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EXTREMAL ORDERS INSIDE SIMPLE ARTINIAN RINGS*

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Abstract – The aim of this paper is to study orders over a valuation ring V with arbitrary rank in a central simple F-algebra Q. The relation between all of the orders is explained with a diagram. It is then shown that inside Bezout order, extremal *V-orders* are precisely semi-hereditary. In the last section, the effect of Henselization on maximal and semi-hereditary orders is examined.

Keywords – Dubrovin valuation rings, extremal orders, Henselization

1. INTRODUCTION

In this paper, all rings are associative with a multiplicative unit and all modules are unitary. If A is a ring, $J(A)$ will denote its Jacobson radical, $U(A)$ its group of units, $Z(A)$ its center, A^* its set of nonzero divisors, and $M_n(A)$ the ring of $n \times n$ matrices over A. The residue ring $A/J(A)$ will be denoted by $\overline{A}$. And Q denotes a simple artinian ring with finite dimension over its center $Z(Q)$, while D denotes a division ring.

In the second section we briefly discuss some of the ring theoretic properties and definitions.

In the third section we will see that semihereditary *V-orders* are extremal *V-orders* and obtain a diagram of maximal *V-orders* when V is a Henselian valuation ring.

In the fourth section we show that inside Bezout orders, extremal *V-orders* are precisely semihereditary, which is a generalization of Proposition 2.1 of [1].

In the last section we will examine the effect of Henselization on maximal and semihereditary orders.

2. DEFINITION AND PRELIMINARIES

In this paper F denotes a field and Q is a central simple *F-Algebra*, i.e., Q is *a F-Algebra* with $[Q:F]<\infty$ and $F=Z(Q)$.

The most successful extension of the classical valuation theory on F to Q is the one introduced by Dubrovin in [2] and [3].

Definition 2. 1. A subring B of a central simple *F-algebra* Q is called a Dubrovin valuation ring in Q if

(1) B has an ideal M such that B/M is a simple artinian ring and
(2) For each $q \in Q \backslash B$ there exist $b, a \in B$ such that $bq, qa \in B \backslash M$.

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The following properties of Dubrovin valuation rings were proved by Dubrovin in [2, 3].

i) The two sided ideals of B are totally ordered by inclusion, where two sided ideals are a *B-bimodule* of Q. Therefore we have $M=J(B)$

ii) Each finitely generated left (resp, right) ideal of B is principal.

iii) (a) Let V be a valuation ring of F, then there exists a Dubrovin valuation ring of B in Q such that $B\bigcap F=V$, [2-4].

(b) If B, and B' are two Dubrovin valuation rings of Q extending V, then $B'=dBd^{-1}$ for some $d\in Q^*$ [5, 6].

Therefore, for every valuation ring V of $F=Z(Q)$, there is a unique (up to conjugate) associated Dubrovin valuation ring B of Q. It is reasonable to expect that B will carry much information about the arithmetic of Q in relation to V, (see [7] Theorem 3.4 and [8] Theorem 3.7).

Definition 2. 2. Let Q be a finite-dimensional *F-Algebra* and V a ring with quotient field F. A subring R of Q is said to be an order in Q if $RF=Q$. If $V=Z(R)$, then R is said to be a *V-order* if, in addition, R is integral over V. If R is maximal with respect to inclusion among *V-order* of Q, then R is said to be a maximal order over V.

a) In the case V is a discrete valuation ring, then by ([9], 18.6 and 18.2) any *V-order* in a central simple *F-algebra* is a finite *V-module*, so for such V, Definition 2.2 agrees with the usual one, as in [10].

b) In this paper we assume V is a commutative valuation ring in F of arbitrary Krull-dimension. The integrality hypothesis in the above definition is used to guarantee the existence of maximal orders for any Q and V. But finitely generated maximal *V-orders* need not exist, (see [7] Proposition 2.3).

c) Let V be a valuation ring of a field F, and Q a central simple *F-Algebra*. If B is an integral Dubrovin extension of V to Q (i.e., B is a Dubrovin valuation ring of Q such that B is integral over V and $V=B\bigcap F$) then B is a maximal *V-order* (by Example 2.2 [7]).

