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ON GENERAL RELATIVELY ISOTROPIC MEAN LANDSBERG METRICS*

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Abstract – The general relatively isotropic mean Landsberg metrics contain the general relatively isotropic Landsberg metrics. A class of Finsler metrics is given, in which the mentioned two concepts are equivalent. In this paper, an interpretation of general relatively isotropic mean Landsberg metrics is found by using C-conformal transformations. Some necessary conditions for a general relatively isotropic mean Landsberg metric, as well as generalized Landsberg metric to be a Riemannian metric are also found.

Keywords – Finsler structure, Cartan connection, (generalized) Landsberg metric, scalar curvature, C2-like space, c-conformal transformation

1. INTRODUCTION

Cartan and Landsberg tensors, respectively C and L, play an important role in Finsler geometry. It is natural to study L/C as the relative rate of change of L along geodesics, leading to a study of *general relative isotropic Landsberg metrics*. The study of general relative isotropic Landsberg metric was initiated by Izumi [1, 2]. Since then, there have been many contributions to this subject such as those by Chen, Mo and Shen [3] and Bacso and Papp [4]. By using C -conformal transformation, an interpretation of a general relative isotropic Landsberg metric was given by Hashiguchi [5].

Mean Cartan and mean Landsberg tensors, respectively I and J, are two important tensors in Finsler geometry. Studying J/I is also natural as the relative rate of change of J along geodesics, which leads to a study of *general relative isotropic mean Landsberg metrics*. This class of Finsler metrics contains the class of general relatively isotropic Landsberg metrics. In Section 3, it is shown that these two cited classes are the same on $C2$-like spaces (Theorem 9). In Section 4, C-conformal transformations are used to obtain an interpretation of general relative isotropic mean Landsberg metrics (Theorem 14).

The authors found some necessary conditions for a general relative isotropic Landsberg metric to be a Riemannian metric [6]. In the last section, the same is done for a general relative isotropic mean Landsberg metric, as well as a generalized Landsberg metric.

Throughout this paper, the Cartan connection is set on Finsler manifolds. The Einstein convention is used, that is, repeated indices with one upper index and one lower index denote summation over their range.

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2. PRELIMINARIES

In this section, mainly background material is presented about the basic tools and notations. Let M be an n-dimensional C^∞ manifold. The tangent space at $x \in M$ is denoted by T_xM, and the tangent bundle of M is denoted by TM. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) = x$. The *pull-back tangent bundle* π^*TM is a vector bundle over TM_0 whose fiber $\pi^*_v TM$ at $v \in TM_0$ is just T_xM, where $\pi(v) = x$. Thus

$$\pi^*TM = \{(x, y, v) \mid y \in T_xM_0, v \in T_xM\}.$$

A *Finsler metric* on a manifold M is a function $F : TM \to [0, \infty)$ with the following properties:
(i) F is C^∞ on TM_0,
(ii) $F(x, \lambda y) = \lambda F(x, y) \quad \forall \lambda > 0$,
(iii) For any tangent vector $y \in T_xM$, the vertical Hessian of $\frac{F^2}{2}$ given by

$$(g_{ij}(x, y)) = \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$

is positive definite.

Definition 1. A Finsler metric F on a manifold M is said to be locally Minkowskian if at every point $x \in M$ and $y \in T_xM$, and there is a local standard coordinate system (x^i, y^i) for TM, such that F has no dependence on the x^i.
A symmetric tensor **C** is defined by

$$\mathbf{C}(U, V, W) = C_{ijk}(y) U^i V^j W^k,$$

where $U = U^i \frac{\partial}{\partial x^i}$, $V = V^i \frac{\partial}{\partial x^i}$, $W = W^i \frac{\partial}{\partial x^i}$ and $C_{ijk} = \frac{1}{4}[F^2]_{y^i y^j y^k}(y)$. $\mathbf{C}$ is called *Cartan* tensor. Further, let $I_k = g^{ij} C_{ijk}$. Then $\mathbf{I}$ is called *mean Cartan tensor*.

