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THE STRUCTURE OF (α,β)-DERIVATIONS OF TRIANGULAR RINGS*

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Abstract – Let R and S be reduced rings with identities whose idempotents are central, and let M be an (R, S)-bimodule such that $\mathrm{ann}_r(M)=0$. In this paper, we determine first the structure of automorphisms of the triangular ring $T=\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$, and then for all automorphisms α,β of T we determine the structure of (α,β)-derivations of T.

Keywords – Triangular ring, automorphism, (α,β) -derivation

1. INTRODUCTION

Recently many authors have considered (α,β)- derivations and generalized (α,β)-derivations of rings. We refer the interested readers to [1] and [2] and the references therein.

Motivated by [3], we describe the (α,β)-derivations of the triangular matrix ring $T=\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$, whose components satisfy certain conditions.

Let R and S be reduced rings with identities whose idempotents are central, M be an (R,S)-bimodule such that $\mathrm{ann}_r(M)=0$, and T be the triangular ring

$$T=\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}=\left\{\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} : r\in R, m\in M, s\in S\right\}.$$

I denotes the identity matrix and $E_{i,j}$ denotes the usual unitary matrix. The zero matrix and the identically zero functions are denoted by 0, and we assume that the ring automorphisms conserve identities. If f and h are automorphisms of R and S respectively, by an (f, h)-automorphism of M we mean an additive bijective mapping g on M such that for all $r\in R, m\in M, s\in S, g(rms)=f(r)g(m)h(s)$.

Clearly, if f and g are the identity automorphisms, then g is an (R,S)-bimodule automorphism. It is easy to see that if f and h are automorphisms of R and S, respectively, and g is an (f, h)-automorphism of M, then the mapping

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$$\Phi : T \to T,\ \begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \mapsto \begin{bmatrix} f(r) & g(m) \\ 0 & h(s) \end{bmatrix}$$

is an automorphism of *T*. Φ is called *the* automorphism *induced by f*, *g*, *and h.* If *A* is an invertible matrix in *T*, then the *inner* automorphism induced by *A* is denoted by Inn_A. If α, β are automorphisms of *T*, then by an $(\alpha, \beta)-derivation$ of *T*, we mean an additive mapping *d* on *T* such that for each $x, y \in T, d(xy) = \alpha(x)d(y) + d(x)\beta(y)$. If α is the identity mapping, *d* is a $\beta-derivation$ and if α, β are both identities, *d* is *a derivation* of *T*. For each $C \in T$, the mapping $I_C : T \to T$ given by $I_C(P) = CP - PC$ is easily seen to be a derivation of *T*. I_C is called the *inner derivation* induced by *C*. We determine first the structure of automorphisms of *T*, and then the structure of (α, β)-derivations of *T*. This result has attracted the attention of mathematicians in the fields of ring theory and functional analysis.

2. THE STRUCTURE OF AUTOMORPHISMS OF T

Theorem 2. 1. Let the ring T be as above and let α be an automorphism of T. Then there exist automorphisms f and h of R and S, respectively, an (f, h)-automorphism g of M, an invertible matrix $A \in T$, and an automorphism Φ of T induced by f, g, and h such that for each $P \in T$,

$$\alpha(P) = Inn_A \circ \Phi(P).$$

Proof: Let $m \in M, \alpha(E_{11}) = \begin{bmatrix} a & m_1 \\ 0 & b \end{bmatrix}$, and $\alpha(mE_{12}) = \begin{bmatrix} k(m) & g(m) \\ 0 & t(m) \end{bmatrix}$, where $a, m_1,$ and *b* are some elements of *R*, *M*, and *S*, respectively, and $k : M \to R, g : M \to M$, and $t : M \to S$ are some functions determined by α. We have

$$\alpha(E_{22}) = I - \alpha(E_{11}) = \begin{bmatrix} 1-a & -m_1 \\ 0 & 1-b \end{bmatrix};$$

$$\begin{bmatrix} a & m_1 \\ 0 & b \end{bmatrix} = \alpha(E_{11}) = \alpha(E_{11}^2) = \alpha(E_{11})\alpha(E_{11}) = \begin{bmatrix} a^2 & am_1 + m_1 b \\ 0 & b^2 \end{bmatrix}. \tag{1}$$