Definition 2. 3. A ring R is said to be *extremal* if for every overring S such that $J(R)\subseteq J(S)$ we have $S=R$. If S is an overring of R, we say that R is extremal in S if R is extremal among all subrings of S. A *V-order R* is said to be an extremal *V-order* (or just extremal when the context is clear) if it is extremal among all *V-orders* in Q.

Definition 2. 4. A ring R is said to right (resp left) Bezout if every finitely generated right (left) ideal is principal. It is called Bezout if it is both right and left Bezout.

If V is a valuation ring, then there exists a Bezout *V-order B* in Q and each Bezout *V-order* is a maximal order by ([7] Theorem 3.4), and if B, and B' are two Bezout *V-orders*, then B, and B' are conjugate (by Theorem 6.12 [4]).

Definition 2. 5. A ring R is said to be right semihereditary (resp right hereditary) if every finitely generated right ideal (resp every right ideal) is projective as a right *R-module*. A ring is said to be semihereditary (resp hereditary) if it is both left and right semihereditary (resp hereditary).

a) If V be Dedekind domain with quotient field F and Q is a central simple *F-Algebra*, where $Q\cong M_n(D)$ and D is a division ring with center F, then R is a hereditary *V-order* if and only if R is an extremal (see 39.14 [10]).

b) Let V be a valuation ring of $F=Z(Q)$ and Q a central simple *F-Algebra*. J.S. Kauta proved that every semihereditary *V-Order* is extremal (see Theorem 1.5 [11]), but the converse is not true. If F is a field, $Q=M_2(F)$, V_n is a discrete valuation ring of dimension n, and R is a maximal V_n-*order* in Q, then there are three possibilities for the isomorphism class of R.

(1) $R \cong M_2(V_m)$, where V_m is the overring of V_n of dimension m. In this case R is a Bezout.

(2) $R \cong \begin{bmatrix} V_m & J(V_p) \\ V_p & V_m \end{bmatrix}$, where $m<p$. In this case R is semihereditary, but not Bezout.

(3) R is primary (i.e., $J(R)$ is a maximal ideal of R) but not Bezout (see [7], Theorem 5.7). Let R be maximal V-*order* in $M_2(F)$ which is primary, but not Bezout. Such an order cannot be semihereditary, since any primary semihereditary order is a Dubrovin valuation ring ([3]: Theorem 4), and hence Bezout.

3. MAXIMAL ORDERS OVER HENSELIAN VALUATION RINGS

In this section D always means a finite dimensional algebra with center F. A subring B of D is said to be a total valuation ring in D if $d \in B$ or $d^{-1} \in B$ for all nonzero $d \in D$.

We recall that a valuation ring V in a field F is Henselian when Hensel's Lemma holds for V, i.e., for every monic polynomial $f \in V[x]$, if its image $\bar{f} \in \bar{V}[x]$, where $\bar{V} = V/J(V)$ has a factorization $\bar{f} = \tilde{g}\tilde{h}$ on $\bar{V}[x]$ with $\tilde{g}, \tilde{h}$ monic and $gcd(\tilde{g}, \tilde{h})=1$, then there exist monic $g, h \in V[x]$ with $f = gh, \bar{g} = \tilde{g}$ *and* $\bar{h} = \tilde{h}$, where $\bar{g}$ and $\bar{h}$ are images g and h respectively.

There are several other equivalent characterizations of the Henselian valuation ring, but the most relevant here is the following.

A valuation ring V in a field F is Henselian if V has a unique extension to each field $F \subset K$ with K algebraic over F (see [9] Coro.16.6 for a proof).

Now let D be a division algebra finite dimensional over its center $Z(D)=F$, and V a Henselian valuation ring of F. Schilling ([12] P.53, Theorem 9) proved that the integral closure V in D forms a ring B. The ring B is a total valuation ring of V and by ([13], Theorem 1) and B is the unique extension V to D. Therefore B is an invariant valuation ring of D (i.e., $dBd^{-1}=B$ for any $d \in D^*$).

Theorem 3. 1. Let D be a division algebra admitting a total valuation ring extending V. Then the integral closure of V in D is the unique extremal V-order (and hence the unique semihereditary V-*order*) in D.