Theorem 2. ([7])The following are equivalent
a) **C**=0,
b) **I**=0,
c) F is Riemannian.
Two symmetric tensors $\mathbf{L}$ and $\mathbf{J}$ on π^*TM are defined as the following

$$\mathbf{L}(U, V, W) = L_{ijk}(y) U^i V^j W^k, \quad \mathbf{J}(U) = J_i(y) U^i,$$

where $L_{ijk} = C_{ijk|s} y^s$ and $J_i = g^{jk} L_{ijk}$. $\mathbf{L}$ is called *Landsberg tensor*, and $\mathbf{J}$ is called *mean Landsberg tensor*.

Definition 3. A Finsler metric is called a *Landsberg metric* (resp. *weakly Landsberg metric*) if $L = 0$ (resp. $J = 0$).
Let $c(t)$ be an arbitrary geodesic in (M, F). Suppose $U(t)$, $V(t)$, and $W(t)$ are arbitrary parallel vector fields along c. It is easy to see that Landsberg and Cartan tensors satisfy the following

$$L_{\dot{c}(t)}(U(t),V(t),W(t))=\frac{d}{dt}[C_{\dot{c}(t)}(U(t),V(t),W(t))].$$

Thus the Landsberg tensor measures the rate of change of the Cartan tensor along geodesics.

Let (M,F) be a Finsler manifold. A global vector field G is induced by F on TM_0. This vector field in a standard coordinate (x^i,y^i) for TM_0 is given by

$$G=y^i\frac{\partial}{\partial x^i}-2G^i(x,y)\frac{\partial}{\partial y^i},$$

where $G^i(x,y)$ are local functions on TM_0 satisfying $G^i(x,\lambda y)=\lambda^2G^i(x,y)$ $\lambda>0$. G is called the associated *spray* to (M,F).

The projection of an integral curve of G is called a *geodesic* in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i+2G^i(\dot{c})=0$. F is said to be *positively complete* (resp. *negatively complete*), if any geodesic on an open interval (a,b) can be extended to a geodesic on (a,∞) (resp. $(-\infty,b)$). F is said to be *complete* if it is positively and negatively complete.

The notion of Riemann curvature for Riemann metrics can be extended to Finsler metrics. For a non-zero vector $y\in T_xM$, the Riemann curvature $R_y:T_xM\to T_xM$ is defined by

$$R_y(u)=R^i_k(y)u^k\frac{\partial}{\partial x^i},$$

where

$$R^i_k(y)=2\frac{\partial G^i}{\partial x^k}-\frac{\partial^2G^i}{\partial x^j\partial y^k}y^j+2G^j\frac{\partial^2G^i}{\partial y^j\partial y^k}-\frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

Suppose $P\subset T_xM$ (flag) is an arbitrary plane and $y\in P$ (flag pole). The flag curvature $K(P,y)$ is defined by

$$K(P,y)=\frac{g_y(R_y(v),v)}{g_y(y,y)g_y(v,v)-g_y(v,y)g_y(v,y)},$$

where v is an arbitrary vector in P such that $P=span(y,v)$.
F is said to be of scalar curvature if for any non-zero vector $y\in P\subset T_xM$, $K(P,y)=\lambda(y)$ is independent of P, or equivalently,

$$R_y=\lambda(y)F^2(y)\{I-g_y(y,.)y\},\quad y\in T_xM,\quad x\in M$$

where $I:T_xM\to T_xM$ denotes the identity map and $g_y(y,.)=\frac{1}{2}[F^2]_{y^i}dx^i$. F is also said to be of constant curvature λ, if the above identity holds for the constant λ [8, 9].

Definition 1. The Funk metric on a strongly convex domain $\Omega\subset R^n$ is a nonnegative function on $T\Omega=\Omega\times R^n$, which satisfies $F_{x^i}=FF_{y^i}$ ([10]).

