ the pull-back of $S$ by $\phi_t$. Then Lie derivation of $S$ with respect to $V$ is a tensor field $\mathcal{L}_v S$ on $M$ defined by

$$\mathcal{L}_v S = \frac{\partial}{\partial t} \phi_t^*(S) |_{t=0} = \lim_{t \to 0} \frac{\phi_t^*(S) - (S)}{t},$$

on the domain of $\phi_t$. The mapping $\mathcal{L}_v$ which maps $S$ to $\mathcal{L}_v(S)$ is called the Lie derivative with respect to $V$.

Suppose that $S$ is a tensor field of type $(n, m)$. Then the components $(\mathcal{L}_v S)_{\alpha_1 \ldots \alpha_n}^{\beta_1 \ldots \beta_m}$ of $\mathcal{L}_v S$ may be expressed as

$$(\mathcal{L}_v S)_{\alpha_1 \ldots \alpha_n}^{\beta_1 \ldots \beta_m} = V^\alpha \partial_\alpha S_{\beta_1 \ldots \beta_m}^{\beta_1 \ldots \beta_m} + \sum_{k=1}^{m} \partial_\alpha V^\alpha S_{\alpha \beta_1 \ldots \beta_k}^{\beta_1 \ldots \beta_m} - \sum_{k=1}^{n} \partial_\alpha S_{\beta_1 \ldots \beta_k}^{\beta_1 \ldots \beta_m},$$

where $S_{\beta_1 \ldots \beta_m}^{\beta_1 \ldots \beta_m}$ and $V^\alpha$ denote the components of $S$ and $V$.

The local expression of the Lie derivative $\mathcal{L}_v(S)$ in terms of covariant derivatives on a Riemannian manifold for a tensor field of type $(1, 2)$ is given by:

$$\mathcal{L}_v S_{\alpha \beta}^{\gamma} = \nabla_\alpha \nabla_\beta S_{\gamma}^{\alpha \beta} - \nabla_\beta \nabla_\alpha S_{\gamma}^{\alpha \beta} + S_{\alpha \beta}^{\gamma} \nabla_\alpha \phi^\alpha + S_{\alpha \beta}^{\gamma} \nabla_\beta \phi^\alpha,$$

where $S_{\alpha \beta}^{\gamma}$ and $\phi^\alpha$ are components of $S$ and $V$, and $\nabla_\alpha S_{\gamma}^{\alpha \beta}$, $\nabla_\alpha \phi^\alpha$ are components of covariant derivatives of $S$ and $V$, respectively [1, 3, 6].

**Lemma 1.** [1], [7] The Lie bracket of adapted frame of $TM$ satisfies the following relations

$$[X_j, X_j] = y^r K_{j \mu}^r X_{\mu},$$

$$[X_i, X_j] = \Gamma_{ij}^r X_{\mu},$$

$$[X_\tau, X_\tau] = 0,$$

where $K_{j \mu}^r$ denotes the components of a Riemannian curvature tensor of $M$.

**Lemma 2.** [1] Let $X$ be a vector field on $TM$ with components $(X^h, X^\pi)$ with respect to the adapted frame $\{X_h, X_\pi\}$. Then $X$ is fiber-preserving vector field on $TM$ if and only if $X^h$ are functions on $M$.

Therefore, every fiber-preserving vector field $X$ on $TM$ induces a vector field $V = X^h \frac{\partial}{\partial v^h}$ on $M$.

**Definition 1.** [1], [3] Let $V$ be a vector field on $M$ with components $V^h$. We have the following vector fields on $TM$ which are called respectively, **complete**, **horizontal** and **vertical** lifts of $V$:
Conformal vector fields on…

$$X^C := V^h X^h + y^m (\Gamma^h_m a V^a + \partial_m V^h) X^a$$,

$$X^H := V^h X^h$$,

$$X^V := V^h X^h$$.

From Lemma 2 we know that $X^C, X^H$ and $X^V$ are fiber-preserving vector fields on $TM$.

