EXTREMAL ORDERS INSIDE SIMPLE ARTINIAN RINGS*

N. H. HALIMI**

84 Manburgh Tce, Darra, QLD 4076, Australia
Email: n_h_halimi@yahoo.com.au

Abstract – The aim of this paper is to study orders over a valuation ring $V$ with arbitrary rank in a central simple $F$-algebra $Q$. The relation between all of the orders is explained with a diagram. It is then shown that inside Bezout order, extremal $V$-orders are precisely semi-hereditary. In the last section, the effect of Henselization on maximal and semi-hereditary orders is examined.

Keywords – Dubrovin valuation rings, extremal orders, Henselization

1. INTRODUCTION

In this paper, all rings are associative with a multiplicative unit and all modules are unitary. If $A$ is a ring, $J(A)$ will denote its Jacobson radical, $U(A)$ its group of units, $Z(A)$ its center, $A^*$ its set of nonzero divisors, and $M_n(A)$ the ring of $n \times n$ matrices over $A$. The residue ring $A/J(A)$ will be denoted by $\overline{A}$. And $Q$ denotes a simple artinian ring with finite dimension over its center $Z(Q)$, while $D$ denotes a division ring.

In the second section we briefly discuss some of the ring theoretic properties and definitions.

In the third section we will see that semihereditary $V$-orders are extremal $V$-orders and obtain a diagram of maximal $V$-orders when $V$ is a Henselian valuation ring.

In the fourth section we show that inside Bezout orders, extremal $V$-orders are precisely semihereditary, which is a generalization of Proposition 2.1 of [1].

In the last section we will examine the effect of Henselization on maximal and semihereditary orders.

2. DEFINITION AND PRELIMINARIES

In this paper $F$ denotes a field and $Q$ is a central simple $F$-algebra, i.e., $Q$ is a $F$-Algebra with $[Q:F]<\infty$ and $F=Z(Q)$.

The most successful extension of the classical valuation theory on $F$ to $Q$ is the one introduced by Dubrovin in [2] and [3].

Definition 2.1. A subring $B$ of a central simple $F$-algebra $Q$ is called a Dubrovin valuation ring in $Q$ if

1. $B$ has an ideal $M$ such that $B/M$ is a simple artinian ring and
2. For each $q \in Q\setminus B$ there exist $b, a \in B$ such that $bq, qa \in B\setminus M$.

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**Corresponding author
The following properties of Dubrovin valuation rings were proved by Dubrovin in [2, 3].

i) The two sided ideals of $B$ are totally ordered by inclusion, where two sided ideals are a $B$-bimodule of $Q$. Therefore we have $M=J(B)$.

ii) Each finitely generated left (resp, right) ideal of $B$ is principal.

iii) (a) Let $V$ be a valuation ring of $F$, then there exists a Dubrovin valuation ring of $B$ in $Q$ such that $B∩F=V$, [2-4].

(b) If $B$, and $B'$ are two Dubrovin valuation rings of $Q$ extending $V$, then $B'=dBd^{-1}$ for some $d ∈ Q^*$ [5, 6].

Therefore, for every valuation ring $V$ of $F=Z(Q)$, there is a unique (up to conjugate) associated Dubrovin valuation ring $B$ of $Q$. It is reasonable to expect that $B$ will carry much information about the arithmetic of $Q$ in relation to $V$, (see [7] Theorem 3.4 and [8] Theorem 3.7).

**Definition 2.2.** Let $Q$ be a finite-dimensional $F$-Algebra and $V$ a ring with quotient field $F$. A subring $R$ of $Q$ is said to be an order in $Q$ if $RF=Q$. If $V=Z(R)$, then $R$ is said to be a $V$-order if, in addition, $R$ is integral over $V$. If $R$ is maximal with respect to inclusion among $V$-order of $Q$, then $R$ is said to be a maximal order over $V$.

a) In the case $V$ is a discrete valuation ring, then by ([9], 18.6 and 18.2) any $V$-order in a central simple $F$-algebra is a finite $V$-module, so for such $V$, Definition 2.2 agrees with the usual one, as in [10].

b) In this paper we assume $V$ is a commutative valuation ring in $F$ of arbitrary Krull-dimension. The integrality hypothesis in the above definition is used to guarantee the existence of maximal orders for any $Q$ and $V$. But finitely generated maximal $V$-orders need not exist, (see [7] Proposition 2.3).

c) Let $V$ be a valuation ring of a field $F$, and $Q$ a central simple $F$-Algebra. If $B$ is an integral Dubrovin extension of $V$ to $Q$ (i.e., $B$ is a Dubrovin valuation ring of $Q$ such that $B$ is integral over $V$ and $V=B∩F$) then $B$ is a maximal $V$-order (by Example 2.2 [7]).

