HYPERRULED SURFACES IN MINKOWSKI 4-SPACE*

R. ASLANER**
Department of Mathematics, İnönü University, 44280 Malatya, Turkey
E-mail: raslaner@inonu.edu.tr

Abstract – In this paper, the time-like hyperruled surfaces in the Minkowski 4-space and their algebraic invariants are worked. Also some characteristic results are found about these algebraic invariants.

Keywords – Ruled surfaces, rulings, main curvature, scalar curvature, time-like vector

1. INTRODUCTION

The Minkowski space is the space $\mathbb{R}^4$ with the Lorentzian inner product

$$g_0 = -dt^2 + dx^2 + dy^2 + dz^2$$

which is denoted by $\mathbb{R}_1^4$. The representation of $g_0$ in the matrix form with respect to the standard basis of $\mathbb{R}_1^4$ is $\eta = \text{diag}(-1,1,1,1)$. Suppose that $\mathbb{R}_1^4$ is a 4-dimensional vector space over the field of real numbers. A symmetric bilinear form $\beta : \mathbb{R}_1^4 \times \mathbb{R}_1^4 \rightarrow \mathbb{R}$ is called

i) positive (resp. negative), definite if and only if $\omega \neq 0$ implies $\beta(\omega, \omega) > 0$ (resp. $\beta(\omega, \omega) < 0$) for all $\omega$ in $\mathbb{R}_1^4$,

ii) non-degenerate if and only if $\beta(\omega, z) = 0$ for all $z$ in $\mathbb{R}_1^4$, implying that $\omega = 0$, and

iii) indefinite if and only if there exists $\omega$ and $z$ in $\mathbb{R}_1^4$ such that $\beta(\omega, \omega) > 0$ and $\beta(z, z) < 0$.

[1]

A non-degenerate, symmetric bilinear form $\beta$ is called a scalar product. A scalar product may be positive definite, negative definite or indefinite. For an indefinite scalar product $\beta$ in $\mathbb{R}_1^4$, a nonzero vector $\omega$ is said to be

i) space-like if and only if $\beta(\omega, \omega) > 0$,

ii) time-like if and only if $\beta(\omega, \omega) < 0$,

iii) null if and only if $\beta(\omega, \omega) = 0$.

The vector $0$ is taken to be space-like. The label space-like, time-like or null is called the causal character of a vector. A curve is called time-like (or space-like) curve if the tangent vector at every point of the curve is a time-like (or space-like) vector. A surface is called time-like surface if each

*Received by the editor April 27, 2004 and in final revised form June 7, 2005
**Corresponding author
tangential bundle of the surface is a time-like subspace of $R^4_1$, [1]. A ruled surface is a surface swept out by a straight line $\ell$ moving along a curve $\alpha$. Such a surface has a parametrization in the ruled form

$$\varphi(t,v) = \alpha(t) + ve_1(t),$$

where $\alpha$ is the base curve and $e_1$ is the director vector of $\ell$. The various positions of the generating line $\ell$ are called the rulings of the surface. If the tangent plane is constant along a fixed ruling, then the ruled surface is called a developable or cylindrical surface. All other ruled surfaces are called skew surfaces [2].

2. TIME-LIKE RULED SURFACES

Let

$$\alpha : I \rightarrow R^4_1$$

$$t \rightarrow \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t))$$

be a differentiable time-like curve in the Minkowski space, where $0 \in I$. A space-like straight line,

$$\ell : R \rightarrow R^4_1$$

$$v \rightarrow \ell(v) = \alpha(t) + ve_1(t),$$

where $e_1(t)$ is the director vector of $\ell$ at the point $\alpha(t)$ such that $e_1(t)$ and the tangent vector of $\alpha$ are linearly independent at every point of the curve $\alpha$. Since $\ell$ is a space-like straight line $<e_1, e_1> = 1$, and $\dot{e}_1$ denotes the derivative of the vector field $e_1$ along the curve $\alpha$, we have $<\dot{e}_1, e_1> = 0$.

When $\ell$ moves along $\alpha$, it generates a ruled surface given by the chart $(I \times R, \varphi)$, where

$$\varphi : I \times R \rightarrow R^4_1$$

$$(t,v) \rightarrow \varphi(t,v) = \alpha(t) + ve_1(t).$$

This ruled surface will be denoted by $M$. Taking the derivatives of $\varphi$ with respect to $t$ and $v$, we have

$$\varphi_t = \dot{\alpha}(t) + v\dot{e}_1(t)$$

and $\varphi_v = e_1(t)$.