Proof: By ([14]: Lemma 2) V has only a finite number of extensions to D. If $B_1,\ldots,B_n$ are all the extensions of V, then B_i and B_j are conjugate for all i,j by ([14]: Theorem 2). Let $T=Int_D(V)$ be the integral closure of V in D. Then $T=\bigcap_{i=1}^{n} B_i$ by ([14]: Theorem 3). Let R be an extremal V-order.

Then $R \subseteq T$, because R is integral over V. But both R and $J(B_i)$ contain $J(V)$. Hence for each i, $R/(J(B_i) \cap R)$ is finite dimensional over $V/J(V)$. But one has the embedding $R/(J(B_i) \cap R) \to B_i/J(B_i)$ and $[B_i/J(B_i): V/J(V)] \leq [D:F] < \infty$ by ([14]: Lemma 3). It follows that $R/(J(B_i) \cap R)$ is division algebra, and hence $J(B_i) \cap R$ is a maximal ideal of R. Hence, $J(R) \subseteq J(B_i) \cap R$.

Let $x \in \bigcap_i J(B_i)$ and $a,b \in J(T)$. Then $1-axb \in U(B_i)$ for all i, and thus $1-axb \in U(T)$. Therefore $x \in J(T)$. Hence $J(R) \subseteq \bigcap_i J(B_i) \subseteq J(T)$. Since R is extremal, we must have $R=T$.

On the other hand, T is a Bezout V-order by ([7]: Theorem 3.4) and every such T is a semihereditary V-order in D.

Corollary 3. 2. Let V be a valuation ring of F, and D suppose admits and invariant valuation ring B extending V. Then B is the unique extremal (and hence the unique semihereditary) V-order in D.

Proof: Since the extensions of V to D are conjugate, B is the unique extension of V to D. So the corollary follows from Theorem 3.1.

In the rest of the section we assume V to be a Henselian valuation ring of F, and D be a finite dimensional division algebra over its center $Z(D)=F$.

Let B be the unique extension of V to D, and let β be the set of all nonzero B-submodules of D. Then β is totally ordered. For if I and J are two B-submodules of D such that $I\not\subset J$, there exists an $a\in I-J$. Then if $b\in J$, then $ab^{-1}\notin B$; thus $ba^{-1}\in B$, and hence $b\in Ba\subset I\Rightarrow J\subseteq I$.

Definition 3. 3. Let I be a B-submodule of D. We define I^{-1} to be $\{d\in D: dI\subseteq B\}$.

Definition 3. 4. Let $Q=M_n(D)$. An order $R=\begin{bmatrix} B, B_{1,2},\ldots\ldots\ldots, B_{1,n} \\ B_{2,1}, B, B_{2,3},\ldots, B_{2,n} \\ ,\ldots\ldots\ldots\ldots\ldots \\ . \\ . \\ B_{n,1}, B_{n,2},\ldots B_{n,n-1}, B \end{bmatrix}$ is said to be of *type* Φ H if

i) $B_{i,j}\in\beta$.
ii) If $d\notin B_{i,j}$, then $d^{-1}\in B_{j,i}$ for all $d\neq 0\in D$. (Morandi's condition).
iii) $B_{r,j}B_{j,s}\subseteq B_{r,s}$, for all $1\leq r,s,j\leq n$.
We denote R by $(B_{i,j})$

Lemma 3. 5. (a) R is a ring and $RF=RD=Q$, i.e., R is an order.
(b), $B_{i,j}\subseteq B\subseteq B_{j,i}$ *or* $B_{j,i}\subseteq B\subseteq B_{i,j}$ for all i,j.

Proof: (a) by (iii) R is a ring, because $B_{i,j}\neq 0$ for all i,j, therefore $RF=RD=Q$.
For (b) since β is totally ordered, we have $B_{i,j}\subset B$ or $B\subseteq B_{i,j}$. If $B_{i,j}\subset B$, then $1\notin B_{i,j}$, and hence, $1\in B_{ij}$ by (ii). Thus $B=B1\subseteq B_{j,i}$, and so $B_{i,j}\subseteq B\subseteq B_{j,i}$.