Also,

$$\begin{bmatrix} k(m) & g(m) \\ 0 & t(m) \end{bmatrix} = \alpha(mE_{12}) = \alpha(E_{11}mE_{12}) = \alpha(E_{11})\alpha(mE_{12})$$

$$= \begin{bmatrix} a & m_1 \\ 0 & b \end{bmatrix} \begin{bmatrix} k(m) & g(m) \\ 0 & t(m) \end{bmatrix}$$

$$= \begin{bmatrix} ak(m) & ag(m) + m_1 t(m) \\ 0 & bt(m) \end{bmatrix}; \tag{2}$$

$$\begin{bmatrix} k(m) & g(m) \\ 0 & t(m) \end{bmatrix} = \alpha(mE_{12}) = \alpha(mE_{12}E_{22}) = \alpha(mE_{12})\alpha(E_{22})$$

$$=\begin{bmatrix} k(m) & g(m) \\ 0 & t(m) \end{bmatrix}\begin{bmatrix} 1-a & -m_1 \\ 0 & 1-b \end{bmatrix}$$

$$=\begin{bmatrix} k(m)-k(m)a & -k(m)m_1+g(m)-g(m)b \\ 0 & t(m)-t(m)b \end{bmatrix}. \tag{3}$$

From (1), (2), and (3) it follows that

$$a^2=a \ , \ b^2=b \ , \ k(m)=ak(m) \ , \ t(m)=bt(m),$$

$$k(m)a=0 \ , \ t(m)b=0.$$

Since the idempotents of R and S are central, then these relations imply that $k=t=0$. So $\alpha(mE_{12})=g(m)E_{12}$.

Next, assume that $\alpha\left(\begin{bmatrix} r_1 & m_2 \\ 0 & s_1 \end{bmatrix}\right)=E_{11}$ for some $r_1\in R, m_2\in M$, and $s_1\in S$. Then we have

$$\alpha\left(\begin{bmatrix} r_1 & m_2 \\ 0 & 0 \end{bmatrix}\right)=\alpha\left(E_{11}\begin{bmatrix} r_1 & m_2 \\ 0 & s_1 \end{bmatrix}\right)$$

$$=\begin{bmatrix} a & m_1 \\ 0 & b \end{bmatrix}E_{11}=aE_{11}.$$

Hence,

$$\alpha(s_1E_{22})=\alpha\left(\begin{bmatrix} r_1 & m_2 \\ 0 & s_1 \end{bmatrix}-\begin{bmatrix} r_1 & m_2 \\ 0 & 0 \end{bmatrix}\right)$$

$$=\alpha\left(\begin{bmatrix} r_1 & m_2 \\ 0 & s_1 \end{bmatrix}\right)-\alpha\left(\begin{bmatrix} r_1 & m_2 \\ 0 & 0 \end{bmatrix}\right)=(1-a)E_{11}. \tag{4}$$

Now, let $t\in M$ be arbitrary. Then

$$0=g(t)E_{12}(1-a)E_{11}=\alpha(tE_{12}s_1E_{22})=\alpha(ts_1E_{12}).$$

So, $ts_1=0$ for all $t\in M$. Thus by assumption, $s_1=0$, and, by (4), $a=1$.

Next, we prove that g is onto. Let $m\in M$ and assume that

$$\alpha\left(\begin{bmatrix} x & m' \\ 0 & z \end{bmatrix}\right)=mE_{12}.$$

Then

$$\alpha\left(\begin{bmatrix} x^2 & xm'+m'z \\ 0 & z^2 \end{bmatrix}\right)=\alpha\left(\begin{bmatrix} x & m' \\ 0 & z \end{bmatrix}^2\right)=\left(\alpha\begin{bmatrix} x & m' \\ 0 & z \end{bmatrix}\right)^2=0.$$

Since α is one-to-one, we have $x^2 = 0$, $z^2 = 0$, and since R and S are reduced, $x = 0$ and $z = 0$. Thus, $g(m')E_{12} = \alpha(m'E_{12}) = mE_{12}$, proving that g is onto.