Note that $\text{rank}[\varphi_t, \varphi_v] = \text{rank}[\dot{\alpha} + v\dot{e}_1, e_1] = 2$

So $M$ is 2-manifold in the Minkowski space $R^4_1$.

3. TIME-LIKE HYPERRULED SURFACES IN THE MINKOWSKI SPACE $R^4_1$

Throughout this section we assume that

$$1 \leq i, j \leq 2 \text{ and } 0 \leq m, n \leq 2.$$
then $M$ is a 3-manifold in $R^4_1$. In this case $M$ is called a hyperruled surface and can be (locally) represented by the chart $(U, \varphi)$, where $U = I \times R^2$ and

$$\varphi: I \times R^2 \rightarrow R^4_1$$

$$(t,v) \rightarrow \varphi(t,v) = \alpha(t) + v^i e_i(t), \quad v = (v^1,v^2).$$

(4)

Suppose that the base curve $\alpha$ is an orthogonal trajectory of the generating plane $E_2(t)$. If

$$\text{rank}[e_0, e_1, e_2, \dot{e}_1, \dot{e}_2] = 4 - k$$

(5)

Then

i) if $k = 0$ in (5), then $M$ is called non-developable,

ii) if $k = 1$ in (5), then $M$ is called developable,

where $e_0$ is the unit tangent vector field of the base curve $\alpha$, which is a time-like curve, and $\dot{e}_i$ is the derivative of the vector fields $e_i$ along $\alpha$.

We begin with some properties of a general pseudo-Riemann manifold $M$. Suppose that $\overline{D}$ is the Levi-Civita connection on $R^4_1$, while $\not{D}$ is the Levi-Civita connection of $M$. Then, for any vector fields $X, Y$ on $M$, we have the Gauss equation:

$$\overline{D}_X Y = D_X Y + V(X,Y)$$

(6)

where $V$ is the second fundamental form of $M$.

If the $\overline{\xi}$ is the unit normal vector field on $M$, we have the Weingarten equation giving the tangential and normal components of $\overline{D}_X \overline{\xi}$:

$$\overline{D}_X \overline{\xi} = -A_{\overline{\xi}}(X) + D^\perp_X \overline{\xi},$$

(7)

where $A_{\overline{\xi}}$ is determined at each point of a self-adjoint linear map on $\chi(M)$, and $D^\perp$ is a metric connection in the normal bundle of $M$ [3].

Let $X, Y \in \chi(M)$ and $\overline{\xi} \in \chi(M^\perp)$. Then, by combining (6), (7) and the Minkowski inner product on $R^4_1$, denoted by $<, >$, yield that

$$<V(X,Y), \overline{\xi}> = <Y, A_{\overline{\xi}}(X)>. $$

(8)

Assume that $\{e_0, e_1, e_2\}$ is an orthonormal base field of the tangential bundle of $M$ and $\overline{\xi}$ is the unit normal vector field of $M$. Then we have the following Weingarten equation

$$\overline{D}_{e_0} \overline{\xi} = a^n_m e_n + b^n_m \overline{\xi},$$

(9)

where the Einstein summation is used. $a^n_m$'s are coefficients of the matrix $A_{\overline{\xi}}$, and

$$a^n_m = <\overline{D}_{e_0} \overline{\xi}, e_n> = -<\overline{\xi}, \overline{D}_{e_0} e_n>. $$

Since the generating space $E_2(t)$ of $M$ is a space-like subspace in $R^4_1$, we have that $<e_i, e_j> = \delta_{ij}$ and $\overline{D}_{e_i} e_j = 0$, which imply that $a_i' = 0$ and

$$a_0^n = <\overline{D}_{e_0} \overline{\xi}, e_n> = -<\overline{\xi}, \overline{D}_{e_0} e_n> = -<\overline{\xi}, \dot{e}_n> = -a_n,$$

so we may write the matrix $A_{\overline{\xi}}$ as
Lemma 3.1. Consider the orthonormal base fields $e_0, e_1, e_2$ of $M$. Then the Riemannian curvature $\kappa_\sigma(e_i, e_0)$ in the two-dimensional direction $\sigma$ of $\chi(M)$, spanned by the vector fields $e_i$ and $e_0$, is given by

$$\kappa_\sigma(e_i, e_0) = -\langle D_{e_i} e_0, D_{e_i} e_0 \rangle. \quad (10)$$