**Definition 2.3.** A ring $R$ is said to be extremal if for every overring $S$ such that $J(R) ⊆ J(S)$ we have $S=R$. If $S$ is an overring of $R$, we say that $R$ is extremal in $S$ if $R$ is extremal among all subrings of $S$. A $V$-order $R$ is said to be an extremal $V$-order (or just extremal when the context is clear) if it is extremal among all $V$-orders in $Q$.

**Definition 2.4.** A ring $R$ is said to right (resp left) Bezout if every finitely generated right (left) ideal is principal. It is called Bezout if it is both right and left Bezout.

If $V$ is a valuation ring, then there exists a Bezout $V$-order $B$ in $Q$ and each Bezout $V$-order is a maximal order by ([7] Theorem 3.4), and if $B$, and $B'$ are two Bezout $V$-orders, then $B$, and $B'$ are conjugate (by Theorem 6.12 [4]).

**Definition 2.5.** A ring $R$ is said to be right semihereditary (resp right hereditary) if every finitely generated right ideal (resp every right ideal) is projective as a right $R$-module. A ring is said to be semihereditary (resp hereditary) if it is both left and right semihereditary (resp hereditary).

a) If $V$ be Dedekind domain with quotient field $F$ and $Q$ is a central simple $F$-Algebra, where $Q ≅ M_n(D)$ and $D$ is a division ring with center $F$, then $R$ is a hereditary $V$-order if and only if $R$ is an extremal (see 39.14 [10]).

b) Let $V$ be a valuation ring of $F=Z(Q)$ and $Q$ a central simple $F$-Algebra. J.S. Kauta proved that every semihereditary $V$-Order is extremal (see Theorem 1.5 [11]), but the converse is not true. If $F$ is
a field, $Q=M_2(F)$, $V_n$ is a discrete valuation ring of dimension $n$, and $R$ is a maximal $V_n$-order in $Q$, then there are three possibilities for the isomorphism class of $R$.

(1) $R \cong M_2(V_m)$, where $V_m$ is the overring of $V_n$ of dimension $m$. In this case $R$ is a Bezout.

(2) $R \cong \begin{bmatrix} V_m & J(V_p) \\ V_p & V_m \end{bmatrix}$, where $m < p$. In this case $R$ is semihereditary, but not Bezout.

(3) $R$ is primary (i.e., $J(R)$ is a maximal ideal of $R$) but not Bezout (see [7], Theorem 5.7). Let $R$ be maximal $V$-order in $M_2(F)$ which is primary, but not Bezout. Such an order cannot be semihereditary, since any primary semihereditary order is a Dubrovin valuation ring ([3]: Theorem 4), and hence Bezout.

### 3. MAXIMAL ORDERS OVER HENSELIAN VALUATION RINGS

In this section $D$ always means a finite dimensional algebra with center $F$. A subring $B$ of $D$ is said to be a total valuation ring in $D$ if $d \in B$ or $d^i \in B$ for all nonzero $d \in D$.

We recall that a valuation ring $V$ in a field $F$ is Henselian when Hensel’s Lemma holds for $V$, i.e., for every monic polynomial $f \in V[x]$, if its image $\tilde{f} \in \overline{V}[x]$, where $\overline{V} = V/J(V)$ has a factorization $\tilde{f} = \tilde{g}\tilde{h}$ on $\overline{V}[x]$ with $\tilde{g},\tilde{h}$ monic and $\gcd(\tilde{g},\tilde{h})=1$, then there exist monic $g,h \in V[x]$ with $f = gh$, $\overline{g} = \tilde{g}$ and $\overline{h} = \tilde{h}$, where $\overline{g}$ and $\overline{h}$ are images $g$ and $h$ respectively.

There are several other equivalent characterizations of the Henselian valuation ring, but the most relevant here is the following.

A valuation ring $V$ in a field $F$ is Henselian if $V$ has a unique extension to each field $F \subseteq K$ with $K$ algebraic over $F$ (see [9] Corol.16.6 for a proof).

Now let $D$ be a division algebra finite dimensional over its center $Z(D)=F$, and $V$ a Henselian valuation ring of $F$. Schilling ([12] P.53, Theorem 9) proved that the integral closure $V$ in $D$ forms a ring $B$. The ring $B$ is a total valuation ring of $V$ and by ([13], Theorem 1) and $B$ is the unique extension $V$ to $D$. Therefore $B$ is an invariant valuation ring of $D$ (i.e., $dBD = B$ for any $d \in D^*$).