If $B\subseteq B_{i,j}$, then $B_{i,j}B_{j,i}\subseteq B_{i,i}=B\Rightarrow B_{j,i}=B_{j,i}1\subseteq B$, and hence $B_{j,i}\subseteq B\subseteq B_{i,j}$.

Lemma 3. 6. (Morandi) Let $Q=M_n(D)$ and $R=(B_{i,j})$. Then xR is projective as a R-module for all $x\in Q$.

Proof: We first suppose xR is projective for all $x\in e_{i,i}R$ for any i. We prove xR is projective for any x (where $e_{i,i}$ is matrix $n\times n$ with 1 in (i,i) entry and zero in the others). We do this by showing that e_ixR is projective, where $e_i=e_{1,1}+e_{2,2}+\ldots+e_{i,i}$. We use induction on i, the case i=1 is true by assumption (because if $x=(d_{i,j})$ then $xe_{1,1}R=(xe_{1,1})R$, and since $xe_{1,1}=d_{1,1}e_{1,1}$ and $d_{1,1}\in B_{i,j}$ or $d_{1,1}\in B_{j,i}$, therefore $xe_{1,1}\in e_{1,1}R)$. So suppose $e_{i-1}xR$ is projective for all $x\in e_{ii}R$. We have the exact sequence of R-modules.

$0\to e_ixR\cap(1\text{-}e_{i-1})R\to e_ixR\to e_{i-1}e_ixR\to 0$, where $1=e_{1,1}+e_{2,2}+\dots+e_{n,n}=e_n$. Now $e_{i-1}e_ixR=e_{i-1}xR$ and $e_ixR\cap(1\text{-}e_{i-1})R\subseteq e_iR\cap(1\text{-}e_{i-1})R=e_iR$ (because $1\text{-}e_{i-1}=e_{i,i}+\dots+e_{n,n}$). Since $e_{i-1}xR$ is projective by the induction of the sequence splits. So $e_ixR\cong e_{i-1}xR\oplus(e_ixR\cap(1\text{-}e_{i-1})R)$.

Thus $e_{i-1}xR\oplus(e_ixR\cap(1\text{-}e_{i-1})R)$ is a cyclic right R-module and is a submodule of $e_{i,i}R$. Hence it is projective by assumption. Therefore we obtain e_ixR as a sum of two projective modules, thus it is projective. Thus by induction, e_ixR is projective for all i. Setting $i=n$, then $e_nxR=xR$ is a projective.

We now show that xR is projective for all $x\in e_{ii}M_n(D)$. Recall that xR is projective if and only if the annihilator $ann_R(x)=eR$ for some idempotent $e\in R$. This holds for $x\in Q$, not just for $x\in R$ as $RF=Q$ and $ann_R(x)=ann_R(x\alpha)$ for any $\alpha\in F^*$.

Say $x=\sum_{j=1}^{i}x_je_{i,j}\in e_{i,i}M_n(D)$ with $x_j\in D$. If $x=0$ then $ann_R(x)=R$ and we are done.

Also, by Lemma 2.5 of [7] there is an i_o with $x_jx_{i_0}^{-1}\in B_{i_0,j}$ for all j, and so $x_{i_0}^{-1}x_j\in B_{i_0,j}$ for all j. Let e be the permutation matrix which switches the $i_o th$ and i th rows. Let

$$e=I_n\text{-}x_{i_0}^{-1}(Ex)=\begin{bmatrix}1, & 0, & 0,\dots\dots,0, & 0\\ 0, & 1, & 0,\dots\dots,0, & 0\\ . & & \\ -x_1x_{i_o}^{-1},\dots,-x_{i_o-1}x_{i_o},0,-x_{i_o+1}x_{i_o},\dots,-x_nx_{i_o}^{-1} & & \\ 0, & & 0\quad,1,\quad 0,\dots\quad ,0\\ . & & \\ . & & \\ 0, & 0, & 0,\dots\dots,0, & 1\end{bmatrix}.$$

We have $e\in R$ *since* $x_jx_{i_o}^{-1}\in B_{i_o\,j}$. Also $xe=xI_n-xx_{i_0}^{-1}(Ex)=x-x=0$ $xe=xI_n\text{-}xx^{-1}_{i_o}(Ex)=x\text{-}x=0$, and so $e\in ann_R(x)$.