Definition 2. A Finsler metric F is said to be *generalized Landsberg metric* if the Riemannian curvatures of Berwald and Chern connections coincide [11].

Remark 2. Every Landsberg metric is a generalized Landsberg metric, but the converse is an open problem.

Definition 3. A Finsler metric F on a manifold M is said to be *general relative isotropic Landsberg metric* (GRI Landsberg metric), if

$$L_{ijk} + \lambda C_{ijk} = 0,$$

where λ is a positively 1-homogeneous scalar function on TM_0.

Definition 4. A Finsler metric F is said to be *general relatively isotropic mean Landsberg metric* (GRI mean Landsberg metric), if

$$J_i + \lambda I_i = 0.$$

Remark 3. A simple manipulation yields $\lambda = \frac{J^m I_m}{I^m I_m}$.

Remark 4. It is obvious that every GRI Landsberg metric is GRI mean Landsberg metric, but the converse is not true in general.

3. C2-LIKE SPACES

Definition 5. A Finsler metric F is said to be *semi-C-reducible* if the Cartan tensor is written in the following form

$$C_{ijk} = \frac{p}{1+n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\} + \frac{q}{C^2}I_i I_j I_k,$$

where $h_{ij} = g_{ij} - \ell_i \ell_j$ is the angular metric tensor, $p = 1-q$ and $C^2 = I^m I_m$. Moreover, if $p = 0$, F is said to be a $C2$*-like* Finsler metric and if $q = 0$, F is said to be a C-reducible Finsler metric [12, 13].

Remark 5. Every 2-dimensional Finsler metric is C-reducible.

Theorem 6. Let F be a C2-like Finsler metric. Then the following are equivalent
1) F is GRI Landsberg metric.
2) F is GRI mean Landsberg metric.

Proof: It is sufficient to show that if F is a GRI mean Landsberg metric, then it is a GRI Landsberg metric. By definition of GRI mean Landsberg metric we have

$$J_k = \lambda I_k.$$

Since F is C2-like, then

$$C_{ijk} = \frac{1}{I^m I_m} I_i I_j I_k.$$

Taking horizontal covariant derivative $"|s"$ from the above relation and contracting it with y^s yield

$$L_{ijk} = \frac{-(J^m I_m + I^m J_m)}{(I^m I_m)^2} I_i I_j I_k + \frac{1}{I^m I_m}(J_i I_j I_k + I_i J_j I_k + I_i I_j J_k).$$

Plugging $J_k = \lambda I_k$ into the last relation implies that

$$L_{ijk} = -\frac{2\lambda}{I^m I_m} I_i I_j I_k + \frac{3\lambda}{I^m I_m} I_i I_j I_k .$$

Thus

$$L_{ijk} = \lambda C_{ijk} .$$

This means that F is a GRI Landsberg metric.

Remark 6. Singh and Gupta show that on C2-like Finsler space, Landsberg metrics and mean Landsberg metrics are equivalent [14]. Therefore Theorem 9 can be considered as an extension of their result.

4. C-CONFORMAL MEAN LANDSBERG METRICS

Definition 7. Two Finsler metrics F and $\tilde{F}$ on M are called *conformal* if $\tilde{g}_{ij} = \varphi g_{ij}$, where φ is a positive scalar function on TM . Furthermore, when φ is a constant, they are called *homothetic*.

Remark 7. Knebelman's theorem states that φ falls into, at most, a point function. Thus we can assume $\varphi = e^{2\alpha}$, where α is a scalar function on M .
We put $\alpha_i = \frac{\partial \alpha}{\partial x^i}$, $C^i_j := C^{ir}_j \alpha_r$ and $\alpha_0 = \alpha_i y^i$. Then we have the following well-known results for two conformal Finsler metrics

$$\tilde{F} = e^{\alpha} F,$$

$$\tilde{g}^{ij} = e^{-2\alpha} g^{ij},$$

$$\tilde{C}_{ijk} = e^{2\alpha} C_{ijk},$$

$$\tilde{C}^i_{jk} = C^i_{jk}.$$

$$\tilde{L}^i_{jk} = L^i_{jk} + V^i_{jk}$$

where $V^i_{jk} = F^2(C^i_{j|k} + C^{ir}_j C_{rk}) + C^i_j y_k + C^i_k y_j + C_{jk} y^i + \alpha_0 C^i_{jk}$.