Now, we show that $b = 0$. Let $m \in M$. Since $a = 1$,

$$g(m)E_{12} = \alpha(mE_{12}) = \alpha(mE_{12}E_{22}) = \alpha(mE_{12})\alpha(E_{22})$$

$$= g(m)E_{12}\begin{bmatrix} 0 & -m_1 \\ 0 & 1-b \end{bmatrix} = g(m)(1-b)E_{12}.$$

Therefore, $g(m)b = 0$ for all $m \in M$. Since g is onto and ann_r (M)=0, it follows that b=0. Consequently, for each $m \in M$ we have

$$\alpha(E_{11}) = \begin{bmatrix} 1 & m_1 \\ 0 & 0 \end{bmatrix}, \ \alpha(E_{22}) = \begin{bmatrix} 0 & -m_1 \\ 0 & 1 \end{bmatrix}, \ \alpha(mE_{12}) = g(m)E_{12}. \qquad (5)$$

Let $x \in R$ and set $\alpha(xE_{11}) = \begin{bmatrix} x_1 & x_2 \\ 0 & x_3 \end{bmatrix}$. Applying α to $xE_{11} = xE_{11}E_{11}$ and using (5) we find

$$\alpha(xE_{11}) = \begin{bmatrix} x_1 & x_1m_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} f(x) & f(x)m_1 \\ 0 & 0 \end{bmatrix}, \qquad (6)$$

for some function $f : R \to R$. Similarly, applying α to $xE_{22} = E_{22}xE_{22}$, where $x \in S$, we observe that there exists a function h on S such that

$$\alpha(xE_{22}) = \begin{bmatrix} 0 & -m_1h(x) \\ 0 & h(x) \end{bmatrix}. \qquad (7)$$

Since α is additive, then so are f, g, and h.

Therefore, by (5), (6), and (7), for each $\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \in T$ we have

$$\alpha\left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix}\right) = \begin{bmatrix} f(r) & f(r)m_1 - m_1h(s) + g(m) \\ 0 & h(s) \end{bmatrix}. \qquad (8)$$

Let $x \in R$, $y \in S$, and $m' \in M$. Since α is onto, there exists $\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \in T$ such that

$$\begin{bmatrix} x & m' \\ 0 & y \end{bmatrix} = \alpha\left(\begin{bmatrix} r & m \\ o & s \end{bmatrix}\right) = \begin{bmatrix} f(r) & f(r)m_1 - m_1h(s) + g(m) \\ 0 & h(s) \end{bmatrix},$$

proving that f and g are onto. Relation (8) and the fact that α is one-to-one imply that f, g, h are one-to-one.

Our next step is to show that f and h are homorphisms and g is an (f, g)-automorphism. Since α is additive, so are f, g, h. Let $x, y \in R$. Then

$$\alpha(xyE_{11})\alpha(xE_{11}yE_{11}) = \begin{bmatrix} f(x) & f(x)m_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} f(y) & f(y)m_1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} f(x)f(y) & f(x)f(y)m_1 \\ 0 & 0 \end{bmatrix}.$$

So, $f(xy) = f(x)f(y)$. Similarly, for each $x, y \in S$, $h(xy) = h(x)h(y)$. Now, let $r \in R, m \in M$, and $s \in S$. Then

$$g(rms)E_{12} = \alpha(rmsE_{12}) = \alpha(rE_{11}mE_{12}sE_{22}) = \alpha(rE_{11})\alpha(mE_{12})\alpha(sE_{22})$$

$$= \begin{bmatrix} f(r) & f(r)m_1 \\ 0 & 0 \end{bmatrix} g(m)E_{12} \begin{bmatrix} 0 & m_1h(s) \\ 0 & h(s) \end{bmatrix}$$

$$= f(r)g(m)h(s)E_{12},$$

Hence $g(rms) = f(r)g(m)h(s)$.

Finally, for each $\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \in T$ we have

$$\alpha\left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix}\right) = \begin{bmatrix} f(r) & f(r)m_1 - m_1h(s) + g(m) \\ 0 & h(s) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -m_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f(r) & g(m) \\ 0 & h(s) \end{bmatrix} \begin{bmatrix} 1 & m_1 \\ 0 & 1 \end{bmatrix}$$

$$= A\Phi\left(\begin{bmatrix} r & m \\ o & s \end{bmatrix}\right) A^{-1} = Inn_A \circ \Psi \left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix}\right),$$

where $A = \begin{bmatrix} 1 & -m_1 \\ 0 & 1 \end{bmatrix}$ and $\Phi : T \to T$ is given by

$$\Phi\left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix}\right) = \begin{bmatrix} f(r) & g(m) \\ 0 & h(s) \end{bmatrix}.$$

Using the properties of *f*, *g*, and *h* one can easily verify that Φ is an automorphism of *T* induced by the automorphisms *f*, *g*, and *h*. This completes the proof.