Let $a\in ann_R(x)$, then $ea=(I_n\text{-}x_{i_0}^{-1}(Ex))a=a\text{-}0=a$. Thus $e^2=e$, and $ann_R(x)=eR$ is generated by an idempotent. Therefore xR is projective.

Theorem 3. 7. (J.S. KAUTA) R is a semihereditary V-order if and only if R is conjugate to an order of type Φ H. Therefore orders of type Φ H are extremal. (See Theorem 4.7 [7] and 39.14 (ii) [10] for special cases of this theorem.)

Proof: Suppose R is a semihereditary *V-order*. Then R contains a full set of primitive orthogonal idempotents. After a conjugation, if necessary, we may assume all the standard idempotents $e_{1,1},e_{2,2},\dots,e_{n,n}\in R$. Since R is integral over V, $e_{i,i}Re_{i,i}$ is integral over V. Also $e_{i,i}Re_{i,i}F=e_{i,i}RFe_{i,i}=e_{i,i}De_{i,i}=D$, therefore $e_{i,i}Re_{i,i}$ is a V-order; indeed, $e_{i,i}Re_{i,i}$ is a semihereditary V-order in D. Hence $e_{i,i}Re_{i,i}=B$ (because B is an invariant valuation ring extending V; therefore B is the unique extremal and hence the unique semihereditary V-order in D). Set $B_{i,j}=e_{i,i}Re_{j,j}$. Then $B_{i,j}\neq 0$, since R is an order in Q. Since $B\subseteq R$, we have $Be_{i,i}Re_{j,j}=e_{i,i}BRe_{j,j}=e_{i,i}R\,e_{j,j}=e_{i,i}R\,e_{j,j}B$, therefore $BB_{i,j}=B_{i,j}B=B_{i,j}$ and so $B_{i,j}$ is a B-bisubmodule of D. Now R is a ring and $Re_{j,j}e_{jj}R=Re_{j,j}R\subseteq R$; so $B_{k,j}B_{j,l}\subseteq B_{k,l}$, where $B_{k,j}=e_{k,k}Re_{j,j}$ *and* $B_{j,l}=e_{j,j}Re_{l,l}$ holds. We only have to show Morandi's condition holds.

Suppose $\exists i_0, j_0$ and an $0 \neq \alpha \in D$ such that $\alpha \notin B_{i_0 j_0}$ and $\alpha^{-1} \notin B_{j_0, i_0}$. Since B is an invariant valuation ring, $i_0 \neq j_0$. Let $\Gamma = (e_{i_0,i_0} + e_{j_0,j_0})R(e_{i_0,i_0} + e_{j_0,j_0}) \cong \begin{bmatrix} B & B_{j_0,i_0} \\ B_{i_0,j_0} & B \end{bmatrix}$. Then Γ is a semihereditary order in $M_2(D)$ by [15]. Consider $x=\begin{bmatrix} \alpha & 1 \\ 0 & 0 \end{bmatrix} \in M_2(D)$.

Then $\text{ann}_\Gamma(x)=\left\{\begin{bmatrix} t & r \\ -\alpha t & -\alpha r \end{bmatrix} \text{such that } t, \alpha r \in B, r \in B_{j_0,i_0}, \alpha t \in B_{i_0,j_0} \right\}$ (see the proof of Theorem 1.5 [11]). We have $\alpha t \in B_{i_0,j_0}$ and $t \in B$. But $\alpha \notin B_{i_0,j_0}$. So $t \in J(B)$. Since Γ is a semihereditary order in $M_2(D)$, $\text{ann}_\Gamma(x)$ is generated by an idempotent $\begin{bmatrix} a & b \\ -\alpha a & -\alpha b \end{bmatrix} = \begin{bmatrix} a & b \\ -\alpha a & -\alpha b \end{bmatrix}^2$. So $1=a-b\alpha$.

But $a \in J(B)$, so $b\alpha$ is a unit in B. Hence αb is also a unit in B. But $b \in B_{j_0,i_0} \supseteq \alpha b B = B$ since αb is a unit in B, hence $\alpha^{-1} \in B_{j_0,i_0}$, a contradiction, and so we have Morandi's condition.