Proposition 8. The following conditions are equivalent [5]

a) $C_{jk} = 0$,

b) $C^k_j = 0$,

where $C_{jk} = g_{jm} C^m_k$.

Definition 9. Two conformal Finsler metrics F and $\tilde{F}$ on M are called *C-conformal* if their conformal transformation is non-homothetic and satisfies

$$C_{jk} = 0.$$

Proposition 10. Let $\tilde{F}$ and F be two homothetic Finsler metrics on M. Then F is mean Landsberg metric if and only if $\tilde{F}$ is so.

Proof: By definition α is constant. This leads to $V^{\ i}_{jk} = 0$. Hence, we have $\tilde{L}^{\ i}_{jk} = L^i_{jk}$. Contracting the last relation with $\tilde{g}^{jk}$ yields $\tilde{J}^i = e^{-2\alpha} J^i$, which concludes the proof.

Theorem 11. Let $\tilde{F}$ and F be two C-conformal Finsler metrics on M. Suppose $\tilde{F}$ is mean Landsberg metric. Then F is a GRI mean Landsberg metric.

Proof: By definition of C-conformal and Proposition 11, we get $V^{\ i}_{jk} = \alpha_0 C^{\ i}_{jk}$. Consequently

$$\tilde{L}^{\ i}_{jk} = L^i_{jk} + V^{\ i}_{jk} = L^i_{jk} + \alpha_0 C^{\ i}_{jk}.$$

Contraction with $\tilde{g}^{ij}$ yields

$$\tilde{J}_k = e^{2\alpha}(J_k + \alpha_0 I_k).$$

Since $\tilde{F}$ is mean Landsberg metric, we get $J_k = \lambda I_k$, where $\lambda = -\alpha_0$.

Corollary 12. Let F be a mean Landsberg metric. If F is unchanged by a C-conformal transformation, then F is Riemannian. Especially if a Minkowski space is C-conformal to a mean Landsberg space, then the space is Euclidean

5. REDUCTION TO RIEMANNIAN METRIC

Bejancu and Farran have shown that if a generalized Landsberg metric F is of non-zero scalar curvature, then F must be Riemannian [11]. Here, we state another condition on a generalized Landsberg metric to be Riemannian.

Theorem 1. Let F be a generalized Landsberg metric. Suppose F is also GRI mean Landsberg metric and $\lambda_0 := \lambda_{|s} y^s = 0$, $\lambda \neq 0$. Then F is Riemannian.

Proof: By definition of generalized Landsberg metric, we get

$$L_{ijl|k} - L_{ijk|l} + L_{isk} L^s_{jl} - L_{isl} L^s_{jk} = 0.$$

Contraction with g^{ij} yields

$$J_{l|k} - J_{k|l} + L^j_{sk} L^s_{jl} - L^j_{sl} L^s_{jk} = 0.$$

Contraction with y^l implies that

$$\dot{J}_k = 0.$$

Since F is GRI mean Landsberg metric and $\lambda_0 = 0$, we get

$$\dot{J}_k + \lambda J_k = 0.$$

Thus we have $I_k = 0$. By the Diecke Theorem F is Riemannian.

Authors show that every $\mathbf{R}$-flat and non-zero constant relative Landsberg metric is a Riemannian metric [6]. Here we extend this result to GRI mean Landsberg metrics.

Theorem 2. Let F be a R-flat Finsler metric. Suppose F is GRI mean Landsberg metric and $\lambda_0 = 0$, $\lambda \neq 0$. Then F is Riemannian.