**Theorem 3.1.** Let $D$ be a division algebra admitting a total valuation ring extending $V$. Then the integral closure of $V$ in $D$ is the unique extremal $V$-order (and hence the unique semihereditary $V$-order) in $D$.

**Proof:** By ([14]: Lemma 2) $V$ has only a finite number of extensions to $D$. If $B_1,...,B_n$ are all the extensions of $V$, then $B_i$ and $B_j$ are conjugate for all $i,j$ by ([14]: Theorem 2). Let $T=\text{Int}_D(V)$ be the integral closure of $V$ in $D$. Then $T=\bigcap_{i=1}^n B_i$ by ([14]: Theorem 3). Let $R$ be an extremal $V$-order.

Then $R \subseteq T$, because $R$ is integral over $V$. But both $R$ and $J(B_i)$ contain $J(V)$. Hence for each $i$, $R/(J(B_i) \cap R)$ is finite dimensional over $V/J(V)$. But one has the embedding $R/(J(B_i) \cap R) \rightarrow B_i/J(B_i)$ and $[B_i/J(B_i): V/J(V)] \leq [D:F] < \infty$ by ([14]: Lemma 3). It follows that $R/(J(B_i) \cap R)$ is division algebra, and hence $J(B_i) \cap R$ is a maximal ideal of $R$. Hence, $J(R) \subseteq J(B_i) \cap R$.

Let $x \in \bigcap_i J(B_i)$ and $a,b \in J(T)$. Then $1-axb \in U(B_i)$ for all $i$, and thus $1-axb \in U(T)$. Therefore $x \in J(T)$. Hence $J(R) \subseteq \bigcap_i J(B_i) \subseteq J(T)$. Since $R$ is extremal, we must have $R=T$. 

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On the other hand, $T$ is a Bezout $V$-order by ([7]: Theorem 3.4) and every such $T$ is a semihereditary $V$-order in $D$.

**Corollary 3.2.** Let $V$ be a valuation ring of $F$, and $D$ suppose admits and invariant valuation ring $B$ extending $V$. Then $B$ is the unique extremal (and hence the unique semihereditary) $V$-order in $D$.

**Proof:** Since the extensions of $V$ to $D$ are conjugate, $B$ is the unique extension of $V$ to $D$. So the corollary follows from Theorem 3.1.

In the rest of the section we assume $V$ to be a Henselian valuation ring of $F$, and $D$ be a finite dimensional division algebra over its center $\mathbb{Z}(D)=F$.

Let $B$ be the unique extension of $V$ to $D$, and let $\beta$ be the set of all nonzero $B$-submodules of $D$. Then $\beta$ is totally ordered. For if $I$ and $J$ are two $B$-submodules of $D$ such that $I \nsubseteq J$, there exists an $a \in I-J$. Then if $b \in J$, then $ab^{-1} \notin B$; thus $ba^{-1} \in B$, and hence $b \in Ba \subseteq I$. Thus $J \subseteq I$.

**Definition 3.3.** Let $I$ be a $B$-submodule of $D$. We define $I^1$ to be \{ $d \in D$: $dI \subseteq B$ \}.

**Definition 3.4.** Let $Q=M_n(D)$. An order $R=(Bi,j)$ is said to be of type $\Phi H$ if

\[
\begin{bmatrix}
B_i \cdots & B_i \ldots \\
& \vdots & \ddots & \vdots \\
& \ddots & \ddots & B_i \\
B_{i,1} & B_{i,2} & \cdots & B_{i,n}
\end{bmatrix}
\]

i) $Bi,j \in \beta$.

ii) If $d \notin Bi,j$, then $d^{-1} \in Bi,j$ for all $d \neq 0 \in D$. (Morandi’s condition).

iii) $Bi,jBi,s \subseteq Bi,s$ for all $1 \leq r,s,j \leq n$.

We denote $R$ by $(Bi,j)$.

**Lemma 3.5.** (a) $R$ is a ring and $RF=RD=Q$, i.e., $R$ is an order.

(b) $Bi,j \subseteq B \subseteq Bi,j$ or $Bi,j \subseteq B \subseteq Bi,j$ for all $i,j$.

**Proof:** (a) by (iii) $R$ is a ring, because $Bi,j \neq 0$ for all $i,j$, therefore $RF=RD=Q$.

For (b) since $\beta$ is totally ordered, we have $Bi,j \subset B$ or $B \subseteq Bi,j$. If $Bi,j \subset B$, then $1 \notin Bi,j$, and hence, $1 \in Bi,j$.