**Proof:** Suppose that $R$ is the curvature tensor of $M$, then

$$\kappa_\sigma(e_i, e_0) = \langle e_i, R(e_i, e_0) e_0 \rangle.$$  

But we see from the Gauss equation that

$$\langle e_i, R(e_i, e_0) e_0 \rangle = \langle V(e_i, e_0), V(e_0, e_0) \rangle - \langle V(e_i, e_0), V(e_i, e_0) \rangle$$

and we know that $V(e_i, e_i) = 0$. Moreover, we have

$$\langle D_{e_i} e_0, e_j \rangle = \langle e_0, D_{e_i} e_j \rangle = 0 \Rightarrow D_{e_i} e_0 \perp e_j$$

and

$$\langle D_{e_i} e_0, e_0 \rangle = \langle e_0, D_{e_i} e_0 \rangle = 0 \Rightarrow D_{e_i} e_0 \perp e_0.$$  

This means that $D_{e_i} e_0$ is a normal vector field or

$$D_{e_i} e_0 = V(e_i, e_0) \quad (11)$$

which completes the proof.

### 4. THE ALGEBRAIC INVARIANTS OF THE HYPERRULED SURFACES IN THE SPACE $R^4_1$

Let $M$ be a time-like hyperruled surface in the Minkowski 4-space $R^4_1$. Then the space of tangent vector fields of $M$ denoted by $\chi(M)$, is a time-like vector subspace of $R^4_1$ over the field of real numbers. Let $A$ be linear operator on $\chi(M)$. A characteristic value of $A$ is a scalar $\lambda$ in $R$ such that there exists a non-zero vector field $X$ in $\chi(M)$, with $A(X) = \lambda X$, where $X$ is called the characteristic vector of $A$ corresponding to $\lambda$. The set of all $X$’s is called the characteristic space of $A$.

The function $f(\lambda) = \det(A - \lambda I)$ is called the characteristic polynomial of $A$, where $I = \text{diag}(-1,1,1)$ is the matrix of the induced metric on $\chi(M)$. In order to find the roots of the characteristic polynomial we must solve the characteristic equation $\det(A - \lambda I) = 0$

$$| a_0 + \lambda - a_1 - a_2 | 
| a_1 - \lambda 0 0 | = (a_0 + \lambda)\lambda^2 - a_1^2\lambda + a_2^2\lambda = 0$$

$$| a_2 0 - \lambda |$$
or

\[ \lambda (\lambda^2 + a_0 \lambda - a_1^2 - a_2^2) = 0. \]

This implies that

\[ \lambda = 0 \quad \text{and} \quad \lambda^2 + a_0 \lambda - (a_1^2 + a_2^2) = 0 \]

Since \( \Delta = a_0^2 + 4(a_1^2 + a_2^2) > 0 \), the solution of the characteristic equation are

\[ \lambda_1 = 0, \quad \lambda_2 = -\frac{1}{2}(a_0 + \sqrt{\Delta}) \quad \text{and} \quad \lambda_3 = \frac{1}{2}(-a_0 + \sqrt{\Delta}). \]

Thus we may give the following result:

**Result 4.1.** Let \( M \) be a time-like hyperruled surface in \( \mathbb{R}_1^4 \). If \( \lambda_2 = \lambda_3 \), then \( M \) is minimal and developable.

**Proof:** Let \( \lambda_2 = \lambda_3 \), then \( \Delta = 0 \), which implies that \( a_0 = a_1 = a_2 = 0 \).

Thus \( a_0 = 0 \) implies that \( \text{tr} A = 0 \), and so \( M \) is minimal.

By lemma 1, \( a_i = 0 \) implies that \( \kappa(e_i, e_0) = 0 \) and so \( M \) is developable.