On the other hand, let $R = (B_{i,j})$ be of type ΦH .We want to show that R is a semihereditary V-order in $Q=M_2(D)$. By Lemma 2.5, R is a ring with the identity element of Q, and $FR=Q$. By the proof of ([7], Proposition 4.3), R is a V-order. But $M_r(R)$ is of type ΦH whenever R is. Hence Lemma 2.6 shows that for each r, every principal right ideal of $M_r(R)$ is projective. So R is right Semihereditary by [12]. Similarly, R is left semihereditary and hence it is semihereditary.

Proposition 3. 8. Every Bezout *V-order* is a semihereditary *V-order*, but the converse does not hold.

Proof: Suppose

$$R=\begin{bmatrix} B \supset J(B_{1,2}) \supset,\ldots\ldots\ldots, \supset J(B_{1,n}) \\ \cap \quad \cap \qquad\qquad \cap \\ B_{2,1} \supset B \supset,\ldots\ldots\ldots\ldots, \supset J(B_{2,n}) \\ \cap \quad \cap \quad ,\ldots, \qquad \cap \\ .,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \cap,\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\cap \\ B_{n,1} \supset B_{n,2} \supset,\ldots\ldots, \quad \supset B \end{bmatrix},$$

where $B_{i,j}$ is an overring B for all i,j and $B_{i,j} \neq B$ for some i,j. By Theorem 2.7 and Theorem 2.6 of [11] R is semihereditary maximal V-*order*. But $B_{n,1} \supset B$ by assumption. Let $W=B_{n,1} \cap F$, then $RW \subset M_n(B_{n,1})$, since $WB \subset WB_{n,1}=B_{n,1}$. If R is a Bezout, then $R \cong M_n(B)$ by Corollary 3.5 of [7]. But RW would be a Dubrovin valuation ring over W and $RW \subset M_n(B_{n,1})$. Therefore $RW=M_n(B_{n,1})$, a contradiction.

If R is a Bezout *V-order*, by Proposition 1.8 and Example 1.15 of [16], then R is semihereditary and also more examples of semihereditary orders can be found in [17].

Therefore we have the following diagram in general.

Integral Dubrovin valuation rings $\Rightarrow$ *Bezout V-orders* $\Rightarrow$ *Maximal V-orders*

$\Downarrow \qquad\qquad\qquad \Downarrow$

(if V is Henselian) type ΦH $\Leftrightarrow$ *semihereditary V-orders* $\Rightarrow$ *Extremal V-orders.*

4. SEMIHEREDITARY ORDERS INSIDE BEZOUT ORDERS

Let V be a discrete valuation ring of F and Q a central simple F-algebra. By Wedderburn structure theorem $Q \cong M_n(D)$, where D is a division algebra with center F.

By (10-4) Corollary of [10] every *V-order* in Q is contained in a maximal *V-order* in Q. If V be complete valuation ring, then the integral closure V in D, i.e., $\Delta = \text{int}_D(D)$ is the unique maximal *V-order* in D. let R be an *V-order* in Q. Then by Theorem (39-14) of [10], R is a hereditary order if R is an Extremal *V-order.*

In this case R is precisely,

$$R = \begin{bmatrix} (\Delta)(P)(P),.........,(P) \\ (\Delta)(\Delta)(P),..........,(P) \\ . \\ . \\ . \\ (\Delta)(\Delta),...............,(\Delta) \end{bmatrix}^{(n_1, n_2, \ldots, n_r)} ,$$

where $P = J(\Delta)$ and $n_1+n_2+\ldots+n_r=n$.

Now we assume V is a Henselian valuation ring of F, not necessarily discrete. Let R be an Extremal *V-order* inside an integral Dubrovin valuation ring of B with $B \cap F = V$. We know the integral closure V in D i.e., $\Delta = \text{int}_D(V)$ is a unique maximal *V-order* in D, and so $B \cong M_n(\Delta)$ is a Dubrovin valuation ring and we can consider $R \subset M_n(\Delta)$. By (Proposition [1]) R is semihereditary. So in this case we have

$$R = \begin{bmatrix} (\Delta),(J(\Delta)),..............,(J(\Delta)) \\ (\Delta),(\Delta),(J(\Delta)),........,(J(\Delta)) \\ . \\ . \\ . \\ (\Delta),(\Delta),.....................,(\Delta) \end{bmatrix}^{(n_1, n_2, \ldots, n_r)}$$

where $n_1 + n_2 + \ldots + n_r = n$ and R=B if $J(R) = J(\Delta)R$ if $J^{-1}(\Delta) = \Delta$.