**Lemma 3.** [1] Let $X$ be a fiber-preserving vector field on $TM$. Then the Lie derivative of the adapted frame and its dual basis are given by:

I) $\mathcal{L}_X X^h = (-\partial_h X^e) X^e_a + (y^h X^c K_{eh}^{cb} - X^c \Gamma^h_{cb} X^e) X^e$

II) $\mathcal{L}_X X^e = \{X^h \Gamma^e_{bh} - X^e (X^h)\} X^e$

III) $\mathcal{L}_X dx^h = (\partial_m X^h) dx^m$, 

IV) $\mathcal{L}_X \delta y^h = -\{y^h X^c K_{mcb} h - X^c \Gamma^h_{cb} - X^c (X^h)\} dz^m - \{X^h \Gamma^h_{cm} - X^c (X^h)\} \delta y^m$.

**Lemma 4.** [8] Let $X$ be a fiber-preserving vector field on $TM$, which induces a vector field $V$ on $M$. Then Lie derivatives $\mathcal{L}_X g_1, \mathcal{L}_X g_2$ and $\mathcal{L}_X g_3$ are given by:

I) $\mathcal{L}_X g_1 = (\mathcal{L}_v g_1) dx^i dx^j$

II) $\mathcal{L}_X g_2 = 2g_{jm} \{y^h X^c K_{mcb} - X^c \Gamma^h_{cb} - X^c (X^h)\} dx^i dx^j$

III) $\mathcal{L}_X g_3 = -2g_{jm} \{y^h X^c K_{mcb} - X^c \Gamma^h_{cb} - X^c (X^h)\} dy^i dy^j$

where $\mathcal{L}_v g_1$ and $\nabla_i X^m$ denote the components of $\mathcal{L}_v g$ and the covariant derivative of $V$ respectively.

6. MAIN RESULTS

**Proposition 1.** Let $X$ be a complete (resp. horizontal or vertical) lift conformal vector field on $TM$. Then the scalar function $\Omega(x, y)$ in $\mathcal{L}_X \tilde{g} = 2\Omega \tilde{g}$ is a function of position alone (resp. $\Omega = 0$).

**Proof:** Let $TM$ be the tangent bundle over $M$ with Riemannian metric $\tilde{g}$ and $X$ be a complete (resp. horizontal or vertical) lift conformal vector field on $TM$. By definition, there is a scalar function $\Omega$ on $TM$ such that

$$\mathcal{L}_X \tilde{g} = 2\Omega \tilde{g}.$$

Since the complete horizontal and vertical lift vector fields are fiber preserving, by applying $\mathcal{L}_X$ to the definition of $\tilde{g}$, using Lemma 4 and the fact that $dx^i dx^j, dx^i dy^j$ and $dy^i dy^j$ are linearly independent, we have following three relations

$$a(\mathcal{L}_v g_{ij} - 2\Omega g_{ij}) = bg_{jm} (y^h X^c K_{mcb} - X^c \Gamma^h_{cb} - X^c (X^h))$$

$$+ g_{jm} (y^h X^c K_{mcb} - X^c \Gamma^h_{cb} - X^c (X^h))$$

$$b(\mathcal{L}_v g_{ij} - 2\Omega g_{ij}) = cg_{jm} (y^h X^c K_{mcb} - X^c \Gamma^h_{cb} - X^c (X^h))$$

$$+ cg_{jm} (y^h X^c K_{mcb} - X^c \Gamma^h_{cb} - X^c (X^h))$$

$$c(\mathcal{L}_v g_{ij} - 2\Omega g_{ij}) = \tilde{c}g_{jm} (y^h X^c K_{mcb} - X^c \Gamma^h_{cb} - X^c (X^h))$$

$$+ \tilde{c}g_{jm} (y^h X^c K_{mcb} - X^c \Gamma^h_{cb} - X^c (X^h))$$
Using relation 1, we have $L_{\gamma} g_{ij} = \nabla_i V_j + \nabla_j V_i$, from which we obtain
\[2\Omega g_{ij} = g_{mij} X_{T}(X^m) + g_{ni} X_{T}(X^m). \tag{4}\]

Applying $X_{T}$ to the relation 4 and using the fact that $g_{ij}$ is a function of position alone, we have
\[2g_{ij} X_{T}(\Omega) = g_{mij} X_{T}(X^m) + g_{ni} X_{T}(X^m). \tag{5}\]