If $B \subseteq Bi,j$, then $Bi,jBi,j \subseteq Bi,j=B \Rightarrow Bi,j=Bi,jI \subseteq B$, and hence $Bi,j \subseteq B \subseteq Bi,j$.

**Lemma 3.6.** (Morandi) Let $Q=M_n(D)$ and $R=(Bi,j)$. Then $xR$ is projective as a $R$-module for all $x \in Q$.

**Proof:** We first suppose $xR$ is projective for all $x \in e_iR$ for any $i$. We prove $xR$ is projective for any $x$ (where $e_{i,j}$ is matrix $n \times n$ with 1 in $(i,j)$ entry and zero in the others). We do this by showing that $e_{i,xR}$ is projective, where $e_{i}x=e_{i,1}+e_{i,2}+\ldots+e_{i,n}$. We use induction on $i$, the case $i=1$ is true by assumption (because if $x=(d_{i,j})$ then $xe_{1,i}R=(xe_{1,i})R$, and since $xe_{1,i}=d_{1,i}e_{1,1}$ and $d_{1,i} \in B_{i,i}$ or $d_{1,i} \in B_{i,j}$, therefore $xe_{1,i} \in e_{1,i}R$). So suppose $e_{i,xR}$ is projective for all $x \in e_iR$. We have the exact sequence of $R$-modules.
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$0 \rightarrow e_0xR \cap (1-e_0)xR \rightarrow e_0xR \rightarrow e_0xR \rightarrow 0$, where $1 = e_{1,1} + e_{2,2} + \ldots + e_{n,n}$. Now $e_{i,1}e_0xR = e_{i,1}xR$ and $e_0xR \cap (1-e_{i,1})xR \subseteq e_0xR \cap (1-e_{i,1})xR$ (because $1-e_{i,1} = e_{i,1} + \ldots + e_{n,n}$). Since $e_{i,1}xR$ is projective by the induction of the sequence splits. So $e_0xR \cong e_{i,1}xR \oplus (e_0xR \cap (1-e_{i,1})xR)$.

Thus $e_{i,1}xR \oplus (e_0xR \cap (1-e_{i,1})xR)$ is a cyclic right $R$-module and is a submodule of $e_{i,1}xR$. Hence it is projective by assumption. Therefore we obtain $e_0xR$ as a sum of two projective modules, thus it is projective. Thus by induction, $exR$ is projective for all $i$. Setting $i = n$, then $e_nxR = xR$ is a projective.

We now show that $xR$ is projective for all $x \in e_{i,j}M_n(D)$. Recall that $xR$ is projective if and only if the annihilator $\text{ann}_R(x) = eR$ for some idempotent $e \in R$. This holds for $x \in Q$, not just for $x \in R$ as $RF = Q$ and $\text{ann}_R(x) = \text{ann}_R(x\alpha)$ for any $\alpha \in F^*$.

Say $x = \sum_{j=1}^{n} x_{ij}e_{i,j} \in e_{i,j}M_n(D)$ with $x_{ij} \in D$. If $x = 0$ then $\text{ann}_R(x) = R$ and we are done.

Also, by Lemma 2.5 of [7] there is an $i_0$ with $x_{i_0,j} \in B_{i_0,j}$ for all $j$, and so $x_{i_0,j}^{-1}x_j \in B_{i_0,j}$ for all $j$. Let $e$ be the permutation matrix which switches the $i_0$th and $i$th rows. Let

$$e = I_n - x_{i_0,j}^{-1}(Ex) = \begin{pmatrix} 1, & 0, & 0, \ldots, & 0 & 0 \\ 0, & 1, & 0, \ldots, & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -x_{i_0,j}^{-1}, & \ldots, & -x_{i_0,j}^{-1}, & 0, & -x_{i_0,j}^{-1}x_{i_0,j}, \ldots, -x_{i_0,j}^{-1}x_{i_0,j} \\ 0, & 0, & 1, & 0, \ldots, & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0, & 0, & 0, \ldots, & 0 & 1 \end{pmatrix}.$$ 

We have $e \in R$ since $x_{i_0,j}^{-1} \in B_{i_0,j}$. Also $xe = xI_n - xx_{i_0,j}^{-1}(Ex) = x - x = 0$ $xe = xI_n - xx_{i_0,j}^{-1}(Ex) = x - x = 0$, and so $e \in \text{ann}_R(x)$.