If V isn't Henselian, then $B_h = B \otimes_v V_h$ is a Dubrovin valuation ring. Therefore $J(B) \otimes_v V_h \subseteq R \otimes_v V_h = R_h$. Hence we have

$$\begin{array}{ccc} B/J(B) & \cong & B_h/J(B_h) \\ \bigcup & & \bigcup \\ R/J(B) & \cong & R_h/J(B_h) \end{array}$$

, thus R_h is semihereditary and so R is semihereditary by ([11] Proposition 3.3). Thus inside an integral Dubrovin valuation ring, extremal *V-orders* are precisely the semihereditary *V-orders*.

Corollary 4. 1. Let R be an extremal *V-order* inside a Dubrovin valuation ring of B, and if $R \subseteq R' \subseteq B$, then R' is extremal V-order in B.

Proof: Since R is semihereditary, R' is a semihereditary *V-order* (by Lemma 4.10 of [7]), and so R' is an extremal V-order.

Corollary 4. 2. Let R be an extremal *V-order* inside an integral Dubrovin valuation ring with $J(B)$ a non-principal ideal of B. Then $R=B$ *if* $J(R)=J(V)R$.

Now the generalization of Proposition 2.1 of [1] is given.

Theorem 4. 3. Let R be an Extremal *V-order* sitting inside a Bezout *V-order* B.Then R is a semihereditary *V-order*.

Proof: By induction on $[Q: F]$. If $[Q: F]=1$, then B is an integral Dubrovin valuation ring and so R is a semihereditary.

Now we assume B is not a Dubrovin valuation ring. Then there exists an integral Dubrovin valuation ring T of Q, with center $W \supset V$ such that

$$\text{i) } T \supset B \quad ii)\ J(T) \subseteq J(B) \subseteq J(R) \quad iii)\ \tilde{R} = R / J(T), \tilde{B} = B / J(T)$$

are $V / J(W)$-orders in $\overline{T} = T / J(T)$, and $(iv)[\overline{T} : Z(\overline{T})] < [Q : F]$. By induction, $\widetilde{R}$ is semihereditary and so R is semihereditary (by Lemma 4.11 of [7]).

5. THE HENSELIZATION

We now consider V to be a valuation ring of a field F of arbitrary rank which need not be Henselian. One aim of this section is to examine the effect of Henselization on Bezout and maximal semihereditary *V-orders*.

Let (V_h, F_h) be the Henselization of (V,F) (see [9] for definition).

Let Q be a central simple F-algebra, then $Q \otimes_F F_h$ is a central simple F_h-algebra and by ([10] Corollary 7.8) and also by Wedderburn's Theorem $Q \otimes_F F_h \cong M_n(D)$ for some n, where D is a division algebra finite dimension over F_h.

Let R be a *V-order* in Q. Clearly if $R \otimes_V V_h$ is a maximal V_h-*order*, then R is a maximal *V-order.* Thus the difficulty lies in proving the converse.

If V be a discrete valuation ring, then a *V-order* R of Q is a maximal order if R is a Dubrovin valuation ring ([6]: Example 1.15). Therefore, in this case $R \otimes_v V_h$ is a Dubrovin valuation ring of $Q \otimes_F F_h$, which is integral over V_h. Thus $R \otimes_V V_h$ is a maximal V_h-order.

On the other hand, there exists a Bezout maximal V-order R such that $R \otimes_V V_h$ is a semihereditary maximal order, but is not Bezout, (see [7] Example 4.14).

P. Morandi [7] mentioned two questions.

(1) Suppose R is a maximal *V-order* in a central simple F-algebra Q. Let (F_h, V_h) be the Henselization of (V,F).Then $R \otimes_V V_h$ is a V_h-*order* in $Q \otimes_F F_h$. Is $R \otimes_V V_h$ a maximal order?