Let us find the characteristic vector corresponding to characteristic values \( \lambda_1, \lambda_2, \lambda_3 \) of the matrix \( A \). The vector field \( X_1 \) corresponding to \( \lambda_1 \) is obtained by the solution of the equation

\[ AX_1 = 0 \Leftrightarrow X_1(t) = \left(0, t, -\frac{a_1}{a_2} t\right). \]

Similarly, the vector fields \( X_2 \) and \( X_3 \), corresponding to \( \lambda_2 \) and \( \lambda_3 \), are obtained by the solutions of the equations

\[ AX_2 = \lambda_2 X_2 \Leftrightarrow X_2(t) = \left(t, -\frac{a_1}{\lambda_2} t, -\frac{a_2}{\lambda_2} t\right) \]

and

\[ AX_3 = \lambda_3 X_3 \Leftrightarrow X_3(t) = \left(t, -\frac{a_1}{\lambda_3} t, -\frac{a_2}{\lambda_3} t\right) \]

Since the vector fields \( X_k(t), k = 1, 2, 3 \) have one arbitrary parameter, the dimension of the characteristic space is equal to 1. Therefore, we can choose an orthonormal base field \( \phi = \{X_1, X_2, X_3\} \) of \( \chi(M) \) corresponding to characteristic values \( \lambda_1, \lambda_2, \lambda_3 \).

If we denote the matrix of the linear map \( A \) with respect to the orthogonal base \( \phi \) by \( S \), then we observe that

\[ S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \]

\( S \) is called as the *Weingarten* (or *Shape*) operator of \( M \) with respect to the base \( \phi \).

Thus we can state the following results:
**Result 4.2.** Let $M$ be a time-like hyperruled surface in $R_1^4$, and $S$ be the shape operator of $M$. Then

i) The main curvature of $M$ is $\|H\| = -\frac{d_\varphi}{3}$.

ii) The Gauss curvature of $M$ is $\kappa = 0$.

**Definition 4.1.** Let $M$ be a time-like hyperruled surface with curvature tensor $R$ in $R_1^4$. If $\{e_0, e_1, e_2\}$ is an orthonormal base field of $\chi(M)$, then the Ricci curvature tensor $S$ is defined by

$$S: \chi(M) \times \chi(M) \rightarrow R$$

$$(X, Y) \rightarrow S(X, Y) = \sum_m e_m < R(e_m, X), e_m >$$

where

$$e_m = \begin{cases} -1, & m = 0 \\ 1, & m = 1,2. \end{cases}$$

The scalar curvature of $M$ is defined by

$$r = \sum_m S(e_m, e_m),$$

and the scalar normal curvature of $M$ is defined by

$$r_n = \sum_{\sigma, \nu} M(\xi_\sigma, \xi_\nu - \xi_\nu, \xi_\sigma) \xi_\sigma, \nu \in \{1,2\}, [4].$$

Thus we can find the following results for the time-like hyperruled surfaces in the Minkowski 4-space $R_1^4$:

**Result 4.3.** Let $M$ be a time-like hyperruled surface with a base curve $\alpha$ and the generating space $E_2(t)$ spanning by the vectors $e_i(t)$ in the Minkowski 4-space $R_1^4$. Then the scalar curvature of $M$ is

$$r = -2\sum_i a_i^2,$$

where $a_i = \langle \xi, \dot{e}_i \rangle$ and $\xi \in \chi(M^\perp)$.

**Proof:** Let $\{e_0, e_1, e_2\}$ be an orthonormal base field of $M$. Then

$$r = \sum_m S(e_m, e_m) = S(e_0, e_0) + \sum_i S(e_i, e_i)$$

$$S(e_0, e_0) = \sum_m < R(e_m, e_0), e_0, e_m >$$

$$= \sum_i < R(e_i, e_0), e_0, e_i >$$

$$= \sum_i \kappa(e_i, e_0) = -\sum_i a_i^2$$

$$S(e_i, e_i) = \sum_m < R(e_m, e_i), e_i, e_m > = \kappa(e_i, e_0) = -a_i^2.$$
which implies that $S(e_0, e_0) = -\sum S(e_i, e_i)$.

So we have $r = -2\sum S(e_i, e_i) = -2\sum a_i^2$.

Thus we derive the following results for a time-like hyperruled surface in $R^4_1$:

i) $r = 0$ if $M$ is developable,

ii) $r = 0$ and $M$ is minimal if $M$ is hyperplane,

iii) The scalar normal curvature of $M$ is always zero.

**Acknowledgements** - The author would like to express his pleasure to Prof. Dr. Feyzi Başar for his valuable help during the revision of this paper. The author also wishes to thank Prof. Dr. Sadık Keleş, for suggesting this problem.

**REFERENCES**