Let $a \in \text{ann}_R(x)$, then $ea = (I_n - xx_{i_0,j}^{-1}(Ex))a = -a - 0 = a$. Thus $e^2 = e$, and $\text{ann}_R(x) = eR$ is generated by an idempotent. Therefore $xR$ is projective.

Theorem 3.7 (J.S. KAUTA) $R$ is a semihereditary $V$-order if and only if $R$ is conjugate to an order of type $\Phi H$. Therefore orders of type $\Phi H$ are extremal. (See Theorem 4.7 [7] and 39.14 (ii) [10] for special cases of this theorem.)

Proof: Suppose $R$ is a semihereditary $V$-order. Then $R$ contains a full set of primitive orthogonal idempotents. After a conjugation, if necessary, we may assume all the standard idempotents $e_{1,1}, e_{2,2}, \ldots, e_{n,n} \in R$. Since $R$ is integral over $V$, $e_{ij}R_{e_{ij}}$ is integral over $V$. Also $e_{ij}R_{e_{ij}}F = e_{ij}RF_{e_{ij}} = e_{ij}De_{ij} = D$, therefore $e_{ij}R_{e_{ij}}$ is a $V$-order; indeed, $e_{ij}R_{e_{ij}}$ is a semihereditary $V$-order in $D$. Hence $e_{ij}R_{e_{ij}} = B$ (because $B$ is an invariant valuation ring extending $V$; therefore $B$ is the unique extremal and hence the unique semihereditary $V$-order in $D$). Set $B_{ij} = e_{ij}R_{e_{ij}}$. Then $B_{ij} \neq 0$, since $R$ is an order in $Q$. Since $B \subseteq R$, we have $Be_{ij}R_{e_{ij}} = e_{ij}B_{e_{ij}} = e_{ij}e_{ij}R_{e_{ij}} = e_{ij}R_{e_{ij}}B$, therefore $BB_{ij} = B_{ij}B = B_{ij}$ and so $B_{ij}$ is a $B$-bisubmodule of $D$. Now $R$ is a ring and $Re_{ij}e_{ij}R_{e_{ij}} \subseteq R$; so $B_{k,l}B_{ij} \subseteq B_{k,l}$, where $B_{k,l} = e_{ij}R_{e_{ij}}$ and $B_{ij} = e_{ij}R_{e_{ij}}$ holds. We only have to show Morandi's condition holds.
Suppose \( \exists i_0, j_0 \) and an \( 0 \neq \alpha \in D \) such that \( \alpha \notin B_{i_0, j_0} \) and \( \alpha^{-1} \notin B_{j_0, i_0} \). Since \( B \) is an invariant valuation ring, \( t_0 \neq j_0 \). Let \( \Gamma = (e_{i_0, j_0} + e_{j_0, i_0})R(e_{i_0, j_0} + e_{j_0, i_0}) \cong \begin{bmatrix} B & B_{j_0, i_0} \\ B_{i_0, j_0} & B \end{bmatrix} \). Then \( \Gamma \) is a semihereditary order in \( M_2(D) \) by [15]. Consider \( x = \begin{bmatrix} \alpha & 1 \\ 0 & 0 \end{bmatrix} \in M_2(D) \).

Then \( \text{ann}_R(x) = \begin{bmatrix} t & r \\ -\alpha t & -\alpha r \end{bmatrix} \) such that \( t, \alpha r \in B, r \in B_{j_0, i_0}, \alpha t \in B_{i_0, j_0} \) (see the proof of Theorem 1.5 [11]). We have \( \alpha t \notin B_{i_0, j_0} \) and \( t \in B \). But \( \alpha \notin B_{i_0, j_0} \). Hence \( \alpha^{-1} \notin B_{j_0, i_0} \), a contradiction, and so we have Morandi’s condition.

But \( a \in J(B) \), so \( b \alpha \) is a unit in \( B \). Hence \( ab \) is also a unit in \( B \). But \( b \in B_{j_0, i_0} \supseteq abB = B \) since \( ab \) is a unit in \( B \), hence \( \alpha^{-1} \notin B_{j_0, i_0} \), a contradiction, and so we have Morandi’s condition.

On the other hand, let \( R = (B_{i,j}) \) be of type \( \Phi \). We want to show that \( R \) is a semihereditary \( V \)-order in \( Q = M_2(D) \). By Lemma 2.5, \( R \) is a ring with the identity element of \( Q \), and \( FR = Q \). By the proof of ([7], Proposition 4.3), \( R \) is a \( V \)-order. But \( MR(R) \) is of type \( \Phi \) whenever \( R \) is. Hence Lemma 2.6 shows that for each \( r \), every principal right ideal of \( MR(R) \) is projective. So \( R \) is right semihereditary by [12]. Similarly, \( R \) is left semihereditary and hence it is semihereditary.