3. THE STRUCTURE OF (α,β)-DERIVATIONS OF THE RING *T*

Let the ring *T* be as above and consider the automorphisms

$$\alpha = Inn_A \circ \Phi, \quad \beta = Inn_B \circ \Psi$$

of *T*, where $A = \begin{bmatrix} 1 & m_1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & m_2 \\ 0 & 1 \end{bmatrix}$ are some (invertible) matrices in *T*, Φ is an automorphism of *T* induced by automorphisms *f*, *g*, *h* of *R*, *M*, *S*, respectively, and Ψ is an automorphism of *T* induced by automorphisms f', g', h', of *R*, *M*, *S*, respectively.

Theorem 3. 1. Assume that the automorphisms α, β of the ring *T* are as above. If *d* is an (α,β)-derivation of *T*, then there exists a (Φ,Ψ)-derivation ϕ of *T* and a matrix *C* in *T* such that for each $P \in T$,

$$d(P) = \phi(P) + A\phi(P) - \phi(P)B + C\Psi(P) - \Phi(P)C.$$

In particular, if *d* is an (α,α)-derivation of T, then

$$d(P) = \phi(P) + I_A \circ \phi(P) + I_C \circ \Phi(P).$$

Proof: Recall that for every $x, y \in T$,

$$d(xy) = \alpha(x)d(y) + d(x)\beta(y). \tag{9}$$

Assume that $d(E_{11}) = \begin{bmatrix} x_1 & x_2 \\ 0 & x_3 \end{bmatrix}$. Applying *d* to $E_{11} = E_{11}^2$ and noting that $d(I) = 0$, it follows that

$$d(E_{11}) = x_2 E_{12} \text{ and } d(E_{22}) = -d(E_{11}) = -x_2 E_{12}. \tag{10}$$

Let $m \in M$ and set $d(mE_{12}) = \begin{bmatrix} y_1 & y_2 \\ 0 & y_3 \end{bmatrix}$. Since $mE_{12} = E_{11}mE_{12} = mE_{12}E_{22}$, using (9) and (10) we infer that

$$d(mE_{12}) = y_2 E_{12} = \tau(m)E_{12}, \tag{11}$$

for some function $\tau : M \to M$.

Let $x \in R$ and set $d(xE_{11}) = \begin{bmatrix} z_1 & z_2 \\ 0 & z_3 \end{bmatrix}$. Since $xE_{11} = E_{11}xE_{11} = xE_{11}E_{11}$, using (9) and previous calculations we arrive at

$$d(xE_{11}) = \begin{bmatrix} \delta(x) & -\delta(x)m_2 + f(x)x_2 \\ 0 & 0 \end{bmatrix}, \tag{12}$$

for some function $\delta : R \to R$. Since *d* is additive, then so is δ. Let $x, y \in R$. Then the identity $xyE_{11} = xE_{11}yE_{11}$ and (9) imply that

$$\delta(xy) = f(x)\delta(y) + \delta(x)f'(y).$$

That is, δ is an (f, f')-derivation of *R*. Similar computations show that there exists an (h, h')-derivation γ of *S* such that for each $x \in S$,

$$d(xE_{22}) = \begin{bmatrix} 0 & m_1\gamma(x) - x_2 h'(x) \\ 0 & \gamma(x) \end{bmatrix}. \tag{13}$$