Proof: For R-flat Finsler metrics we have

$$L_{ijk|l} = L_{ijl|k} + L_{sjk}(L^s_{il} - A^s_{il}) + L_{isk}(L^s_{jl} - A^s_{jl}) - L_{sjl}(L^s_{ik} - A^s_{ik}) - L_{isl}(L^s_{jk} - A^s_{jk})$$

Contraction with $y^l g^{ij}$ yields

$$\dot{J}_k = 0.$$

Since F is GRI mean Landsberg metric and $\lambda_0 = 0$, we get

$$\dot{J}_k + \lambda J_k = 0.$$

Thus we have $I_k = 0$, which means that F is Riemannian.

Theorem 3. Let (M, F) be a complete non-zero constant relatively mean Landsberg manifold with bounded Cartan torsion. Then (M, F) is a Riemannian manifold.

Proof: Let p be an arbitrary point of M, and $y, u \in T_pM$. Let $c: (-\infty, \infty) \to M$ be the unit speed geodesic passing through p and let $\frac{dc}{dt}(0) = y$. If $U(t)$ is the parallel vector field along c with $U(0) = u$, we put $I(t) = I(U(t))$ and $\dot{I}(t) = \dot{I}(U(t))$. By definition, we have

$$\dot{I}(t) = c_0 I(t),$$

of which its general solution is

$$I(t) = I(0)e^{c_0 t}.$$

Using $\| I \| < \infty$, and letting $t \to +\infty$ or $t \to -\infty$, we have $I(0) = I(u) = 0$, so $I = 0$ i.e. (M, F) is a Riemannian manifold.

Corollary 4. Every non-zero constant relatively compact mean Landsberg metric is a Riemannian metric.

Remark 8. The completeness condition in the above theorem can not be replaced by positively complete or negatively complete. For example, Funk metric on $B^n \subseteq R^n$ satisfies all conditions of the above theorem but completeness, more precisely Funk metric is a non-Riemannian positively complete Finsler metric.

Numata has shown that a Landsberg metric of non-zero scalar curvature is Riemannian [15]. Izumi extended Numata's result to GRI Landsberg metrics. More precisely

Proposition 5. [2] Let F be a Finsler metric of constant curvature, say K. Suppose $L = \lambda C$ and

$F^2K + \lambda_{|0} + \lambda^2 \neq 0$. Then F is Riemannian metric.

Here, we extend Izumi's result to GRI mean Landsberg metrics.

Theorem 6. Let F be a Finsler metric of constant curvature say K. Suppose $J = \lambda I$ and $F^2K + \lambda_{|0} + \lambda^2 \neq 0$. Then F is Riemannian metric.

Proof: Let F be a Finsler metric of scalar curvature $K = K(x, y)$. Then

$$3B_{lijk|0} - 2y_i(3KC_{jkl} + K_j h_{kl} + K_k h_{jl} + K_l h_{jk}) + F(h_{ij}K_{kl} + h_{ik}K_{jl} + h_{il}K_{jk}) = 0.$$

Contraction of the above identity with y^i yields

$$-6L_{jkl|0} - 2F^2(3KC_{jkl} + K_j h_{kl} + K_k h_{jl} + K_l h_{jk}) = 0,$$

contraction of the above equation with g^{jk} yields

$$-6J_{l|0} - 2F^2\{3KI_l + K_j(\delta_l^j - \ell^j \ell_l) + K_k(\delta_l^k - \ell^k \ell_l) + K_l(n-1)\} = 0,$$

or

$$3J_{l|0} + F^2(3KI_l + (n+1)K_l) = 0.$$

Now if we suppose F is of constant curvature, then we get the following

$$J_{l|0} + F^2KI_l = 0.$$

The assumption that F is GRI mean Landsberg metric implies that

$$(F^2K + \lambda^2 + \lambda_{|0})I_l = 0.$$

If $F^2K + \lambda^2 + \lambda_{|0} \neq 0$, then $I_l = 0$, by the Deicke theorem F is Riemannian metric.

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