**Proposition 3.8.** Every Bezout \( V \)-order is a semihereditary \( V \)-order, but the converse does not hold.

**Proof:** Suppose

\[
R = \begin{bmatrix}
B & J(B_{1,2}) & \cdots & \cdots & J(B_{1,n}) \\
\cap & \cap & \cdots & \cdots & \cap \\
B_{2,1} & J(B_{2,2}) & \cdots & \cdots & J(B_{2,n}) \\
\cap & \cap & \cdots & \cdots & \cap \\
\vdots & \vdots & \cdots & \cdots & \vdots \\
B_{n,1} & B_{n,2} & \cdots & \cdots & \vdash B
\end{bmatrix},
\]

where \( B_{i,j} \) is an overring \( B \) for all \( i,j \) and \( B_{i,j} \neq B \) for some \( i,j \). By Theorem 2.7 and Theorem 2.6 of [11] \( R \) is semihereditary maximal \( V \)-order. But \( B_{n,1} \supseteq B \) by assumption. Let \( W = B_{n,1} \cap F \), then \( RW \subset M_n(B_{n,1}) \), since \( WB \subset WB_{n,1} = B_{n,1} \). If \( R \) is a Bezout, then \( R \cong M_n(B) \) by Corollary 3.5 of [7]. But \( RW \) would be a Dubrovin valuation ring over \( W \) and \( RW \subset M_n(B_{n,1}) \). Therefore \( RW = M_n(B_{n,1}) \), a contradiction.

If \( R \) is a Bezout \( V \)-order, by Proposition 1.8 and Example 1.15 of [16], then \( R \) is semihereditary and also more examples of semihereditary orders can be found in [17].

Therefore we have the following diagram in general.

\[
\text{Integral Dubrovin valuation rings} \Rightarrow \text{Bezout } V\text{-orders} \Rightarrow \text{Maximal } V\text{-orders}
\]

\[
\text{(if } V \text{ is Henselian) type } \Phi \text{ } H \Leftrightarrow \text{semihereditary } V\text{-orders} \Rightarrow \text{Extremal } V\text{-orders}.
\]
4. SEMIHEREDITARY ORDERS INSIDE BEZOUT ORDERS

Let \( V \) be a discrete valuation ring of \( F \) and \( Q \) a central simple \( F \)-algebra. By Wedderburn structure theorem \( Q \cong M_n(D) \), where \( D \) is a division algebra with center \( F \).

By (10-4) Corollary of [10] every \( V \)-order in \( Q \) is contained in a maximal \( V \)-order in \( Q \). If \( V \) be complete valuation ring, then the integral closure \( V \) in \( D \), i.e., \( \Delta = \text{int}_D(V) \) is the unique maximal \( V \)-order in \( D \). Let \( R \) be an \( V \)-order in \( Q \). Then by Theorem (39-14) of [10], \( R \) is a hereditary order if \( R \) is an Extremal \( V \)-order.

In this case \( R \) is precisely,

\[
R = \begin{bmatrix}
(\Delta)(P),(P),\ldots,(P) \\
(\Delta)(\Delta)(P),\ldots,(P) \\
\vdots \\
(\Delta)(\Delta),\ldots,(\Delta)
\end{bmatrix}^{(n_1,n_2,\ldots,n_r)}
\]

where \( P = J(\Delta) \) and \( n_1+n_2+\ldots+n_r=n \).

Now we assume \( V \) is a Henselian valuation ring of \( F \), not necessarily discrete. Let \( R \) be an Extremal \( V \)-order inside an integral Dubrovin valuation ring of \( B \) with \( B \cap F = V \). We know the integral closure \( V \) in \( D \) i.e., \( \Delta = \text{int}_D(V) \) is a unique maximal \( V \)-order in \( D \), and so \( B \cong M_n(\Delta) \) is a Dubrovin valuation ring and we can consider \( R \subseteq M_n(\Delta) \). By (Proposition [1]) \( R \) is semihereditary. So in this case we have

\[
R = \begin{bmatrix}
(\Delta),(J(\Delta)),\ldots,(J(\Delta)) \\
(\Delta),(\Delta),(J(\Delta)),\ldots,(J(\Delta)) \\
\vdots \\
(\Delta),(\Delta),\ldots,(\Delta)
\end{bmatrix}^{(n_1,n_2,\ldots,n_r)}
\]

where \( n_1+n_2+\ldots+n_r=n \) and \( R=\text{B if J(R)=J(\Delta)R if J^{-1}(\Delta)=\Delta} \).