By means of definition 1 for complete lift vector fields, and by replacing the value of $X^m$ in relation 5, we have
\[2g_{ij} X_{T}(\Omega) = g_{mij} X_{T}(y^j(\Gamma^m_{i} V^a + \partial_i V^m)) + g_{ni} X_{T}(y^j(\Gamma^m_{i} V^a + \partial_i V^m)). \]

Since the coefficients of the Riemannian connection on $M$, and components of vector field $V$ are functions of position alone, the right hand side of the above relation becomes zero, from which we have $X_{T}(\Omega) = 0$. This means that the scalar function $\Omega(x,y)$ on $TM$ depends only on the variables $(x^h)$.

Similarly, for vertical lift vector fields, by using the fact that the components of $V$ are functions of position alone and from relation 4, we have $\Omega = 0$. Finally, for horizontal lift vector field by means of relation 4, we have $\Omega = 0$.

**Proposition 2.** Let $M$ be a connected manifold and $X$ be a complete lift conformal vector field on $TM$. Then the scalar function $\Omega(x,y)$ in $L_{\gamma} \tilde{g} = 2\Omega \tilde{g}$ is constant.

**Proof:** Let $X$ be a complete lift conformal vector field on $TM$ with components $(X^h, X^g)$, with respect to the adapted frame $\{X_h, X_g\}$.

Let us put
\[A^m_a = \Gamma^m_{h} X^h + \partial_a X^m.\]

The coordinate transformation rule implies that $A^m_a$ are the components of $(1, 1)$ tensor field $A$. Then its covariant derivative is
\[\nabla_i A^m_a = \partial_i A^m_a + \Gamma^m_{i k} A^k_a - \Gamma^k_{i a} A^m_k,\]
where $\nabla_i A^m_a$ is the component of the covariant derivative of tensor field $A$.

From definition 1, $X^m = A^m_{a} y^a$. By means of relation 3, we have
\[b[L_{\gamma} g_{ij} - 2\Omega g_{ij} - g_{ni}(\nabla_i X^m - A^m_j)] = c g_{mj}[y^a X^c K^m_{ica} - \Gamma^m_{k} A^k_a y^a - \Gamma^m_{k} A^k_a y^a - X_i(A^m_{a} y^h)]]\]

Note that the components of $A$ are functions of position alone, from which the right hand side of this relation becomes
\[cg_{mj}[y^a X^c K^m_{ica} - \Gamma^m_{k} A^k_a y^a - \Gamma^m_{k} A^k_a y^a + \Gamma^m_{k} A^m_k y^a] = cg_{mj}(X^c K^m_{ica} - g_{mj} \nabla_i A^m_a).\]
Thus we have

\[ b(E, g_{ij} - 2\Omega g_{ij} - g_{mn}(\nabla_i X^m - A^m) = cy^a(X^c K_{ia} - g_{aj} \nabla_i A^a) \]

By means of Proposition 1 the left hand side of the above relation is a function of position alone. Applying \( X \frac{\partial}{\partial y} \) to this relation gives

\[ X^c K_{ia} - g_{aj} \nabla_i A^a = 0, \]

Or

\[ X^c K_{ia} = \nabla_i A_{ja}. \]

From which

\[ \nabla_i A_{ja} + \nabla_j A_{ia} = 0. \]

(6)

Now by replacing \( X \) in relation 4

\[ 2\Omega g_{ij} = g_{mj} X^i \begin{cases} (\Gamma^m_{ij} X^a + \partial_i X^m) \end{cases} + g_{mi} X^j \begin{cases} (\Gamma^m_{ij} X^a + \partial_j X^m) \end{cases} \]

\[ = g_{mj} (\Gamma^m_{ij} X^a + \partial_i X^m) + g_{mi} (\Gamma^m_{ij} X^a + \partial_j X^m) \]

\[ = g_{mj} A^m_i + g_{mi} A^m_j. \]

Applying covariant derivation \( \nabla_k \) to this relation gives

\[ 2g_{ij} \nabla_k \Omega = \nabla_k A_{ji} + \nabla_k A_{ij}. \]

From relation 6, we get \( \nabla_k \Omega = \frac{\partial}{\partial x^k} \Omega = 0 \).