(2) If R is semihereditary, then $R \otimes_V V_h$ is a V_h-*order* in $Q \otimes_F F_h$. Is $R \otimes_V V_h$ semihereditary?

Now we assume that B is an invariant valuation ring extension of V_h to D and $R \cong (B_{i,j})$, an order of type ΦH in $Q \otimes_F F_h$.

Theorem 5. 1. Suppose Q is a central simple F-algebra and V is a valuation ring in F. If T is a Bezout *V-order* in Q, then $T \otimes_V V_h$ is conjugate to an order type ΦH such that $B_{i,j}^{-1}=B_{j,i}$ for all i,j and $J(T) \otimes_V V_h = J(B)(T \otimes_V V_h)$.

Moreover, $T \otimes_V V_h$ is a Dubrovin valuation ring if T is a Dubrovin valuation ring. In this case $T \otimes_V V_h$ is conjugate to $M_n(B)$.

Proof: By Theorem17 of [18], $T \otimes_V V_h$ is a semihereditary maximal V_h-*order* in $Q \otimes_F F_h$. Therefore $T \otimes_V V_h$ is conjugate to an order type ΦH. And by Theorem 2.7 of [11] $B_{i,j}^{-1}=B_{j,i}$ for all i,j and $J(T) \otimes_v V_h=J(B)(T \otimes_v V_h)$. Also, $T \otimes_v V_h$ is Bezout if T is Dubrovin valuation ring (see Theorem 17 in [18]). Since V_h is Henselian, $T \otimes_v V_h$ is a Dubrovin valuation ring, and so $T \otimes_v V_h$ is conjugate to $M_n(B)$.

J. S. Kauta ([11]: Theorem 3.4) proved that a V-order R is semihereditary if its Henselization $R \otimes_v V_h$ is a semihereditary. So the answer (2) is yes.

Theorem 5. 2. If R is a maximal V-order in a central simple F-algebra Q, then $R \otimes_v V_h$ is a maximal V_h-*order* in $Q \otimes_F F_h$ if one of the following conditions holds.
(1) R is a Bezout ring.
(2) R is a semihereditary ring.
(3) R is a finitely generated V-module.
(4) $RankV$=1

Proof: If R is a Bezout ring, then by Theorem 17 of [18] $R \otimes_v V_h$ is a maximal V_h-order.
And if R is a semihereditary ring, it follows from Theorem 1 of [19].

Now we suppose that R is a finitely generated V-module. Then R is contained in a Bezout V-order T by ([7], Prop.3). Since $[T/J(T):V/J(V)]<\infty$, there exists $t_1,\dots,t_n \in T$ such that $T=t_1V+\dots+t_nV+J(T)$. But by ([11]: Prop. 1.4) $J(T)\subset R$ (since maximal orders are extremal). Hence T is a finitely generated Bezout V-order. By the maximality of R, we have $T=R$. Therefore R is a Bezout V-order.

(4) Let (V_h, F_h) be the Henselization of (V, F). Then $(V, F)\subseteq(V_h, F_h) \subseteq(V, F)$, where (V, F) is the complement of (V, F) with respect to the metric induced by the valuation corresponding of V. Hence V is dense in V_h and by (Proposition of [19]) we have $R \otimes_V V_h$ as a maximal V_h-order in $Q \otimes_F F_h$.

Let B be a unique extension valuation ring V_h to D, where $Q \otimes_F F_h \cong M_n(D)$ and $R=(B_{i,j})$ is order type ΦH . Then we have the following theorem.

Theorem 5. 3. Suppose Q is a central simple F-algebra and V is a valuation ring in F. If T is a maximal semihereditary V-*order* in Q, then $T \otimes_V V_h$ is conjugate to an order type ΦH such that $B_{i,j}^{-1}=B_{j,i}$ for all i,j.

Proof: By Theorem 5.2, (2) $T \otimes_V V_h$ is a semihereditary maximal V_h-*order,* and by Theorem 3.7 $T \otimes_V V_h$ is conjugate to an order $R=(B_{i,j})$. On the other hand, R is a semihereditary maximal order, and by Theorem 2.6 of [11] we have $B_{i,j}=B_{j,i}^{-1}$ for all i,j.

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