If \( V \) isn't Henselian, then \( B_h = B \otimes_v V_h \) is a Dubrovin valuation ring. Therefore \( B/\text{J}(B) \cong B_h/\text{J}(B_h) \)

\( J(B) \otimes_v V_h \subseteq R \otimes_v V_h = R_h \). Hence we have \( \bigcup R_h = R \) and \( R_h \) is semihereditary if \( J(R) = J(\Delta)R \) if \( J^{-1}(\Delta) = \Delta \).

Corollary 4.1. Let \( R \) be an extremal \( V \)-order inside a Dubrovin valuation ring of \( B \), and if \( R \subseteq R' \subseteq B \), then \( R' \) is extremal \( V \)-order in \( B \).

Proof: Since \( R \) is semihereditary, \( R' \) is a semihereditary \( V \)-order (by Lemma 4.10 of [7]), and so \( R' \) is an extremal \( V \)-order.
Corollary 4.2. Let $R$ be an extremal $V$-order inside an integral Dubrovin valuation ring with $J(B)$ a non-principal ideal of $B$. Then $R=B$ if $J(R)=J(V)R$.

Now the generalization of Proposition 2.1 of [1] is given.

Theorem 4.3. Let $R$ be an Extremal $V$-order sitting inside a Bezout $V$-order $B$. Then $R$ is a semihereditary $V$-order.

Proof: By induction on $[Q:F]$. If $[Q:F]=1$, then $B$ is an integral Dubrovin valuation ring and so $R$ is a semihereditary.

Now we assume $B$ is not a Dubrovin valuation ring. Then there exists an integral Dubrovin valuation ring $T$ of $Q$, with center $W$ such that

$$i)\ T
\supset
B\ ii)\ J(T)
\subseteq
J(B)
\subseteq
J(R)\ iii)\ \tilde{R}
=\ R/J(T),\ \tilde{B}
=\ B/J(T)$$

are $V/J(W)$-orders in $\overline{T}=T/J(T)$, and $(iv)[\overline{T}:Z(\overline{T})]<[Q:F]$. By induction, $\tilde{R}$ is semihereditary and so $R$ is semihereditary (by Lemma 4.11 of [7]).

5. THE HENSELIZATION

We now consider $V$ to be a valuation ring of a field $F$ of arbitrary rank which need not be Henselian. One aim of this section is to examine the effect of Henselization on Bezout and maximal semihereditary $V$-orders.

Let $(V,F)$ be the Henselization of $(V,F)$ (see [9] for definition).

Let $Q$ be a central simple $F$-algebra, then $Q\otimes_F F_h$ is a central simple $F_h$-algebra and by ([10] Corollary 7.8) and also by Wedderburn's Theorem $Q\otimes_F F_h\cong M_n(D)$ for some $n$, where $D$ is a division algebra finite dimension over $F_h$.

Let $R$ be a $V$-order in $Q$. Clearly if $R\otimes_V V_h$ is a maximal $V_h$-order, then $R$ is a maximal $V$-order. Thus the difficulty lies in proving the converse.

If $V$ be a discrete valuation ring, then a $V$-order $R$ of $Q$ is a maximal order if $R$ is a Dubrovin valuation ring ([6]: Example 1.15). Therefore, in this case $R\otimes_V V_h$ is a Dubrovin valuation ring of $Q\otimes_F F_h$, which is integral over $V_h$. Thus $R\otimes_V V_h$ is a maximal $V_h$-order.

On the other hand, there exists a Bezout maximal $V$-order $R$ such that $R\otimes_V V_h$ is a semihereditary maximal order, but is not Bezout, (see [7] Example 4.14).


(1) Suppose $R$ is a maximal $V$-order in a central simple $F$-algebra $Q$. Let $(F_h,V_h)$ be the Henselization of $(V,F)$. Then $R\otimes_V V_h$ is a $V_h$-order in $Q\otimes_F F_h$. Is $R\otimes_V V_h$ a maximal order?

(2) If $R$ is semihereditary, then $R\otimes_V V_h$ is a $V_h$-order in $Q\otimes_F F_h$. Is $R\otimes_V V_h$ semihereditary?

Now we assume that $B$ is an invariant valuation ring extension of $V_h$ to $D$ and $R\cong (B_{i,j})$, an order of type $\Phi H$ in $Q\otimes_F F_h$.