Now we prove some properties of τ. Since *d* is additive, then so is τ. Let $r \in R$, $m \in M$. Then

$$\tau(rm)E_{12} = d(rmE_{12}) = d(rE_{11}mE_{12})$$

$$= \alpha(rE_{11})d(mE_{12}) + d(rE_{11})\beta(mE_{12})$$

$$= \begin{bmatrix} f(r) & -f(r)m_1 \\ 0 & 0 \end{bmatrix}\tau(m)E_{12}$$

$$+\begin{bmatrix}\delta(r) & -\delta(r)m_2+f(r)x_2\\ 0 & 0\end{bmatrix} g'(m)E_{12}$$

$$=(f(r)\tau(m)+\delta(r)g'(m))E_{12}.$$

Hence,

$$\tau(rm)=f(r)\tau(m)+\delta(r)g'(m).$$

Similarly, for each $m\in M$ and $s\in S$ we find

$$\tau(ms)=\tau(m)h'(s)+g(m)\gamma(s).$$

Define the function $\phi: T\to T$ by

$$\phi\left(\begin{bmatrix}r & m\\ 0 & s\end{bmatrix}\right)=\begin{bmatrix}\delta(r) & \tau(m)\\ 0 & \gamma(s)\end{bmatrix}. \tag{14}$$

Let $P=\begin{bmatrix}r & m\\ 0 & s\end{bmatrix}$ and $P'=\begin{bmatrix}r' & m'\\ 0 & s'\end{bmatrix}$ be in *T*. Using previous computations we have

$$\phi(PP')=\phi\left(\begin{bmatrix}rr' & rm'+ms'\\ 0 & ss'\end{bmatrix}\right)=\begin{bmatrix}\delta(rr') & \tau(rm'+ms')\\ 0 & \gamma(ss')\end{bmatrix}$$

$$=\begin{bmatrix}f(r)\delta(r')+\delta(r)f'(r') & f(r)\tau(m')+\delta(r)g'(m')+\tau(m)h'(s')+g(m)\gamma(s')\\ 0 & h(s)\gamma(s')+\gamma(s)h'(s')\end{bmatrix}$$

$$=\begin{bmatrix}f(r) & g(m)\\ 0 & h(s)\end{bmatrix}\begin{bmatrix}\delta(r') & \tau(m')\\ 0 & \gamma(s')\end{bmatrix}+\begin{bmatrix}\delta(r) & \tau(m)\\ 0 & \gamma(s)\end{bmatrix}\begin{bmatrix}f'(r') & g'(m')\\ 0 & h'(s')\end{bmatrix}$$

$$=\Phi(P)\phi(P')+\phi(P)\Psi(P').$$

Therefore, ϕ is a (Φ,Ψ)-derivation of *T*. Finally, define $C=-x_2E_{12}$ and let $P=\begin{bmatrix}r & m\\ 0 & s\end{bmatrix}$ be in *T*. Then by (11), (12), (13), and (14) we have

$$d(P)=\begin{bmatrix}\delta(r) & -\delta(r)m_2+f(r)x_2+\tau(m)+m_1\gamma(s)-x_2h'(s)\\ 0 & \gamma(s)\end{bmatrix}$$

$$=\begin{bmatrix}\delta(r) & \tau(m)\\ 0 & \gamma(s)\end{bmatrix}+\begin{bmatrix}0 & -\delta(r)m_2+f(r)x_2+m_1\gamma(s)-x_2h'(s)\\ 0 & 0\end{bmatrix}$$

$$=\phi(P)+\begin{bmatrix}1 & m_1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\delta(r) & \tau(m)\\ 0 & \gamma(s)\end{bmatrix}-\begin{bmatrix}\delta(r) & \tau(m)\\ 0 & \gamma(s)\end{bmatrix}\begin{bmatrix}1 & m_2\\ 0 & 1\end{bmatrix}$$

$$+\begin{bmatrix}0 & -x_2\\ 0 & 0\end{bmatrix}\begin{bmatrix}f'(r) & g'(m)\\ 0 & h'(s)\end{bmatrix}-\begin{bmatrix}f(r) & g(m)\\ 0 & h(s)\end{bmatrix}\begin{bmatrix}0 & -x_2\\ 0 & 0\end{bmatrix}$$

$$= \phi(P) + A\phi(P) - \phi(P)B + C\Psi(P) - \Phi(P)C.$$

In particular, if $\alpha = \beta$, then A=B and $\Phi = \Psi$. So,

$$d(P) = \phi(P) + I_A \circ \phi(P) + I_C \circ \Phi(P).$$

Remark. The proof of the above theorem shows that when the ring T reduces to the ordinary triangular ring, *i.e*, $R = S = M$, where R is a ring with identity, then the result is in accordance with [3, Theorem, P. 263].

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