Since \( M \) is connected, the scalar function \( \Omega \) is constant.

**Theorem 1.** Let \( M \) be a connected \( n \)-dimensional Riemannian manifold and \( TM \) be its tangent bundle with metric \( \tilde{g} \). Then every complete lift conformal vector field on \( TM \) is homothetic, moreover, every horizontal or vertical lift conformal vector field on \( TM \) is a Killing vector.

**Proof:** Let \( M \) be an \( n \)-dimensional Riemannian manifold, \( TM \) its tangent bundle with the metric \( \tilde{g} \) and \( X \) a complete (resp. horizontal or vertical) lift conformal vector field on \( TM \). Then by means of Proposition 1 the scalar function \( \Omega(x, y) \) in \( E, \tilde{g} = 2\Omega \tilde{g} \) is a function of position alone (resp. \( \Omega = 0 \)), and by means of Proposition 2 it is constant. Thus, every complete lift conformal vector field on \( TM \) is homothetic and every horizontal or vertical lift conformal vector field on \( TM \) is a Killing vector.

**Theorem 2.** Let \( M \) be a connected \( n \)-dimensional Riemannian manifold and \( TM \) be its tangent bundle with metric \( \tilde{g} \). Then every inessential fiber preserving conformal vector field on \( TM \) is homothetic.

**Proof:** Let \( X \) be an inessential fiber preserving conformal vector field on \( TM \) with components \((X^h, X^k)\), with respect to the adapted frame \( \{X_h, X_k\} \). Using the same argument in proof of Proposition 1, it is obvious that we have relations 2, 3 and 4. From relation 4, we have

\[ \Omega g_{hi} = g_{mi} X^i(X^\tau). \]
Since $\Omega(x, y)$ in $\mathcal{L}_\xi \bar{g} = 2 \Omega \bar{g}$ is supposed to be a function of position alone, by applying $X_\xi$ to the above relation we have

$$X_\xi(X_\xi(X^m)) = 0.$$  

Applying $X_\xi$ to relation 4 again and using above relation gives

$$X_\xi(X_\xi(X^m)) = 0.$$  

Thus we can write

$$X^m = \alpha^m_a y^a + \beta^m,$$  

where $\alpha^m_a$ and $\beta^m$ are certain functions of position alone. Replacing relation 7 in relation 3, we have

$$b(L_\xi g_{ij} - 2 \Omega g_{ij}) = bg_{im}(\nabla_j X^m - \alpha^m_j) + cg_{jm}(y^b X^c K^m_{cb} - y^b \alpha^b_x \Gamma^m_{b i} - \beta^b \Gamma^m_{b i} -$$

$$y^a \frac{\partial}{\partial x^a} \alpha^m_a - \frac{\partial}{\partial x^a} \beta^m + y^b \Gamma^k_{a} \alpha^m_k)$$

$$= bg_{im}(\nabla_j X^m - \alpha^m_j) + cg_{jm}(y^b X^c K^m_{cb} - y^a \nabla_j \alpha^m_a) - cg_{jm} \nabla_i \beta^m.$$  

Therefore

$$b(L_\xi g_{ij} - 2 \Omega g_{ij} - g_{im}(\nabla_j X^m - \alpha^m_j)) + cg_{jm} \nabla_i \beta^m = cg_{jm} y^b (X^c K^m_{cb} - \nabla_i \alpha^m_a).$$  

The left hand side of this relation is a function of position alone. From which by applying $X_\xi$ we have

$$X^c K^m_{ic a} = \nabla_i \alpha^m_a.$$  

Replacing relation 7 in relation 4 we find

$$2\Omega g_{ij} = \alpha_{ji} + \alpha_{ij}.$$  

The covariant derivative of this relation and using relation 8 gives

$$\nabla_i \Omega = \frac{\partial}{\partial x^i} \Omega = 0.$$  

Since $M$ is connected, then the scalar function $\Omega$ on $M$ is constant. This completes the proof of Theorem 2.

REFERENCES