Theorem 5.1. Suppose $Q$ is a central simple $F$-algebra and $V$ is a valuation ring in $F$. If $T$ is a Bezout $V$-order in $Q$, then $T\otimes_V V_h$ is conjugate to an order type $\Phi H$ such that $B_{i,j}^{-1}=B_{j,i}$ for all $i,j$ and $J(T)\otimes_V V_h=J(B)(T\otimes_V V_h)$.

Moreover, $T\otimes_V V_h$ is a Dubrovin valuation ring if $T$ is a Dubrovin valuation ring. In this case $T\otimes_V V_h$ is conjugate to $M_n(B)$. 
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Proof: By Theorem 17 of [18], \( T \otimes \nu V_h \) is a semihereditary maximal \( V_h \)-order in \( Q \otimes \nu F_h \). Therefore \( T \otimes \nu V_h \) is conjugate to an order type \( \Phi H \). And by Theorem 2.7 of [11] \( B_{i,j}^{-1} = B_{j,i} \) for all \( i,j \) and \( J(T) \otimes \nu V_h = J(B)(T \otimes \nu V_h) \). Also, \( T \otimes \nu V_h \) is Bezout if \( T \) is Dubrovin valuation ring (see Theorem 17 in [18]). Since \( V_h \) is Henselian, \( T \otimes \nu V_h \) is a Dubrovin valuation ring, and so \( T \otimes \nu V_h \) is conjugate to \( M_n(B) \).

J. S. Kauta ([11]: Theorem 3.4) proved that a \( V \)-order \( R \) is semihereditary if its Henselization \( R \otimes \nu V_h \) is semihereditary. So the answer (2) is yes.

Theorem 5. 2. If \( R \) is a maximal \( V \)-order in a central simple \( F \)-algebra \( Q \), then \( R \otimes \nu V_h \) is a maximal \( V_h \)-order in \( Q \otimes \nu F_h \) if one of the following conditions holds.

(1) \( R \) is a Bezout ring.
(2) \( R \) is a semihereditary ring.
(3) \( R \) is a finitely generated \( V \)-module.
(4) \( \text{Rank} V = 1 \)

Proof: If \( R \) is a Bezout ring, then by Theorem 17 of [18] \( R \otimes \nu V_h \) is a maximal \( V_h \)-order. And if \( R \) is a semihereditary ring, it follows from Theorem 1 of [19].

Now we suppose that \( R \) is a finitely generated \( V \)-module. Then \( R \) is contained in a Bezout \( V \)-order \( T \) by ([7], Prop.3). Since \( [T/J(T):V/J(V)] < \infty \), there exists \( t_1, \ldots, t_n \in T \) such that \( T = t_1V + \ldots + t_nV + J(T) \). But by ([11]: Prop. 1.4) \( J(T) \subseteq R \) (since maximal orders are extremal). Hence \( T \) is a finitely generated Bezout \( V \)-order. By the maximality of \( R \), we have \( T = R \). Therefore \( R \) is a Bezout \( V \)-order.

(4) Let \((V_h, F_h)\) be the Henselization of \((V, F)\). Then \((V, F) \subseteq (V_h, F_h) \subseteq (V, F)\), where \((V, F)\) is the complement of \((V, F)\) with respect to the metric induced by the valuation corresponding of \( V \). Hence \( V \) is dense in \( V_h \) and by (Proposition of [19]) we have \( R \otimes \nu V_h \) as a maximal \( V_h \)-order in \( Q \otimes \nu F_h \).

Let \( B \) be a unique extension valuation ring \( V_h \) to \( D \), where \( Q \otimes \nu F_h \cong M_n(D) \) and \( R = (B_{i,j}) \) is order type \( \Phi H \). Then we have the following theorem.

Theorem 5. 3. Suppose \( Q \) is a central simple \( F \)-algebra and \( V \) is a valuation ring in \( F \). If \( T \) is a maximal semihereditary \( V \)-order in \( Q \), then \( T \otimes \nu V_h \) is conjugate to an order type \( \Phi H \) such that \( B_{i,j}^{-1} = B_{j,i} \) for all \( i,j \).

Proof: By Theorem 5.2, (2) \( T \otimes \nu V_h \) is a semihereditary maximal \( V_h \)-order, and by Theorem 3.7 \( T \otimes \nu V_h \) is conjugate to an order type \( R = (B_{i,j}) \). On the other hand, \( R \) is a semihereditary maximal order, and by Theorem 2.6 of [11] we have \( B_{i,j} = B_{j,i}^{-1} \) for all \( i,j \).

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